Section 5.6 – Integration by Parts

Integration by parts is another technique that we can use to integrate problems. Typically, we save integration by parts as a last resort when substitution will not work. To see where the integration by parts formula comes from, let's examine the product rule for differentiation and then integrate both sides:

"$\frac{d}{dx}[1^{st} \bullet 2^{nd}] = 1^{st} \bullet$ Derivative of the 2^{nd} $+ 2^{nd} \bullet$ Derivative of the 1^{st} "

$$\frac{d}{dx}[u \bullet v] = u \bullet \frac{dv}{dx} + v \bullet \frac{du}{dx}$$

Now, integrate both sides with respect to x:

$$\int \frac{d}{dx}[u \bullet v]\, dx = \int \left(u \bullet \frac{dv}{dx} + v \bullet \frac{du}{dx}\right)dx$$

Integrate term by term on the right side:

$$\int \frac{d}{dx}[u \bullet v]\, dx = \int u \bullet \frac{dv}{dx}\, dx + \int v \bullet \frac{du}{dx}\, dx$$

Pretend that $\frac{dv}{dx}$ and $\frac{du}{dx}$ are fractions & reduce the "dx".

$$\int \frac{d}{dx}[u \bullet v]\, dx = \int u \bullet dv + \int v \bullet du$$

Integrate the left side:

$$u \bullet v = \int u \bullet dv + \int v \bullet du$$

Now subtract $\int v \bullet du$ from both sides:

$$u \bullet v - \int v \bullet du = \int u \bullet dv$$

Finally, switch the sides to give us the formula:

Integration by Parts formula:

$$\int u \bullet dv = u \bullet v - \int v \bullet du$$

In using this formula, we need to examine what we are integrating and identify a function and a derivative. We then differentiate the function and integrate the derivative. Finally, we plug into the Integration by Parts formula and finish the problem. Let's outline the steps:

Integration by Parts:

I) Identify u and dv.

II) Find du by differentiating u and find v by integrating dv. When finding v, do not worry about + c.

III) Plug our results into $\int u \cdot dv = u \cdot v - \int v \cdot du$ & integrate.

Note if $\int v \cdot du$ is worse than the integral we started with, we made the wrong choice for u and dv in step #I.

When integrating in general, one should follow these steps:

1) Check to see if we can integrate using one of the **Rules of Integration** from Sect 5.1.
2) If that does not work, see if we can integrate by **Substitution**.
3) If that fails, then we will use **Integration by Parts**.

Integrate the following:

Ex. 1 $\int xe^{3x}\,dx$.

Solution:
This problem does not fall into one of our rules of integration from sect 5.1. Also, substitution will not help either. Thus, we need to use integration by parts:

I) Let u = x and $dv = e^{3x}\,dx$.

II) Then du = dx and $v = \int e^{3x}\,dx = \frac{e^{3x}}{3}$.

III) Plug into the integration by parts formula & integrate:

$$\int u \cdot dv = u \cdot v - \int v \cdot du$$

$$\int xe^{3x}\,dx = x \cdot \frac{e^{3x}}{3} - \int \frac{e^{3x}}{3} \cdot dx = \frac{xe^{3x}}{3} - \frac{1}{3}\int e^{3x}\,dx$$

$$= \frac{xe^{3x}}{3} - \frac{1}{3} \cdot \frac{e^{3x}}{3} + c = \frac{xe^{3x}}{3} - \frac{e^{3x}}{9} + c.$$

Notice that we use + c at the end of the problem since we have an indefinite integral. We do not need to use + c when find v since we take care of it at the end.

Ex. 2 $\int_1^2 4t \ln(t)\,dt$.

Solution:

First, find the indefinite integral. This problem does not fall into one of our rules of integration from sect 5.1. Also, substitution will not help either. Thus, we need to use integration by parts:

I) Let $u = \ln(t)$ and $dv = 4t\,dt$.

II) Then $du = \frac{1}{t}\,dt$ and $v = \int 4t\,dt = \frac{4t^2}{2} = 2t^2$.

III) Plug into the integration by parts formula and integrate:

$$\int u \cdot dv = u \cdot v - \int v \cdot du$$

$$\int 4t \ln(t)\,dt = \ln(t) \cdot 2t^2 - \int 2t^2 \cdot \frac{1}{t}\,dt$$

$$= 2t^2 \ln(t) - \int 2t\,dt = 2t^2 \ln(t) - t^2 + c.$$

Thus, $\int_1^2 4t \ln(t)\,dt = 2t^2 \ln(t) - t^2 \Big|_1^2$

$= [2(2)^2 \ln(2) - (2)^2] - [1(1)^2 \ln(1) - (1)^2]$

$= [8 \ln(2) - 4] - [0 - 1] = 8 \ln(2) - 3 \approx 2.5452$

Ex. 3 $\int 5e^{-4x}\,dx$

Solution:

Note: $\int 5e^{-4x}\,dx = 5\int e^{-4x}\,dx$. Since this falls under one of our rules of integration from Section 5.1 (#4 $\int e^{kx}\,dx = \frac{e^{kx}}{k} + c,\ k \neq 0$), we will use that to integrate:

$$5\int e^{-4x}\,dx = 5\frac{e^{-4x}}{-4} + c = -1.25e^{-4x} + c.$$

Ex. 4 $\int_0^1 x^5 e^{x^3} dx$

Solution:

First, find the indefinite integral. This problem does not fall into one of our rules of integration from sect 5.1. Also, substitution will not help either. Thus, we need to use integration by parts: We would like to use $e^{x^3} dx$ as our dv, but we are missing a factor of x^2 since the derivative of $x^3 = 3x^2$. But, we can rewrite the integral as:

$$\int x^5 e^{x^3} dx = \int x^3 \tfrac{1}{3}\cdot(3x^2 e^{x^3})dx = \tfrac{1}{3}\int x^3(3x^2 e^{x^3})dx$$

I) Let $u = x^3$ and $dv = 3x^2 e^{x^3} dx$

II) Then $du = 3x^2\, dx$ and $v = \int 3x^2 e^{x^3} dx$.

To find v, we will need to use substitution:
Let $u^* = x^3$, then $du^* = 3x^2\, dx$. Substituting, we get:

$$v = \int 3x^2 e^{x^3} dx = \int e^{u^*} du^* = e^{u^*} + c.$$

Replacing u^* by x^3, we obtain:

$v = e^{x^3}$ (ignore the + c for now).

III) Plug into the integration by parts formula & integrate:

$$\int u\cdot dv = u\cdot v - \int v\cdot du$$

$$\tfrac{1}{3}\int x^3(3x^2 e^{x^3})dx = \tfrac{1}{3}[x^3\cdot e^{x^3} - \int e^{x^3} 3x^2\, dx\,]$$

But, from above, $\int e^{x^3} 3x^2\, dx = e^{x^3} + c$, Thus,

$$\tfrac{1}{3}[x^3\cdot e^{x^3} - \int e^{x^3} 3x^2\, dx\,] = \tfrac{1}{3}[x^3\cdot e^{x^3} - e^{x^3}\,] + c$$

$$= \frac{x^3 e^{x^3}}{3} - \frac{e^{x^3}}{3} + c.$$

Thus, $\int_0^1 x^5 e^{x^3} dx = \left.\frac{x^3 e^{x^3}}{3} - \frac{e^{x^3}}{3}\,\right|_0^1$

$$= \left(\frac{(1)^3 e^{(1)^3}}{3} - \frac{e^{(1)^3}}{3}\right) - \left(\frac{(0)^3 e^{(0)^3}}{3} - \frac{e^{(0)^3}}{3}\right)$$
$$= \left(\frac{e}{3} - \frac{e}{3}\right) - \left(0 - \frac{1}{3}\right) = \frac{1}{3}$$

Ex. 5 $\int x\sqrt{x-3}\ dx$

Solution:

This problem does not fall into one of our rules of integration from sect 5.1. However, we can use substitution to work this problem:

Let $u = x - 3$ and $u + 3 = x$ (add 3 to both sides)

$du = dx$

Substituting in, we get:

$$\int x\sqrt{x-3}\ dx = \int (u + 3)\sqrt{u}\ du$$
$$= \int (u + 3)\ u^{1/2}\ du = \int (u^{3/2} + 3u^{1/2})\ du$$
$$= \frac{u^{5/2}}{\frac{5}{2}} + 3\frac{u^{3/2}}{\frac{3}{2}} + c = \frac{2}{5}\sqrt{u^5} + 2\sqrt{u^3} + c$$

Replacing u by x – 3, we obtain:

$$= \frac{2}{5}\sqrt{(x-3)^5} + 2\sqrt{(x-3)^3} + c.$$

Ex. 6 Find $\int x\sqrt{x-3}\ dx$ using integration by parts

Solution:

Now, we are being forced to use integration by parts:

I) Let $u = x$ and $dv = \sqrt{x-3}\ dx = (x - 3)^{1/2}\ dx$

II) Then $du = dx$ and $v = \int (x - 3)^{1/2}\ dx$.

To find v, we will use substitution:

Let $u^* = (x - 3)$

$du^* = dx$

Substituting, we find:

$$v = \int (x - 3)^{1/2}\ dx = \int u^{*1/2}\ du^* = \frac{u^{*3/2}}{\frac{3}{2}} + c = \frac{2}{3}u^{*3/2} + c.$$

Replacing u* by (x – 3) yields:

$v = \frac{2}{3}(x - 3)^{3/2}$ (ignore the + c for now)

III) Plug into the integration by parts formula & integrate:

$\int u \cdot dv = u \cdot v - \int v \cdot du$

$\int x\sqrt{x - 3}\ dx = x \cdot \frac{2}{3}(x - 3)^{3/2} - \int \frac{2}{3}(x - 3)^{3/2}\ dx$

$= \frac{2}{3}x(x - 3)^{3/2} - \frac{2}{3}\int (x - 3)^{3/2}\ dx$

To find $\int (x - 3)^{3/2}\ dx$, we will need to use substitution again: Let $u^{**} = (x - 3)$

$du^{**} = dx$

Substituting yields:

$v = \int (x - 3)^{3/2}\ dx = \int u^{**3/2}\ du^{**} = \frac{u^{**5/2}}{\frac{5}{2}} + c$

$= \frac{2}{5}u^{**5/2} + c.$

Replacing u^{**} by $(x - 3)$ yields:

$\frac{2}{5}(x - 3)^{5/2}$ (ignore the + c for now)

Hence, $\frac{2}{3}x(x - 3)^{3/2} - \frac{2}{3}\int (x - 3)^{3/2}\ dx$

$= \frac{2}{3}x(x - 3)^{3/2} - \frac{2}{3} \cdot \frac{2}{5}(x - 3)^{5/2} + c$

$= \frac{2}{3}x(x - 3)^{3/2} - \frac{4}{15}(x - 3)^{5/2} + c$

$= \frac{2}{3}x\sqrt{(x - 3)^3} - \frac{4}{15}\sqrt{(x - 3)^5} + c$

Notice that the answer appears to be different from the answer in Example #5 ($\frac{2}{5}\sqrt{(x - 3)^5} + 2\sqrt{(x - 3)^3} + c$). These two answers are actually equivalent. To see how, consider:

Ex. 7 Show that the answers from Example 5 and 6 are equal.

<u>Solution:</u>

We need to show that:

$\frac{2}{3}x\sqrt{(x - 3)^3} - \frac{4}{15}\sqrt{(x - 3)^5} = \frac{2}{5}\sqrt{(x - 3)^5} + 2\sqrt{(x - 3)^3}$

Build each side up so it has a denominator of 15:

$$\frac{5\cdot 2x\sqrt{(x-3)^3}}{5\cdot 3} - \frac{4\sqrt{(x-3)^5}}{15} = \frac{3\cdot 2\sqrt{(x-3)^5}}{3\cdot 5} + \frac{15\cdot 2\sqrt{(x-3)^3}}{15}$$

$$\frac{10x\sqrt{(x-3)^3} - 4\sqrt{(x-3)^5}}{15} = \frac{6\sqrt{(x-3)^5} + 30\sqrt{(x-3)^3}}{15}$$

Factor out $\sqrt{(x-3)^3}$ from the numerators:

$$\frac{\sqrt{(x-3)^3}\,[10x - 4\sqrt{(x-3)^2}\,]}{15} = \frac{\sqrt{(x-3)^3}\,[6\sqrt{(x-3)^2} + 30]}{15}$$

But $\sqrt{(x-3)^2} = |\,x - 3\,| = x - 3$ since $x - 3 \geq 0$ (for the original problem $\int x\sqrt{x-3}\,dx$ to be defined, $x - 3$ has to be greater than or equal to 0).

$$\frac{\sqrt{(x-3)^3}\,[10x - 4(x-3)]}{15} = \frac{\sqrt{(x-3)^3}\,[6(x-3) + 30]}{15}$$

$$\frac{\sqrt{(x-3)^3}\,[10x - 4x + 12]}{15} = \frac{\sqrt{(x-3)^3}\,[6x - 18 + 30]}{15}$$

$$\frac{\sqrt{(x-3)^3}\,[6x+12]}{15} = \frac{\sqrt{(x-3)^3}\,[6x+12]}{15}$$

Hence, they are equal.

Given the choice of substitution or integration by parts, many times it is easier to use substitution as Examples 5 and 6 illustrate. That's why we use integration by parts as a last resort.

Integrate the following:

Ex. 8 $\int_1^5 \frac{x-3}{x}\,dx$

Solution:

If we go ahead and divide, we can then integrate using our rules of integration from sect 5.1:

$$\int_1^5 \frac{x-3}{x}\,dx = \int_1^5 (1 - \tfrac{3}{x})\,dx = \int_1^5 dx \; - 3\int_1^5 \frac{1}{x}\,dx$$

$$= x - 3\ln(|\,x\,|)\,\Big|_1^5 = [5 - 3\ln(5)] - [1 - 3\ln(1)]$$

$$= [5 - 3\ln(5)] - [1 - 0] = 4 - 3\ln(5) \approx -0.8283.$$

Ex. 9 $\int x^2e^x\,dx$

<u>Solution:</u>

This problem does not fall into one of our rules of integration from sect 5.1. Also, substitution will not help either. Thus, we need to use integration by parts:

I) Let $u = x^2$ and $dv = e^x\,dx$.

II) Then $du = 2x\,dx$ and $v = \int e^x\,dx = e^x$.

III) Plug into the integration by parts formula & integrate:

$$\int u\cdot dv = u\cdot v - \int v\cdot du$$

$$\int x^2e^x\,dx = x^2e^x - \int e^x\cdot 2x\,dx = x^2e^x - 2\left[\int xe^x\,dx\right]$$

To integrate $\int xe^x\,dx$, we need to use integration by parts again:

I) Let $u^* = x$ and $dv^* = e^x\,dx$.

II) Then $du^* = dx$ and $v^* = \int e^x\,dx = e^x$.

III) Plug into the integration by parts formula & integrate:

$$\int u^*\cdot dv^* = u^*\cdot v^* - \int v^*\cdot du^*$$

$$\int xe^x\,dx = xe^x - \int e^x\,dx = xe^x - e^x.$$

(Ignore + c for now). Thus,

$$x^2e^x - 2\left[\int xe^x\,dx\right] = x^2e^x - 2\left[xe^x - e^x\right] + c$$

$$= x^2e^x - 2xe^x + 2e^x + c.$$

Ex. 10 $\int \frac{2x}{2x+5}\, dx$

Solution:

This problem does not fall into one of our rules of integration from sect 5.1. However, we can use substitution to work this problem:

Let u = 2x + 5 and u – 5 = 2x

(Subtract 5 from both sides)

du = 2 dx, solving for dx yields:

$\frac{du}{2} = dx$

Substituting in, we get:

$$\int \frac{2x}{2x+5}\, dx = \int \frac{u-5}{u} \frac{du}{2} = \frac{1}{2}\int \frac{u-5}{u}\, du = \frac{1}{2}\int (1 - \frac{5}{u})\, du$$

$$= \frac{1}{2}\int du - \frac{5}{2}\int \frac{1}{u}\, du = \frac{1}{2}u - \frac{5}{2}\ln(|u|) + c.$$

Replacing u by 2x + 5 yields:

$$\frac{1}{2}u - \frac{5}{2}\ln(|u|) + c = \frac{1}{2}(2x + 5) - \frac{5}{2}\ln(|2x + 5|) + c$$

$$= x + \frac{5}{2} - \frac{5}{2}\ln(|2x + 5|) + c.$$

Ex. 11 $\int \ln(x)\, dx$

Solution:

This problem does not fall into one of our rules of integration from sect 5.1. Also, substitution will not help either. Thus, we need to use integration by parts:

I) Let u = ln (x) and dv = dx.

II) Then $du = \frac{1}{x}\, dx$ and $v = \int dx = x$.

III) Plug into the integration by parts formula and integrate:

$$\int u \cdot dv = u \cdot v - \int v \cdot du$$

$$\int \ln(x)\, dx = x\ln(x) - \int x \cdot \frac{1}{x}\, dx = = x\ln(x) - \int 1\, dx$$

$$= x\ln(x) - x + c.$$