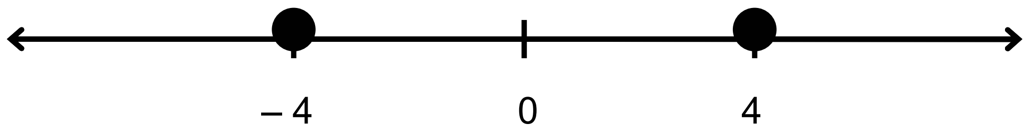
Sect 10.4 - Solving Absolute Values Inequalities

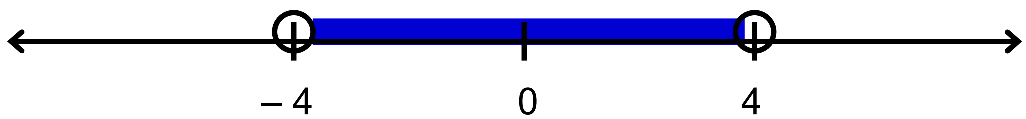
Concept #1 Solving Absolute Value Inequalities using the Definition

In the last section, we saw that the solution to $|x| = 4$ was $x = -4$ and $x = 4$. If we plot these values on a number line, they act as boundary values that we found in section 10.2:

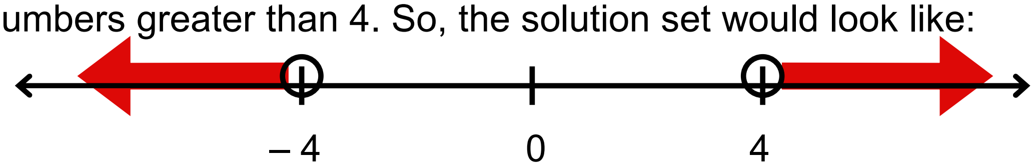


Each of these values is four units away from 0.

In the inequality $|x| < 4$, we are asking what values are less than four units away from 0. This will give us all the numbers between – 4 and 4. So, the solution set would look like:



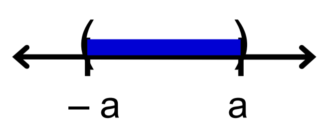
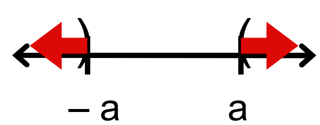
Now, in the inequality $|x| > 4$, we are asking what values are more than four units away from 0. This will give us all the numbers less than – 4 or all the numbers greater than 4. So, the solution set would look like:



This gives us our three cases when solving absolute value equations and inequalities:

Absolute Equations and Inequalities

Let a be a real number such that $a > 0$.

Equation/Inequality	Equivalent Solution	Graph
1) $\lvert x \rvert = a$	$x = a$ or $x = -a$	
2) $\lvert x \rvert < a$	$-a < x < a$	
3) $\lvert x \rvert > a$	$x > a$ or $x < -a$	

Note, if $|x| > -$ number, then the solution will be all real numbers since the absolute value of a number is always positive. Similarly, if $|x| < -$ number, then there will be no solution since the absolute value cannot be negative. Much like solving absolute value equations, when we are solving absolute value inequalities, we first need to isolate the absolute value on one side. We then apply the equivalent solution and solve.

Solve the following:

Ex. 1 $\left|\frac{2}{3}x - 5\right| - 5 > 4$

Solution:

First, isolate the absolute value on one side:

$\left|\frac{2}{3}x - 5\right| - 5 > 4$

$\left|\frac{2}{3}x - 5\right| > 9$

Now, the equivalent solution is:

$\frac{2}{3}x - 5 > 9$ or $\frac{2}{3}x - 5 < -9$

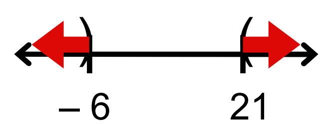
$\frac{2}{3}x > 14$ or $\frac{2}{3}x < -4$

$3(\frac{2}{3}x) > 3(14)$ $3(\frac{2}{3}x) < 3(-4)$

$2x > 42$ or $2x < -12$

$x > 21$ or $x < -6$

$(21, \infty)$ $(-\infty, -6)$



The word "or" means union:

The solution is $(-\infty, -6) \cup (21, \infty)$.

Ex. 2 $|5 - 2x| - 4 \leq 3$

Solution:

First, isolate the absolute value on one side:

$|5 - 2x| - 4 \leq 3$

$|5 - 2x| \leq 7$

Now, the equivalent solution is:

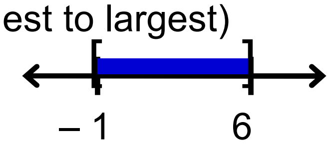
$-7 \leq 5 - 2x \leq 7$ (remember to switch the inequality

$-12 \leq -2x \leq 2$ signs when dividing by a – #)

$6 \geq x \geq -1$ (rewrite from smallest to largest)

$-1 \leq x \leq 6$

The solution is $[-1, 6]$



Ex. 3 $|4x - 5| - 8 < -11$

Solution:

First, isolate the absolute value on one side:

$|4x - 5| - 8 < -11$

$|4x - 5| < -3$ But, the absolute value cannot be negative so there is no solution.

The solution is { }.

Ex. 4 $|3.2x - 5| + 8 > 8$

Solution:

First, isolate the absolute value on one side:

$|3.2x - 5| + 8 > 8$

$|3.2x - 5| > 0$

The absolute value is always positive or zero, so the only real number that is not going to be a solution is when $3.2x - 5 = 0$.

$3.2x - 5 = 0$

$3.2x = 5$

$x = 1.5625$

So, the solution is all real numbers except $x = 1.5625$:

$(-\infty, 1.5625) \cup (1.5625, \infty)$

Ex. 5 $|8 - 3.2x| \geq 4$

Solution:

The absolute value is already isolated on one side:

$|8 - 3.2x| \geq 4$

Now, the equivalent solution is:

$8 - 3.2x \geq 4$	or	$8 - 3.2x \leq -4$
$-3.2x \geq -4$	or	$-3.2x \leq -12$
$x \leq 1.25$	or	$x \geq 3.75$
$(-\infty, 1.25]$		$[3.75, \infty)$

The solution is $(-\infty, 1.25] \cup [3.75, \infty)$.

Ex. 6 $|3 + 4x| > -6$

Solution:

Since the absolute value is always greater than a negative number, then the solution is all real numbers: $(-\infty, \infty)$.

Concept #2 Solving Absolute Value Inequalities using the Test Value.

We can the idea of boundary values established in section 10.2 as an alternative way to solve absolute value inequalities. Although the concept in principal is the same, we will need to make some adjustments to our procedure for the absolute value inequalities:

Test Value Method

1) Isolate the absolute value on one side.
2) Replace the inequality symbol by "=" and solve the absolute value equation.
3) The solutions will be our boundary values.
4) Mark the boundary values on a number line and pick a test value of x in each interval.
5) Plug each x-value into the original inequality see if the inequality is true in that interval. If so, that interval is part of the solution.
6) Use the results to solve the inequality.

<u>Solve the following:</u>

Ex. 7 $5|3x - 8| + 6 \geq 41$

<u>Solution:</u>

1) Isolate the absolute value:

$5|3x - 8| + 6 \geq 41$

$5|3x - 8| \geq 35$

$|3x - 8| \geq 7$

2) Replace ≥ with = and solve:

$|3x - 8| = 7$

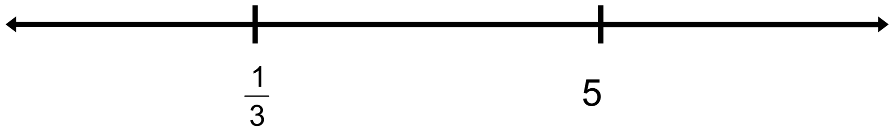
$3x - 8 = 7$ or $3x - 8 = -7$

$3x = 15$ or $3x = 1$

$x = 5$ or $x = \frac{1}{3}$

3) The boundary values are $x = \frac{1}{3}$ and $x = 5$.

4) Mark the boundary values on a number line:



Pick a test value in each interval:

Test: 0 1 6

5) Plug each x-value into the original inequality:

$x = 0$

$5\,|\,3(0) - 8\,|\, + 6 \geq 41$

$5\,|-8|\, + 6 \geq 41$

$5(8) + 6 \geq 41$

$46 \geq 41$ True

$x = 1$

$5\,|\,3(1) - 8\,|\, + 6 \geq 41$

$5\,|-5|\, + 6 \geq 41$

$5(5) + 6 \geq 41$

$31 \geq 41$ False

$x = 6$

$5\,|\,3(6) - 8\,|\, + 6 \geq 41$

$5\,|\,10\,|\, + 6 \geq 41$

$5(10) + 6 \geq 41$

$56 \geq 41$ True

Since the inequality is ≥, we need to include the boundary values.

6) The solution is $(-\infty, \frac{1}{3}] \cup [5, \infty)$.

Ex. 8 $2\left|\frac{3}{7}x - 5\right| - 8 \leq -8$

<u>Solution:</u>

1) Isolate the absolute value:

$2\left|\frac{3}{7}x - 5\right| - 8 \leq -8$

$2\left|\frac{3}{7}x - 5\right| \leq 0$

$\left|\frac{3}{7}x - 5\right| \leq 0$

2) Replace ≤ with = and solve:

$\left|\frac{3}{7}x - 5\right| = 0$

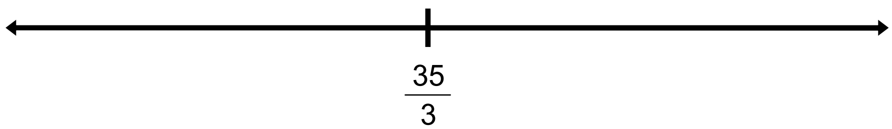
The only solution comes from $\frac{3}{7}x - 5 = 0$:

$$\frac{3}{7}x = 5$$

$$x = \frac{35}{3}$$

3) The boundary value is $\frac{35}{3}$.

4) Mark the boundary values on a number line:



Pick a test value in each interval:

Test: 0 14

5) Plug each x-value into the original inequality:

x = 0	x = 14
$2\left\|\frac{3}{7}(0) - 5\right\| - 8 \le -8$	$2\left\|\frac{3}{7}(14) - 5\right\| - 8 \le -8$
$2\|-5\| - 8 \le -8$	$2\|1\| - 8 \le -8$
$2(5) - 8 \le -8$	$2(1) - 8 \le -8$
$2 \le -8$ False	$-6 \le -8$ False

Since the inequality is ≤, we need to include the boundary value.

6) The solution is $\left\{\frac{35}{3}\right\}$.

Ex. 9 $|8x - 1| < -5$

Solution:

1) The absolute value is already isolated:

$|8x - 1| < -5$

2) Replace < with = and solve:

$|8x - 1| = -5$

No solution since the absolute value cannot be negative.

3) No boundary values.

4) No boundary values to mark:

Pick a test value in the interval

Test: 0

5) Plug the x-value into the original inequality:

$|8(0) - 1| < -5$

$|-1| < -5$

$-1 < -5$ false

6) There is no solution. The solution is { }

Concept #3 Translating to an Absolute Value Expression

Recall the absolute value of a number is the distance a number is from zero. The distance between two values a and b is $|a - b|$.

Express the following without the absolute value:

Ex. 10a $|\pi - 3|$ Ex. 10b $|3 - \pi|$

Solution:

a) The temptation is to write the answer as $\pi + 3$, but the we are asking what is the distance between π and 3. Since $\pi \approx 3.14$, then $\pi - 3 \approx 3.14 - 3 = 0.14$, But, 0.14 is positive so $|0.14| = 0.14$. This means that the absolute value will not change the sign in this case. Thus, $|\pi - 3| = \pi - 3$.

b) Using the same reasoning as in part a, since $3 - \pi \approx 3 - 3.14 = -0.14$, then $|-0.14| = 0.14$. This means the absolute value does change the sign. So, $|3 - \pi| = -(3 - \pi) = \pi - 3$.

Write an absolute value inequality equivalent to the following:

Ex. 11 All real numbers whose distance from – 5 is at least 9.

Solution:

The distance between a number and – 5 is $|x - (-5)| = |x + 5|$.
The phrase "is at least" means ≥. So, our inequality is $|x + 5| \geq 9$.

Ex. 12 In a recent poll, 53% of the respondents favored passing the bond issue. If the poll had a margin of error of 2.5%, then

a) Write an inequality that represents this situation.

b) Solve the inequality and interpret what the solution means.

Solution:

a) Let v be the actual percentage of the people that favor passing the bond issue. Since 53% = 0.53 and 2.5% = 0.025, then the difference in the actual percentage of the people support and the percentage in the poll has to be less than or equal to the margin of error:

$$|v - 0.53| \leq 0.025$$

b) Solve: $| v - 0.53 | \le 0.025$

$- 0.025 \le v - 0.53 \le 0.025$

$\underline{+ 0.53 = \quad + 0.53 = + 0.53}$

$0.505 \le v \le 0.555$

[0505, 0.555]

The actual percentage of people favoring the bond issue is between 50.5% and 55.5% inclusively.

Ex. 13 A three-inch nail will be rejected if its length varies by more than ± 0.02 in.

a) Write an inequality that represents this situation.

b) Solve the inequality and interpret what the solution means.

Solution:

a) Let L be the actual length of the nail. A nail is rejected if the difference between the actual length and 3 inches is more than 0.02 in:

$| L - 3 | > 0.02$

b) Solve: $| L - 3 | > 0.02$

$L - 3 > 0.02$ or $L - 3 < - 0.02$

$L > 3.02$ or $L < 2.98$

$(- \infty, 2.98) \cup (3.02, \infty)$

A nail will be rejected if its length is less than 2.98 inches or more than 3.02 inches.