Sect 11.1 - Definition of the nth Root

Concept #1   Definition of a Square Root.

The square of a number is called a **perfect square**. So, 1, 4, 9, 16, 25, 0.09, and $\frac{16}{25}$ are perfect squares since $1 = 1^2$, $4 = 2^2$, $9 = 3^2$, $16 = 4^2$, $25 = 5^2$, $0.09 = (0.3)^2$ and $\frac{16}{25} = \left(\frac{4}{5}\right)^2$. The reverse operation of squaring a number is finding the square root of a number. We can use the idea of perfect squares to simplify square roots. The square root of a number, a, asks what number times itself is equal to a. For example, the square root of 25 is 5 since 5 times 5 is 25.

Definition of a Square Root
Let a represent a non-negative real number. Then, the **square root** of a, denoted $\sqrt{a}$, is a number whose square is a. $\sqrt{a}$ is called the positive or **principal square root** of a. $-\sqrt{a}$ is called the negative square root of a.

Simplify:

Ex. 1a $-\sqrt{64}$ Ex. 1b $\sqrt{0.25}$

Ex. 1c $\sqrt{\frac{49}{81}}$ Ex. 1d $-\sqrt{6.25}$

Ex. 1e $\sqrt{0}$ Ex. 1f $\sqrt{15}$

Ex. 1g $\sqrt{-25}$ Ex. 1h $\sqrt{\frac{1}{4}}$

Solution:

a)   Since $8^2 = 64$, then $-\sqrt{64} = -1 \bullet \sqrt{64} = -8$

b)   Since $(0.5)^2 = 0.25$, then $\sqrt{0.25} = 0.5$

c)   Since $\left(\frac{7}{9}\right)^2 = \frac{49}{81}$, then $\sqrt{\frac{49}{81}} = \frac{7}{9}$

d)   Since $(2.5)^2 = 6.25$, then $-\sqrt{6.25} = -1 \bullet \sqrt{6.25} = -2.5$

e)   Since $0^2 = 0$, then $\sqrt{0} = 0$

f)   Since 15 is not a perfect square, then need to use a calculator to get approximation: $\sqrt{15} = 3.87298334621... \approx 3.873$

g) $\sqrt{-25}$ is not a real number.

h) $\sqrt{\frac{1}{4}} = \frac{1}{2}$

Concept #2   Definition of an nth Root

The square root of a number is the reverse process of squaring a number. We can also do this with other powers as well. The cube root of a number is the reverse process of cubing a number. The fourth root of a number is the reverse process of raising a number to the fourth power and so forth.

Definition of the nth root
Let a be a real number and let n be a natural number greater than 1
1)   If n is even and a ≥ 0, then $\sqrt[n]{a}$ is the principal nth root of a and
    $-\sqrt[n]{a}$ is the negative nth root of a.
2)   If n is odd, then $\sqrt[n]{a}$ is the principal nth root of a and $-\sqrt[n]{a}$ is the negative nth root of a.

In the radical $\sqrt[n]{a}$, the symbol $\sqrt[n]{}$ is called the **radical sign**, the number a is called the **radicand**, and n is called the **index**. For the square root, the index is 2 and we usually do not write the index. Keep in mind that the even root of a negative number is undefined in the real numbers.

Here is a list of perfect powers in the following table:

Perfect Cubes	Perfect Fourth Powers	Perfect Fifth Powers
$2^3 = 8$	$2^4 = 16$	$2^5 = 32$
$3^3 = 27$	$3^4 = 81$	$3^5 = 243$
$4^3 = 64$	$4^4 = 256$	$4^5 = 1024$
$5^3 = 125$	$5^4 = 625$	$5^5 = 3125$
$6^3 = 216$	$6^4 = 1296$	$6^5 = 7776$
$7^3 = 343$	$7^4 = 2401$	$7^5 = 16,807$
$8^3 = 512$	$8^4 = 4096$	$8^5 = 32,768$
$9^3 = 729$	$9^4 = 6561$	$9^5 = 59,049$
$10^3 = 1000$	$10^4 = 10,000$	$10^5 = 100,000$

Simplify the following:

Ex. 2a $-\sqrt[4]{10,000}$ Ex. 2b $\sqrt[3]{-64}$

Ex. 2c $\sqrt[4]{16}$ Ex. 2d $\sqrt[6]{-1}$

Ex. 2e $-\sqrt[5]{-59,049}$ Ex. 2f $\sqrt[7]{10,000,000}$

Solution:

a) Since 10^4 = 10,000, then $- \sqrt[4]{10,000} = -1 \bullet \sqrt[4]{10,000} = -10$

b) Since $(-4)^3 = -64$, then $\sqrt[3]{-64} = -4$

c) Since $2^4 = 16$, then $\sqrt[4]{16} = 2$

d) $\sqrt[6]{-1}$ is not a real number.

e) Since $(-9)^5 = -59,049$, then $- \sqrt[5]{-59,049} = -1 \bullet \sqrt[5]{-59,049}$
$= -1 \bullet (-9) = 9$

f) Since $10^7 = 10,000,000$, then $\sqrt[7]{10,000,000} = 10$

Concept #3 Roots of Variable Expressions

Simplify the following:

Ex. 3a $\sqrt{(-7)^2}$ Ex. 3b $\sqrt[3]{(-7)^3}$

Ex. 3c $\sqrt[4]{(-7)^4}$ Ex. 3d $\sqrt[5]{(-7)^5}$

Solution:

a) $\sqrt{(-7)^2} = \sqrt{49} = 7$

b) $\sqrt[3]{(-7)^3} = \sqrt[3]{-343} = -7$

c) $\sqrt[4]{(-7)^4} = \sqrt[4]{2401} = 7$

d) $\sqrt[5]{(-7)^5} = \sqrt[5]{-16,807} = -7$

Notice that there is a pattern occurring. If the power and the index are both the same and an odd number, we get the same number for the answer as the number we started with:

$\sqrt[3]{(-7)^3} = -7$ $\sqrt[5]{(-7)^5} = -7$

⌊ Same ↑ ⌊ Same ↑

If the power and the index are both the same and an even number, we get the absolute value of the number we started with for the answer :

$\sqrt{(-7)^2} = 7$ $\sqrt[4]{(-7)^4} = 7$

⌊____↑ ⌊____↑

Absolute Value Absolute Value

In general, $\sqrt[n]{a^n} = a$ if n is odd and $\sqrt[n]{a^n} = |a|$ if n is even.
This illustrates how to take the nth root of variable expressions.

Definition of $\sqrt[n]{a^n}$

Let n be a natural number greater than one. Then

1)   If n is odd, then $\sqrt[n]{a^n} = a$.

2)   If n is even, $\sqrt[n]{a^n} = |a|$.

Simplify the following:

Ex. 3a $\sqrt[3]{(r-s)^3}$

Ex. 3b $\sqrt[4]{(a-3b)^4}$

Ex. 3c $\sqrt{9x^2-30x+25}$

Ex. 3d $\sqrt[5]{a^{15}b^5c^{10}}$

Ex. 3e $\sqrt[6]{\dfrac{64a^6b^{24}}{c^{12}}}$

Solution:

a) $\sqrt[3]{(r-s)^3} = (r-s) = r - s$.

b) $\sqrt[4]{(a-3b)^4} = |(a-3b)| = |a-3b|$.

c)   Since $9x^2 - 30x + 25 = (3x-5)^2$, then

$\sqrt{9x^2-30x+25} = \sqrt{(3x-5)^2} = |(3x-5)| = |3x-5|$

d) $\sqrt[5]{a^{15}b^5c^{10}} = \sqrt[5]{(a^3)^5b^5(c^2)^5} = a^3bc^2$.

e) $\sqrt[6]{\dfrac{64a^6b^{24}}{c^{12}}} = \sqrt[6]{\dfrac{2^6a^6(b^4)^6}{(c^2)^6}} = \left|\dfrac{2ab^4}{c^2}\right|$. But 2, b^4, c^2 are not

negative, so they can be pulled out of the absolute value:

$\left|\dfrac{2ab^4}{c^2}\right| = \dfrac{2b^4}{c^2}\bullet|a| = \dfrac{2|a|b^4}{c^2}$.

Notice that with part d and e, we rewrote the powers as perfect nth powers before applying the nth root. Here are some patterns to look for:

Perfect Squares	Perfect Cubes	Perfect 4th Powers	Perfect 5th Powers
$(x^2)^2 = x^4$	$(x^2)^3 = x^6$	$(x^2)^4 = x^8$	$(x^2)^5 = x^{10}$
$(x^3)^2 = x^6$	$(x^3)^3 = x^9$	$(x^3)^4 = x^{12}$	$(x^3)^5 = x^{15}$
$(x^4)^2 = x^8$	$(x^4)^3 = x^{12}$	$(x^4)^4 = x^{16}$	$(x^4)^5 = x^{20}$
$(x^5)^2 = x^{10}$	$(x^5)^3 = x^{15}$	$(x^5)^4 = x^{20}$	$(x^5)^5 = x^{25}$
$(x^6)^2 = x^{12}$	$(x^6)^3 = x^{18}$	$(x^6)^4 = x^{24}$	$(x^6)^5 = x^{30}$
$(x^7)^2 = x^{14}$	$(x^7)^3 = x^{21}$	$(x^7)^4 = x^{28}$	$(x^7)^5 = x^{35}$
$\left(x^{\frac{n}{2}}\right)^2 = x^n$	$\left(x^{\frac{n}{3}}\right)^3 = x^n$	$\left(x^{\frac{n}{4}}\right)^4 = x^n$	$\left(x^{\frac{n}{5}}\right)^5 = x^n$

Simplify:

Ex. 4a $\quad \sqrt{t^6 v^8}$

Ex. 4b $\quad \sqrt[4]{\dfrac{625a^8}{81b^{16}}}$

Ex. 4c $\quad \sqrt[7]{x^7 y^{21} z^{14}}$

Ex. 4d $\quad -\sqrt[3]{-216x^9 y^3}$

Solution:

a) Since $6 \div 2 = 3$ and $8 \div 2 = 4$, then $\sqrt{t^6 v^8} = \sqrt{(t^3)^2 (v^4)^2}$
$= |\, t^3 v^4 \,|$. But $v^4 \geq 0$, so $|\, t^3 v^4 \,| = v^4 |\, t^3 \,|$.

b) Since $8 \div 4 = 2$ and $16 \div 4 = 4$, then $\sqrt[4]{\dfrac{625a^8}{81b^{16}}} = \sqrt[4]{\dfrac{5^4 (a^2)^4}{3^4 (b^4)^4}}$

$= \left|\, \dfrac{5a^2}{3b^4} \,\right|$. But all the powers are non-negative, so

$\left|\, \dfrac{5a^2}{3b^4} \,\right| = \dfrac{5a^2}{3b^4}$.

c) Since $21 \div 7 = 3$ and $14 \div 7 = 2$, then $\sqrt[7]{x^7 y^{21} z^{14}}$
$= \sqrt[7]{x^7 (y^3)^7 (z^2)^7} = xy^3 z^2$.

d) Since $9 \div 3 = 3$, then $-\sqrt[3]{-216x^9 y^3} = -1 \bullet \sqrt[3]{(-6)^3 (x^3)^3 y^3}$
$= -1(-6x^3 y) = 6x^3 y$.

Concept #4   Pythagorean Theorem

In a right triangle, there is a special relationship between the length of the legs (a and b) and the hypotenuse (c). This is known as the Pythagorean Theorem:
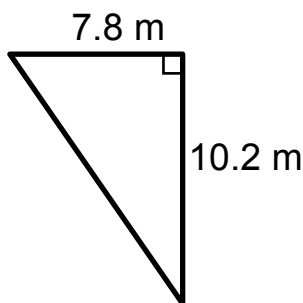
Pythagorean Theorem
In a right triangle, the square of the hypotenuse (c^2)
is equal to the sum of the squares of the legs ($a^2 + b^2$)
$\qquad c^2 = a^2 + b^2$
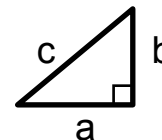
Keep in mind that the hypotenuse is the longest side of the right triangle.

Find the length of the missing sides (to the nearest hundredth):

Ex. 5   7.8 m
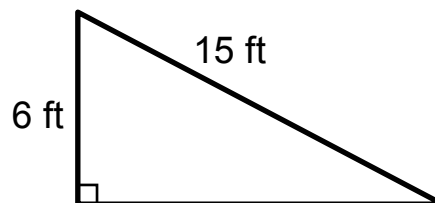
10.2 m

Ex. 6

15 ft

6 ft

Solution:
In this problem, we have the two legs of the triangle and we are looking for the hypotenuse:

$$c^2 = a^2 + b^2$$
$$c^2 = (7.8)^2 + (10.2)^2$$
$$c^2 = 60.84 + 104.04$$
$$c^2 = 164.88$$

To find c, take the square root* of 164.88:

$$c = \pm \sqrt{164.88} = \pm 12.8405...$$
$$c \approx 12.84 \text{ m}$$

Solution:
In this problem, we have one leg and the hypotenuse and we are looking for the other leg:

$$c^2 = a^2 + b^2$$
$$(15)^2 = (6)^2 + b^2$$
$$225 = 36 + b^2 \quad \text{(solve for } b^2\text{)}$$
$$\underline{-36 = -36}$$
$$189 = b^2$$

To find b, take the square root of 189:

$$b = \pm \sqrt{189} = \pm 13.7477...$$
$$b \approx 13.75 \text{ ft}$$

* - The equation c^2 = 164.88 actually has two solutions ≈ 12.84 and ≈ – 12.84, but the lengths of triangles are positive, so we use only the principal square root.

Solve the following:

Ex. 8   Leroy leaves St. Philip's College in his car driving east at 30 mph on Martin Luther King Drive. Juanita leaves at the same time driving south at 40 mph on New Braunfels Avenue. How far apart are they after twelve minutes?
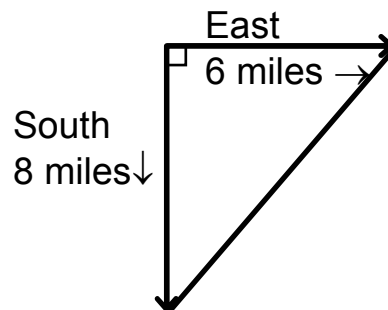
Solution:

Twelve minutes is $\frac{12}{60} = \frac{1}{5}$ of an hour.

Since d = rt, then:

Leroy traveled = $30(\frac{1}{5})$ = 6 miles

Juanita traveled = $40(\frac{1}{5})$ = 8 miles

East
6 miles →

South
8 miles↓

Now, draw a picture:
We have a right triangle with two legs and we are looking for the hypotenuse:

$$c^2 = (8)^2 + (6)^2$$
$$c^2 = 64 + 36$$
$$c^2 = 100$$

To find c, take the square root* of 100:

$c = \pm \sqrt{100} = \pm 10$ Again, we used only the principal square root.
      $c = 10$ miles

Leroy and Juanita are ten miles apart.

Concept #5   Radical Functions

If n is a natural number greater than one, then $f(x) = \sqrt[n]{x}$ is a **radical function**. The domain will depend on the index. If n is odd, the domain will be a real numbers. If the index is even, then the domain will be all values of x that make the radicand greater than or equal to zero.

Find the domain of the following:

Ex. 9a $f(x) = \sqrt{6-4x}$ Ex. 9b $g(x) = \sqrt[3]{5+2x}$

Ex. 9c $r(x) = \dfrac{1}{\sqrt[6]{3x-2}}$ Ex. 9d $h(x) = \dfrac{5}{\sqrt[7]{5x-8}}$

   Solution:
   a)   Since the index is even, the radicand has to be non-negative:
         Solve: $6 - 4x \geq 0$ (switch the inequality sign when
                     $- 4x \geq - 6$ dividing by a negative number)
                     $x \leq 1.5$
         The domain is $(- \infty, 1.5]$,
   b)   Since the index is odd, the radicand is defined for all real
         numbers. The domain is $(- \infty, \infty)$.
   c)   Since the index is even, the radicand has to be non-negative.
         But the radicand is in the denominator, so it also cannot be
         zero:
         Solve: $3x - 2 > 0$
                     $3x > 2$
                     $x > \dfrac{2}{3}$
         Thus, the domain is $(\dfrac{2}{3}, \infty)$.

d) Since the index is odd, the radicand is defined for all real numbers. But the radicand is in the denominator, so it also cannot be zero:

$$5x - 8 \neq 0$$
$$5x \neq 8$$
$$x \neq 1.6$$

So, the function is defined for all real numbers except 1.6. Thus, the domain is $(-\infty, 1.6) \cup (1.6, \infty)$.
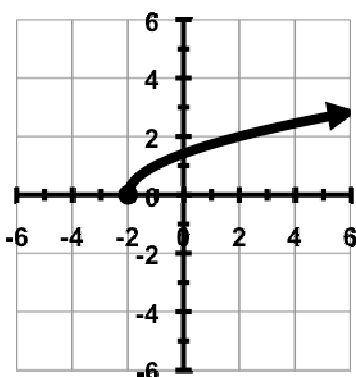
Match the function with the graph:

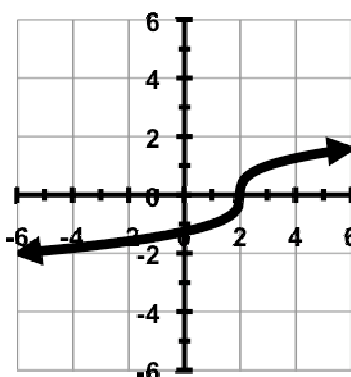Ex. 10a $f(x) = \sqrt[3]{x-2}$ Ex. 10b $h(x) = -\sqrt{x-1}$
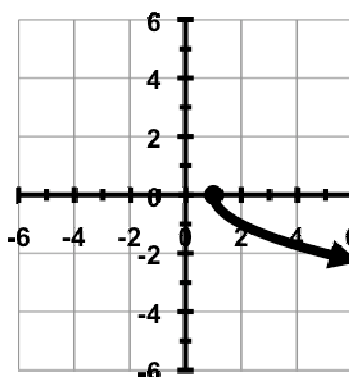
Ex. 10c $g(x) = \sqrt{x+2}$

i)   ii)   iii)

Solution:

a) Since the index is odd, the domain is all real numbers. The only graph that matches it is #ii.

b) Since h(x) is the negative square root, the y-values are negative. This matches #iii.

c) Since g(x) is the positive square root, the y-values are positive. This matches #i.