Sect 10.2 - Polynomial and Rational Inequalities

Concept #1 Solving Inequalities Graphically

Definition

A **Quadratic Inequality** is an inequality that can be written in one of the following forms:

$ax^2 + bx + c \geq 0$ $ax^2 + bx + c \leq 0$

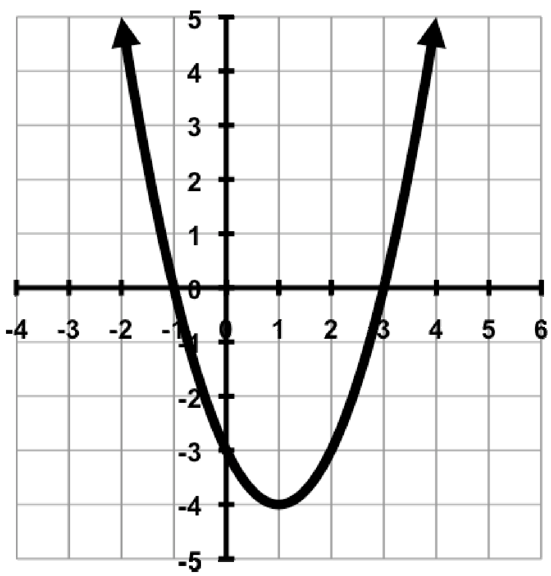
$ax^2 + bx + c > 0$ $ax^2 + bx + c < 0$

where a, b, and c are real numbers and $a \neq 0$.

When graphing quadratic equations in the form $y = ax^2 + bx + c$, we saw that the graph was a parabola. If $a > 0$, then the parabola opens upward and if $a < 0$, then the parabola opens downward. If we are solving the inequality $ax^2 + bx + c > 0$, we are ask for what values of x is y positive or above the x-axis. Similarly, if we are solving the inequality $ax^2 + bx + c < 0$, we are ask for what values of x is y negative or below the x-axis. If we are given the graph of the parabola, we can use this idea to solve a quadratic inequality.

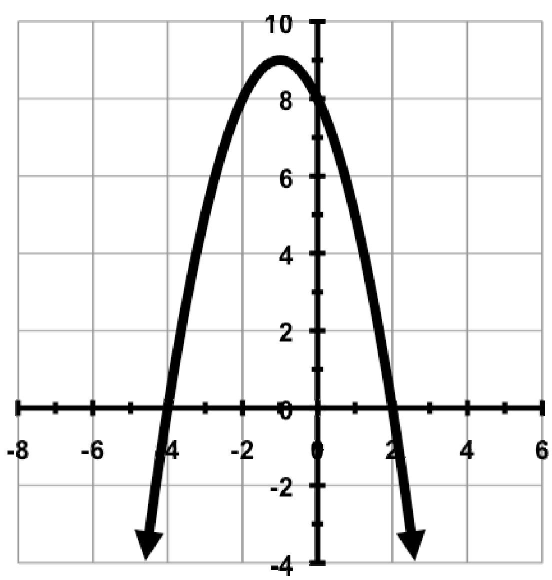
Use the given graphs to solve the following quadratic inequalities:

Ex. 1 $f(x) = x^2 - 2x - 3$



a) $x^2 - 2x - 3 > 0$
b) $x^2 - 2x - 3 \geq 0$
c) $x^2 - 2x - 3 < 0$
d) $x^2 - 2x - 3 \leq 0$

Ex. 2 $f(x) = -x^2 - 2x + 8$



a) $-x^2 - 2x + 8 > 0$
b) $-x^2 - 2x + 8 \geq 0$
c) $-x^2 - 2x + 8 < 0$
d) $-x^2 - 2x + 8 \leq 0$

Solution:
The x-intercepts are (– 1, 0) and (3, 0). The y-values are positive when the graph is above the x-axis and negative when the graph is below the x-axis:

a) $x^2 - 2x - 3 > 0$ on $(-\infty, -1) \cup (3, \infty)$.

b) Now, we include the zeros: $x^2 - 2x - 3 \geq 0$ on $(-\infty, -1] \cup [3, \infty)$

c) $x^2 - 2x - 3 < 0$ on $(-1, 3)$.

d) Now, we include the zeros: $x^2 - 2x - 3 \leq 0$ on $[-1, 3]$

Solution:
The x-intercepts are (– 4, 0) and (2, 0). The y-values are positive when the graph is above the x-axis & negative when the graph is below the x-axis:

a) $-x^2 - 2x + 8 > 0$ on $(-4, 2)$.

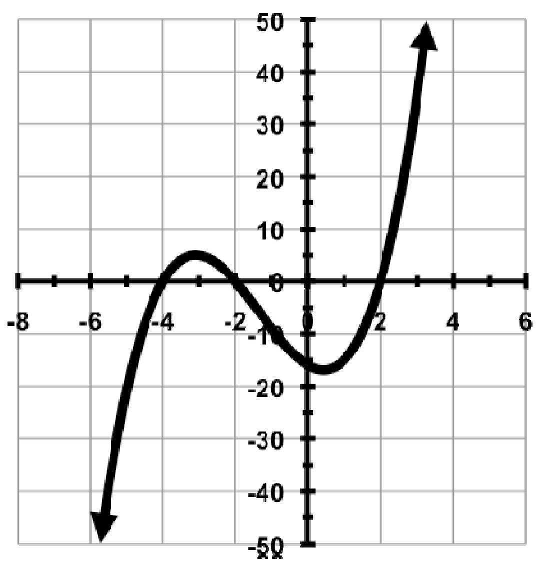
b) Now, we include the zeros: $-x^2 - 2x + 8 \geq 0$ on $[-4, 2]$

c) $-x^2 - 2x + 8 < 0$ on $(-\infty, -4) \cup (2, \infty)$.

d) Now, we include the zeros: $-x^2 - 2x + 8 \leq 0$ on $(-\infty, -4] \cup [2, \infty)$.

Use the given graphs to solve the following inequalities:

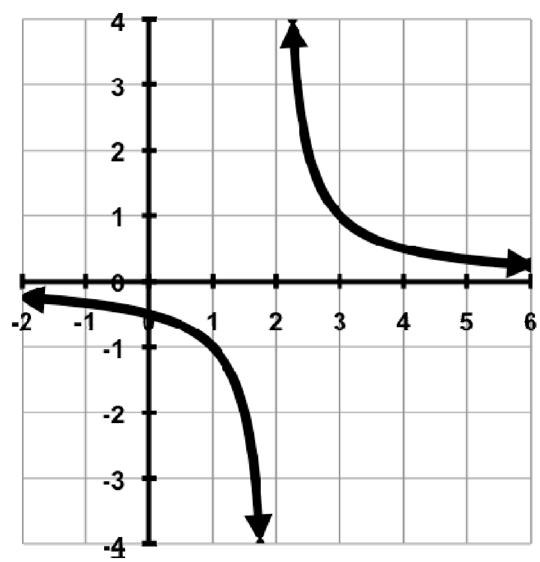
Ex. 3 $f(x) = x^3 + 4x^2 - 4x - 16$



a) $x^3 + 4x^2 - 4x - 16 > 0$

b) $x^3 + 4x^2 - 4x - 16 \leq 0$

Ex. 4 $f(x) = \frac{1}{x-2}$



a) $\frac{1}{x-2} \geq 0$

b) $\frac{1}{x-2} < 0$

Solution:

a) The function is positive when it is above the x-axis. The solution is $(-4, -2) \cup (2, \infty)$

b) The function is negative when it is below the x-axis. We also need to include the zeros. The solution is $(-\infty, -4] \cup [-2, 2]$

Solution:

a) The function is positive when it is above the x-axis. There are no zeros. The solution is $(2, \infty)$.

b) The function is negative when it is below the x-axis. The solution is $(-\infty, 2)$.

Notice that we did not include x = 2 as part of the solution to example #4a. This is because the function $\frac{1}{x-2}$ is undefined x = 2 and not a zero. We did include x = – 4, x = – 2, and x = 2 as part of the solution to example #3b since the polynomial $x^3 + 4x^2 - 4x - 16$ is equal to zero at those values. The values that make a function zero or undefined are called **boundary values**. These values are where the values of the function y-values) can change sign. In other words, the sign of the function will either be always positive or always negative between two consecutive boundary values.

Concepts #2 and #3 Solving Inequalities Algebraically

If we do not have the graph of the function, we can still solve the inequality algebraically. This is done by first finding the boundary values. We use the boundary values to divide the x-values into intervals. We pick a value of x in each of the intervals and plug it into the function. Whatever sign we get for the function value will be the sign of the function in that interval. We then use these results to solve the inequality.

Solving Polynomial and Rational Inequalities Algebraically

1) Get zero on one side of the inequality and simplify.
2) Factor the function on the other side completely.
3) Find the boundary values (the zeros and where the function is undefined).
4) Mark the boundary values on a number line and pick a test value of x in each interval.
5) Plug each x-value into the function to determine the sign of the function is each interval. (a sign chart may be helpful)
6) Use the results to solve the inequality.

Solve the following inequalities:

Ex. 5 $-2x^3 + 4x^2 + 50x > 100$

Solution:

1) Get zero on one side:

$-2x^3 + 4x^2 + 50x > 100$

$-2x^3 + 4x^2 + 50x - 100 > 0$

2) Factor:

$-2x^3 + 4x^2 + 50x - 100 > 0$ (G.C.F. = – 2)

$-2(x^3 - 2x^2 - 25x + 50) > 0$ (factor by grouping)

$-2([x^3 - 2x^2] + [-25x + 50]) > 0$

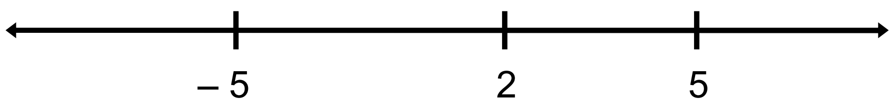
$-2(x^2[x - 2] - 25[x - 2]) > 0$

$-2(x - 2)(x^2 - 25) > 0$ (difference of squares)

$-2(x - 2)(x - 5)(x + 5) > 0$

3) The zeros are x = 2, 5, and – 5. These are the only boundary values.

4) Mark the boundary values on a number line:

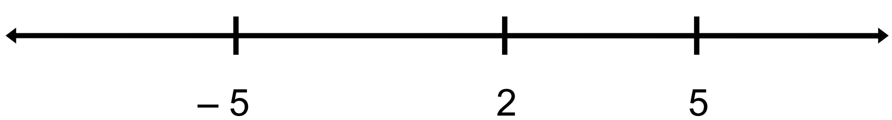


Pick a test value in each interval:

Test: – 6 0 3 6

5) Plug each x-value into the function to determine the sign:

Test:	– 6	0	3	6
– 2	–	–	–	–
(x – 2)	–	–	+	+
(x – 5)	–	–	–	+
(x + 5)	–	+	+	+



Sign	+	–	+	–

6) Since $-2x^3 + 4x^2 + 50x - 100 > 0$, we want the intervals where the function is positive without the zeros.

The solution is $(-\infty, -5) \cup (2, 5)$.

Ex. 6 $(x^2 + 4x + 4)(3x^2 - 5x - 2) \le 0$

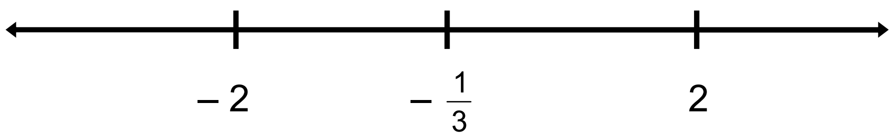
<u>Solution:</u>

1) We already have zero on one side:
$(x^2 + 4x + 4)(3x^2 - 5x - 2) \le 0$

2) Factor:
$(x^2 + 4x + 4)(3x^2 - 5x - 2) \le 0$ (perfect square trinomial)
$(x + 2)^2(3x^2 - 5x - 2) \le 0$ (factor the trinomial)
$(x + 2)^2(3x + 1)(x - 2) \le 0$

3) The zeros are $x = -2, -\frac{1}{3}$, and 2. These are the only boundary values.

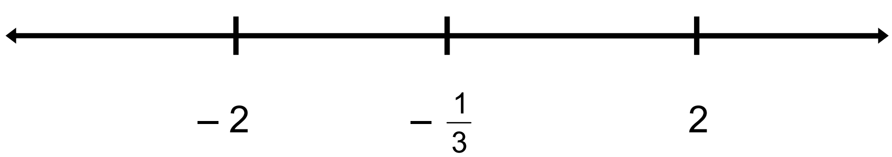
4) Mark the boundary values on a number line:



Pick a test value in each interval:
Test: – 3 – 1 0 3

5) Plug each x-value into the function to determine the sign:

Test:	– 3	– 1	0	3
$(x + 2)^2$	+	+	+	+
$(3x + 1)$	–	–	+	+
$(x - 2)$	–	–	–	+



Sign: + + – +

6) Since $(x + 2)^2(3x + 1)(x - 2) \le 0$, we want the intervals where the function is negative with the zeros.
The solution is $[-\frac{1}{3}, 2]$.

To solve rational inequalities, we can use the exact same procedure. The only difference is there is ≥ or ≤ symbol in the rational inequality, we only include the boundary values that are zeros of the function. The boundary values that make the rational function undefined are not included in the solution.

Solve the following inequalities:

Ex. 7 $\frac{4x^2-9}{x^3+27} \geq 0$

Solution:

1) We already have zero on one side:

$$\frac{4x^2-9}{x^3+27} \geq 0$$

2) Factor:

$$\frac{4x^2-9}{x^3+27} \geq 0 \qquad \text{(difference of squares)}$$

$$\frac{(2x-3)(2x+3)}{x^3+27} \geq 0 \qquad \text{(sum of cubes)}$$

$$\frac{(2x-3)(2x+3)}{(x+3)(x^2-3x+9)} \geq 0$$

3) The zeros are x = 1.5 and – 1.5. The function is undefined at x = – 3. The boundary values are – 3. – 1.5, and 1.5.

4) Mark the boundary values on a number line:

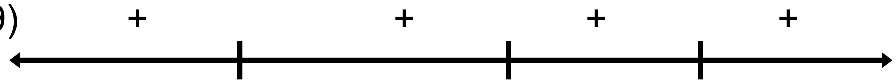
– 3 – 1.5 1.5

Pick a test value in each interval:

Test: – 4 – 2 0 2

5) Plug each x-value into the function to determine the sign:

Test:	– 4	– 2	0	2
(2x – 3)	–	–	–	+
(2x + 3)	–	–	+	+
(x + 3)	–	+	+	+
$(x^2 – 3x + 9)$	+	+	+	+



– 3 – 1.5 1.5

Sign	–	+	–	+

6) Since $\frac{(2x-3)(2x+3)}{(x+3)(x^2-3x+9)} \geq 0$, we want the intervals where the function is positive with the zeros.

The solution is (– 3, – 1.5] ∪ [1.5, ∞).

Ex. 8 $\frac{3x-4}{2x+5} < 1$

<u>Solution:</u>

1) Get zero on one side and simplify:

$\frac{3x-4}{2x+5} < 1$

$\frac{3x-4}{2x+5} - 1 < 0$

$\frac{3x-4}{2x+5} - \frac{2x+5}{2x+5} < 0$

$\frac{3x-4-2x-5}{2x+5} < 0$

$\frac{x-9}{2x+5} < 0$

2) No factoring needed:

$\frac{x-9}{2x+5} < 0$

3) The zero is x = 9. The function is undefined at x = – 2.5
The boundary values are – 2.5 and 9.

4) Mark the boundary values on a number line:

– 2.5 9

Pick a test value in each interval:

Test: – 3 0 10

5) Plug each x-value into the function to determine the sign:

	Test: – 3	0	10
(x – 9)	–	–	+
(2x + 5)	–	+	+

– 2.5 9

Sign	+	–	+

6) Since $\frac{x-9}{2x+5} < 0$, we want the intervals where the function is negative without the zeros.
The solution is (– 2.5, 9).

Concept #4 Special Cases

Solve the following inequalities:

Ex. 9 $x(x - 26) < -169$

Solution:

1) Get zero on one side and simplify:

$x(x - 26) < -169$

$x(x - 26) + 169 < 0$

$x^2 - 26x + 169 < 0$

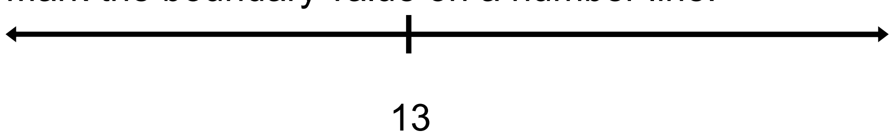
2) Factor:

$x^2 - 26x + 169 < 0$ (perfect square trinomial)

$(x - 13)^2 < 0$

3) The zero is x = 13. The boundary value is 13.

4) Mark the boundary value on a number line:



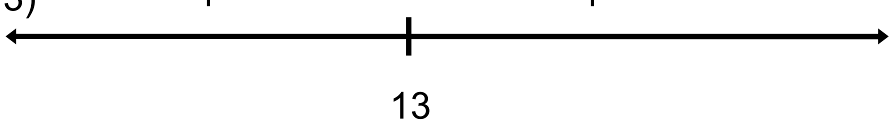
Pick a test value in each interval:

Test: 0 14

5) Plug each x-value into the function to determine the sign:

Test: 0 14

$(x - 13)^2$ + +



Sign + +

6) Since $(x - 13)^2 < 0$, we want the intervals where the function is negative without the zeros, but there are no such intervals. The solution is { }.

Ex. 10 $\frac{-4}{x^2+25} < 0$

Solution:

1) We already have zero on one side:

$\frac{-4}{x^2+25} < 0$

2) There is no factoring that can be done:

$\frac{-4}{x^2+25} < 0$

3) There are no zeros and the function defined for all real numbers. There are no boundary values.

4) There are no boundary values to mark on a number line:

Pick a test value in the interval:

Test: 0

5) Plug each x-value into the function to determine the sign:

Test: 0

$\frac{-4}{x^2+25}$ –

Sign –

6) Since $\frac{-4}{x^2+25} < 0$, we want the intervals where the function is negative without the zeros, but this is the entire number line. The solution is $(-\infty, \infty)$.

Ex. 11 $x^2 + 8x > -16$

Solution:

1) Get zero on one side:

$x^2 + 8x > -16$

$x^2 + 8x + 16 > 0$

2) Factor:

$x^2 + 8x + 16 > 0$ (perfect square trinomial)

$(x + 4)^2 > 0$

3) The zero is $x = -4$. The boundary value is -4.

4) Mark the boundary value on a number line:

– 4

Pick a test value in each interval:

Test: – 5 0

5) Plug each x-value into the function to determine the sign:

Test:	– 5	0
$(x + 4)^2$	+	+

– 4

Sign	+	+

6) Since $(x + 4)^2 > 0$, we want the intervals where the function is positive without the zeros, which is both intervals.
The solution is $(-\infty, -4) \cup (-4, \infty)$.

Ex. 12 $\frac{9}{x^2+9} \geq 1$

Solution:

1) Get zero on one side and simplify:

$$\frac{9}{x^2+9} \geq 1$$

$$\frac{9}{x^2+9} - 1 \geq 0$$

$$\frac{9}{x^2+9} - \frac{x^2+9}{x^2+9} \geq 0$$

$$\frac{-x^2}{x^2+9} \geq 0$$

2) There is no factoring that can be done:

$$\frac{-x^2}{x^2+9} \geq 0$$

3) The zero is x = 0. The boundary value is 0.

4) Mark the boundary value on a number line:

0

Pick a test value in each interval:

Test:	– 2	2

5) Plug each x-value into the function to determine the sign:

	Test:	-2	2
$-x^2$		$-$	$-$
$x^2 + 9$		$+$	$+$
		← 0 →	
Sign		$-$	$-$

6) Since $\frac{-x^2}{x^2+9} \geq 0$, we want the intervals where the function is positive with the zeros. There are no intervals where the function is positive so our only solution is the zero.
The solution is {0}.