Sect 12.1 – The Square Root Property and Completing the Square

Concept #1 Solving Quadratic Equations by Using the Square Root Property.

When we solved quadratic equations before, we had to get zero on one side, factor the polynomial on the other side and then set each factor equal to zero and solve. We were only able to do this if the polynomial was factorable. In this section and the next, we will learn techniques to solve quadratic equations regardless if they are factorable or not. Consider the following example:

$x^2 = 64$	(get zero on one side)
$x^2 – 64 = 0$	(factor)
$(x – 8)(x + 8) = 0$	(set each factor = 0)
$x – 8 = 0$ or $x + 8 = 0$	(solve)
$x = 8$ or $x = – 8$	

Notice that 8 and – 8 are the principal square root and the negative square root of 64 respectively. This means that if $x^2 = 64$, we will get $\sqrt{64} = 8$ and $-\sqrt{64} = -8$ as our answers. This leads us to the **Square Root Property**

The Square Root Property
If d is a real number and $x^2 = d$, then $x = \sqrt{d}$ or $x = -\sqrt{d}$. Sometimes this is written as $x = \pm\sqrt{d}$.

The key to using this property is to have a perfect square isolated on one side of the equation.

Solve the following:

Ex. 1a $y^2 = 121$ Ex. 1b $7x^2 + 567 = 0$
Ex. 1c $3(e – 8)^2 = 60$ Ex. 1d $(2r – 3)^2 – 7 = – 12$

Solution:

a) $y^2 = 121$ (apply the square root property)
$y = \pm\sqrt{121}$ (simplify)
$y = \pm 11$
The solutions are {– 11, 11}

b) $7x^2 + 567 = 0$ (solve for x^2)
$7x^2 = -567$
$x^2 = -81$ (apply the square root property)
$x = \pm\sqrt{-81}$ (apply the definition of $\sqrt{-m}$)
$x = \pm i\sqrt{81}$ (simplify)
$x = \pm 9i$
The solutions are $\{-9i, 9i\}$.

c) $3(e-8)^2 = 60$ (solve for $(e-8)^2$)
$(e-8)^2 = 20$ (apply the square root property)
$(e-8) = \pm\sqrt{20}$ (simplify)
$e - 8 = \pm\sqrt{2^2 \bullet 5}$
$e - 8 = \pm 2\sqrt{5}$ (solve for e)
$e = 8 \pm 2\sqrt{5}$
So, $e = 8 - 2\sqrt{5}$ or $e = 8 + 2\sqrt{5}$
The solutions are $\{8 - 2\sqrt{5}, 8 + 2\sqrt{5}\}$.

d) $(2r-3)^2 - 7 = -12$ (solve for $(2r-3)^2$)
$(2r-3)^2 = -5$ (apply the square root property)
$(2r-3) = \pm\sqrt{-5}$ (apply the definition of $\sqrt{-m}$)
$2r - 3 = \pm i\sqrt{5}$ (solve for r)
$2r = 3 \pm i\sqrt{5}$
$r = \frac{3 \pm i\sqrt{5}}{2} = \frac{3}{2} \pm \frac{\sqrt{5}}{2}i$
The solutions are $\left\{\frac{3}{2} - \frac{\sqrt{5}}{2}i, \frac{3}{2} + \frac{\sqrt{5}}{2}i\right\}$.

Notice that in example #1c and #1d, our answers contain components that are irrational and hence cannot be solved by factoring. But, example #1a can be solved very easily using factoring and we can even solve #1b by factoring: $7x^2 + 567 = 0$
$7(x^2 + 81) = 0$
$x^2 + 81$ is prime in the real numbers, but not in the complex numbers:
$7(x^2 - (-81)) = 0$ (now, apply the difference of squares)
$7(x - 9i)(x + 9i) = 0$
$7 \neq 0$, $x - 9i = 0$, $x + 9i = 0$ (solve)
No Soln, $x = 9i$, $x = -9i$ The solution is $\{-9i, 9i\}$

Concept #2: Solving the Quadratic Equations by Completing the Square.

To solve a quadratic equation that does not already have a perfect square in it, we will need to make a perfect square. The process for doing this is called **Completing the Square**. To see how this process works, let's look at several examples of perfect square trinomials:

$(x + 4)^2 = (x + 4)(x + 4) = x^2 + 8x + 16$
$(x – 5)^2 = (x – 5)(x – 5) = x^2 – 10x + 25$
$(x + p)^2 = (x + p)(x + p) = x^2 + 2px + p^2$
$(x – p)^2 = (x – p)(x – p) = x^2 – 2px + p^2$

Notice that there is a relationship between the coefficient of the linear term and the last term. If we divide the **coefficient of the linear term** by two (or multiply it by a half) and then square the result, we get the *last term*. Also, when we divide the coefficient by two, the result is also the number p that goes into the factored form.

So, for $x^2 + \mathbf{8}x + \mathit{16}$, $\mathbf{8} \div 2 = \underline{4}$, $(\underline{4})^2 = \mathit{16}$. Also, $x^2 + \mathbf{8}x + \mathit{16} = (x + \underline{4})^2$

For $x^2 \mathbf{– 10}x + \mathit{25}$, $\mathbf{– 10} \div 2 = \underline{– 5}$, $(\underline{– 5})^2 = \mathit{25}$. Also, $x^2 \mathbf{– 10}x + \mathit{25} = (x \underline{– 5})^2$

This process only works if the coefficient of the squared term is 1.
Let's try some examples.

Find the value L that completes the square. Then factor the result:

Ex. 2a $x^2 – 14x + L$ Ex. 2b $x^2 + 2.2x + L$

Ex. 2c $y^2 + 17y + L$ Ex. 2d $v^2 – \frac{3}{13}v + L$

Solution:

a) To get p, divide the coefficient of the linear term by 2:
$p = – 14 \div 2 = – 7$
Now, square p to get L: $L = (– 7)^2 = 49$
Thus, $x^2 – 14x + L = x^2 – 14x + 49$
Now, factor into $(x + p)^2$ form:
$x^2 – 14x + 49 = (x – 7)^2$

b) To get p, divide the coefficient of the linear term by 2:
$p = 2.2 \div 2 = 1.1$
Now, square p to get L: $L = (1.1)^2 = 1.21$
Thus, $x^2 + 2.2x + L = x^2 + 2.2x + 1.21$
Now, factor into $(x + p)^2$ form:
$x^2 + 2.2x + 1.21 = (x + 1.1)^2$

c) To get p, divide the coefficient of the linear term by 2:
$p = 17 \div 2 = \frac{17}{2}$
Now, square p to get L: $L = \left(\frac{17}{2}\right)^2 = \frac{289}{4}$
Thus, $y^2 + 17y + L = y^2 + 17y + \frac{289}{4}$
Now, factor into $(y + p)^2$ form:
$y^2 + 17y + \frac{289}{4} = (y + \frac{17}{2})^2$

d) To get p, multiply the coefficient of the linear term by a half:
$p = -\frac{3}{13} \bullet \frac{1}{2} = \frac{3}{26}$
Now, square p to get L: $L = \left(-\frac{3}{26}\right)^2 = \frac{9}{676}$
Thus, $v^2 - \frac{3}{13}v + L = v^2 - \frac{3}{13}v + \frac{9}{676}$
Now, factor into $(v + p)^2$ form:
$v^2 - \frac{3}{13}v + \frac{9}{676} = \left(v - \frac{3}{26}\right)^2$

Now, let's use completing the square to solve quadratic equations. In each case, we will follow these steps:

Completing the Square to Solve a Quadratic Equation

1) Simplify each side of the equation.
2) Isolate the variable terms on one side of the equation.
3) Divide both sides by the coefficient a of the squared term if $a \neq 1$.
4) Divide the coefficient of the linear term by 2 (or multiply it by a half) to get p. Then square p and add the result to both sides of the equation.
5) Rewrite the perfect square as $(x + p)^2$ and use the square root property to solve.

<u>Solve the following:</u>

Ex. 3a $b^2 - 12b - 5 = 0$ Ex. 3b $m(m + 7) + 14 = 4$
Ex. 3c $2y(y + 1) = y - 7$ Ex. 3d $(4x - 5)(x - 3) = -x + 47$

<u>Solution:</u>

a) $b^2 - 12b - 5 = 0$ (each side is already simplified)
$b^2 - 12b - 5 = 0$ (isolate the variable terms on one side)
$b^2 - 12b = 5$ (a is already equal to one)

$b^2 - 12b = 5$ (p = – 12 ÷ 2 = – 6. Add $(-6)^2 = 36$
$b^2 - 12b + 36 = 5 + 36$ to both sides and simplify)
$b^2 - 12b + 36 = 41$ (rewrite in the form $(b + p)^2$)
$(b - 6)^2 = 41$ (use the square root property)
$b - 6 = \pm\sqrt{41}$ (solve)
$b = 6 \pm \sqrt{41}$

The solutions are $\{6 - \sqrt{41}, 6 + \sqrt{41}\}$.

b) $m(m + 7) + 14 = 4$ (simplify each side)
$m^2 + 7m + 14 = 4$ (isolate the variable terms on one side)
$m^2 + 7m = -10$ (a is already equal to one)
$m^2 + 7m = -10$ ($p = 7 \div 2 = \frac{7}{2}$. Add $\left(\frac{7}{2}\right)^2 = \frac{49}{4}$
$m^2 + 7m + \frac{49}{4} = -\frac{40}{4} + \frac{49}{4}$ to both sides and simplify)
$m^2 + 7m + \frac{49}{4} = \frac{9}{4}$ (rewrite in the form $(m + p)^2$)
$\left(m + \frac{7}{2}\right)^2 = \frac{9}{4}$ (use the square root property)
$m + \frac{7}{2} = \pm\sqrt{\frac{9}{4}} = \pm\frac{3}{2}$ (solve)
$m = -\frac{7}{2} \pm \frac{3}{2}$
$m = -\frac{7}{2} - \frac{3}{2} = -\frac{10}{2} = -5$ or $m = -\frac{7}{2} + \frac{3}{2} = -\frac{4}{2} = -2$

The solutions are $\{-5, -2\}$.

c) $2y(y + 1) = y - 7$ (simplify each side)
$2y^2 + 2y = y - 7$ (isolate the variable terms on one side)
$2y^2 + y = -7$ (since a = 2, divided both sides by 2)
$y^2 + \frac{1}{2}y = -\frac{7}{2}$ ($p = \frac{1}{2} \bullet \frac{1}{2} = \frac{1}{4}$. Add $\left(\frac{1}{4}\right)^2 = \frac{1}{16}$
$y^2 + \frac{1}{2}y + \frac{1}{16} = -\frac{56}{16} + \frac{1}{16}$ to both sides and simplify)
$y^2 + \frac{1}{2}y + \frac{1}{16} = -\frac{56}{16}$ (rewrite in the form $(y + p)^2$)
$\left(y + \frac{1}{4}\right)^2 = -\frac{56}{16}$ (use the square root property)
$y + \frac{1}{4} = \pm\sqrt{-\frac{55}{16}} = \pm\frac{\sqrt{55}}{4}i$ (solve)
$y = -\frac{1}{4} \pm \frac{\sqrt{55}}{4}i$ or $\frac{-1 \pm i\sqrt{55}}{4}$

The solutions are $\left\{\frac{-1-i\sqrt{55}}{4}, \frac{-1+i\sqrt{55}}{4}\right\}$.

d) $(4x - 5)(x - 3) = -x + 47$ (simplify each side)
$4x^2 - 17x + 15 = -x + 47$ (isolate the variable terms)
$4x^2 - 16x = 32$ (since a = 4, divided both sides by 4)
$x^2 - 4x = 8$ ($p = -4 \div 2 = -2$. Add $(-2)^2 = 4$)
$x^2 - 4x + 4 = 8 + 4$ to both sides and simplify)
$x^2 - 4x + 4 = 12$ (rewrite in the form $(x + p)^2$)
$(x - 2)^2 = 12$ (use the square root property)
$x - 2 = \pm\sqrt{12} = \pm\sqrt{2^2 \bullet 3} = \pm 2\sqrt{3}$ (solve)
$x = 2 \pm 2\sqrt{3}$
The solutions are $\{2 - 2\sqrt{3}, 2 + 2\sqrt{3}\}$.

Concept #3 Literal Equations

Solve the following:

Ex. 4 Given the formula for the volume of cone: $V = \frac{1}{3}\pi r^2 h$
a) Solve the formula for r. Assume $r > 0$.
b) Find r if h = 8.5 meters and V = 20.0175 m^3. Use $\pi \approx 3.14$.

Solution:

a) $V = \frac{1}{3}\pi r^2 h$ (multiply both sides by 3)
$3 \bullet V = 3 \bullet \frac{1}{3} r^2 h$
$3V = \pi r^2 h$ (divide both sides by πh)
$\frac{3V}{\pi h} = r^2$ (use the square root property)
$r = \pm\sqrt{\frac{3V}{\pi h}}$ (reject the negative root and rationalize)
$r = \sqrt{\frac{3V}{\pi h}} \bullet \frac{\sqrt{\pi h}}{\sqrt{\pi h}} = \frac{\sqrt{3V\pi h}}{\pi h}$
Thus, $r = \frac{\sqrt{3V\pi h}}{\pi h}$.

b) $r = \frac{\sqrt{3V\pi h}}{\pi h} = \frac{\sqrt{3(20.0175)(3.14)(8.5)}}{(3.14)(8.5)} = \frac{\sqrt{1602.801225}}{26.69} = \frac{40.035}{26.69}$
$= 1.5$ m
Thus, the radius is 1.5 meters.