Sect 12.3 - Equations in Quadratic Form

Concept #2 Solving Equations Reducible to a Quadratic.

We will examine solving equations that produce a quadratic equation in the process.

Solve the following:

Ex. 1 $\frac{3c}{2c-1} - \frac{1}{c+2} = \frac{5}{2c^2+3c-2}$

Solution:

$2c – 1$ and $c + 2$ are prime and $2c^2 + 3c – 2 = (2c – 1)(c + 2)$

Thus, L.C.D. = $(2c – 1)(c + 2)$

Our restrictions are $c \neq 0.5$ and $c \neq –2$.

Now, multiply both sides by $(2c – 1)(c + 2)$:

$$\frac{3c}{2c-1} - \frac{1}{c+2} = \frac{5}{2c^2+3c-2}$$

$$\frac{3c}{(2c-1)} \bullet \frac{(2c-1)(c+2)}{1} - \frac{1}{(c+2)} \bullet \frac{(2c-1)(c+2)}{1} = \frac{5}{(2c-1)(c+2)} \bullet \frac{(2c-1)(c+2)}{1}$$

$$\frac{3c}{1} \bullet \frac{(c+2)}{1} - \frac{1}{1} \bullet \frac{(2c-1)}{1} = \frac{5}{1} \bullet \frac{1}{1}$$

$3c(c + 2) – 1(2c – 1) = 5$

$3c^2 + 6c – 2c + 1 = 5$

$3c^2 + 4c + 1 = 5$ (get zero on one side)

$3c^2 + 4c – 4 = 0$ (factor)

$(3c – 2)(c + 2) = 0$

$(3c – 2) = 0$ or $(c + 2) = 0$

$c = \frac{2}{3}$ or $c = –2$

But $c = –2$ matches our restriction that $c \neq –2$ and it must be excluded from the solution. Thus, $c = \frac{2}{3}$ is the only value left.

Hence, our solution is $\left\{\frac{2}{3}\right\}$.

Ex. 2 $\frac{3}{2y+5} - \frac{4}{y-3} = 7$

Solution:

$2y + 5$ and $y – 3$ are prime, then our L.C.D. = $(2y + 5)(y – 3)$

Our restrictions are $y \neq –2.5$ and $y \neq 3$.

Now, multiply both sides by $(2y + 5)(y – 3)$:

$\frac{3}{2y+5} - \frac{4}{y-3} = 7$

$\frac{3}{2y+5} \bullet \frac{(2y+5)(y-3)}{1} - \frac{4}{y-3} \bullet \frac{(2y+5)(y-3)}{1} = 7 \bullet \frac{(2y+5)(y-3)}{1}$

$3(y – 3) – 4(2y + 5) = 7(2y + 5)(y – 3)$

$3y – 9 – 8y – 20 = 7(2y^2 – y – 15)$

$– 5y – 29 = 14y^2 – 7y – 105$ (get zero on one side)

$0 = 14y^2 – 2y – 76$ (use the quadratic formula)

a = 14, b = – 2, and c = – 76

$y = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{-(-2) \pm \sqrt{(-2)^2-4(14)(-76)}}{2(14)} = \frac{2 \pm \sqrt{4+4256}}{28} = \frac{2 \pm \sqrt{4260}}{28}$

$= \frac{2 \pm \sqrt{2^2 \bullet 1065}}{28} = \frac{2 \pm 2\sqrt{1065}}{28} = \frac{2(1 \pm \sqrt{1065})}{28} = \frac{1 \pm \sqrt{1065}}{14}$

Neither of these values match our restrictions, so the solutions are $\left\{\frac{1-\sqrt{1065}}{14}, \frac{1+\sqrt{1065}}{14}\right\}$.

Ex. 3 $3x = 4 + \sqrt{2x+11}$

Solution:

$3x = 4 + \sqrt{2x+11}$ (isolate the square root)

$3x – 4 = \sqrt{2x+11}$ (square both sides)

$(3x – 4)^2 = (\sqrt{2x+11})^2$ (simplify)

$9x^2 – 24x + 16 = 2x + 11$ (get zero on one side)

$9x^2 – 26x + 5 = 0$ (use the quadratic formula)

a = 9, b = – 26, and c = 5

$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{-(-26) \pm \sqrt{(-26)^2-4(9)(5)}}{2(9)} = \frac{26 \pm \sqrt{676-180}}{18} = \frac{26 \pm \sqrt{496}}{18}$

$= \frac{26 \pm \sqrt{4^2 \bullet 31}}{18} = \frac{26 \pm 4\sqrt{31}}{18} = \frac{2(13 \pm 2\sqrt{31})}{18} = \frac{13 \pm 2\sqrt{31}}{9}$

Our potential solutions are $\frac{13-2\sqrt{31}}{9}$ and $\frac{13+2\sqrt{31}}{9}$.

To check, approximate each potential solution:

Check: $x = \frac{13-2\sqrt{31}}{9} \approx 0.207$ and $x = \frac{13+2\sqrt{31}}{9} \approx 2.682$

$3(0.207) \approx 4 + \sqrt{2(0.207)+11}$? $3(2.682) \approx 4 + \sqrt{2(2.682)+11}$?

$0.621 \approx 4 + \sqrt{11.414}$? $8.046 \approx 4 + \sqrt{16.364}$?

$0.621 \approx 7.378$ False $8.046 \approx 8.045$ True

So, the solution is $\left\{\frac{13+2\sqrt{31}}{9}\right\}$.

Ex. 4 $\sqrt{2x+7} - 9 = \sqrt{12x+2} - 4$

<u>Solution:</u>

$\sqrt{2x+7} - 9 = \sqrt{12x+2} - 4$ (isolate $\sqrt{12x+2}$)

$\sqrt{2x+7} - 5 = \sqrt{12x+2}$ (square both sides)

$(\sqrt{2x+7} - 5)^2 = (\sqrt{12x+2})^2$ (simplify)

$(\sqrt{2x+7})^2 - 2(\sqrt{2x+7})(5) + (5)^2 = 12x + 2$

$(2x + 7) - 10\sqrt{2x+7} + 25 = 12x + 2$

$2x + 32 - 10\sqrt{2x+7} = 12x + 2$ (isolate $\sqrt{2x+7}$)

$-10\sqrt{2x+7} = 10x - 30$ (divide by -10)

$\sqrt{2x+7} = -x + 3$ (square both sides)

$(\sqrt{2x+7})^2 = (-x + 3)^2$ (simplify)

$2x + 7 = x^2 - 6x + 9$ (get zero on one side)

$0 = x^2 - 8x + 2$ (use the quadratic formula)

$a = 1$, $b = -8$, and $c = 2$

$$x = \frac{-b\pm\sqrt{b^2-4ac}}{2a} = \frac{-(-8)\pm\sqrt{(-8)^2-4(1)(2)}}{2(1)} = \frac{8\pm\sqrt{64-8}}{2} = \frac{8\pm\sqrt{56}}{2}$$

$$= \frac{8\pm\sqrt{2^2\bullet 14}}{2} = \frac{8\pm 2\sqrt{14}}{2} = \frac{2(4\pm\sqrt{14})}{2} = 4 \pm \sqrt{14}$$

To check, approximate each potential solution:

Check: $x = 4 - \sqrt{14} \approx 0.258$

$\sqrt{2(0.258)+7} - 9 \approx \sqrt{12(0.258)+2} - 4$?

$\sqrt{7.516} - 9 \approx \sqrt{5.096} - 4$?

$-6.258 \approx -1.743$ False

Check: $x = 4 + \sqrt{14} \approx 7.742$

$\sqrt{2(7.742)+7} - 9 \approx \sqrt{12(7.742)+2} - 4$?

$\sqrt{22.484} - 9 \approx \sqrt{94.904} - 4$?

$-4.258 \approx 5.742$ False

Thus, there is no solution.

Concept #1 Solving Equations by Substitution

Sometimes is difficult to solve an equation that is in quadratic form directly. What we can do is to notice a pattern, make a substitution into the equation which gives us an easier equation to solve. After solving the easier equation, we substitute back in the original expression and solve.

Solve the following:

Ex. 5 $(3x^2 - 5)^2 + 4(3x^2 - 5) - 5 = 0$

Solution:

Notice we have a $(3x^2 - 5)$ squared term, $(3x^2 - 5)$ term and then a constant term. So, let u = $\mathbf{(3x^2 - 5)}$. Our equation becomes:

$\mathbf{(3x^2 - 5)}^2 + 4\mathbf{(3x^2 - 5)} - 5 = 0$

$\mathbf{u}^2 + 4\mathbf{u} - 5 = 0$ (factor)

$(u + 5)(u - 1) = 0$ (solve)

$u + 5 = 0$ or $u - 1 = 0$

$u = -5$ or $u = 1$

Now, replace u = $\mathbf{(3x^2 - 5)}$:

$\mathbf{3x^2 - 5} = -5$ or $\mathbf{3x^2 - 5} = 1$ (solve)

$3x^2 = 0$ or $3x^2 = 6$

$x^2 = 0$ or $x^2 = 2$ (square root property)

$x = 0$ or $x = \pm\sqrt{2}$

Thus, the solutions are $\{-\sqrt{2}, 0, \sqrt{2}\}$.

Ex. 6 $3\sqrt{x} + 10\sqrt[4]{x} = 8$

Solution:

$3\sqrt{x} + 10\sqrt[4]{x} = 8$ (get zero on one side)

$3\sqrt{x} + 10\sqrt[4]{x} - 8 = 0$

Notice that $\sqrt{x} = (\sqrt[4]{x})^2$, so our equation looks like:

$3(\sqrt[4]{x})^2 + 10(\sqrt[4]{x}) - 8 = 0$

Let u = $(\sqrt[4]{x})$. Our equation becomes:

$3u^2 + 10u - 8 = 0$ (factor)

$(3u - 2)(u + 4) = 0$ (solve)

$3u - 2 = 0$ or $u + 4 = 0$

$u = \frac{2}{3}$ or $u = -4$

Now, replace u = $\sqrt[4]{x}$

$\sqrt[4]{x} = \frac{2}{3}$ or $\sqrt[4]{x} = -4$ (raise both sides to the 4th power)

$(\sqrt[4]{x})^4 = \left(\frac{2}{3}\right)^4$ or $(\sqrt[4]{x})^4 = (-4)^4$

$x = \frac{16}{81}$ or $x = 256$

Now, we need to check the results.

Check: $x = \frac{16}{81}$ | $x = 256$

$3\sqrt{\frac{16}{81}} + 10\sqrt[4]{\frac{16}{81}} = 8$?

$3\left(\frac{4}{9}\right) + 10\left(\frac{2}{3}\right) = 8$?

$\frac{4}{3} + \frac{20}{3} = 8$?

$8 = 8$ True

$3\sqrt{256} + 10\sqrt[4]{256} = 8$?

$3(16) + 10(4) = 8$?

$48 + 40 = 8$?

$48 = 8$ False

So, the solution is $\left\{\frac{16}{81}\right\}$.

In the last example, you may have noticed that $\sqrt[4]{x} = -4$ cannot be true since $\sqrt[4]{x} \geq 0$ and rejected that answer at that point. Just be sure to check the other answer.

Ex. 7 $m^{2/3} - 2m^{1/3} = 4$

Solution:

$m^{2/3} - 2m^{1/3} = 4$ (get zero on one side)

$m^{2/3} - 2m^{1/3} - 4 = 0$

Notice that $m^{2/3} = (m^{1/3})^2$.

$(m^{1/3})^2 - 2(m^{1/3}) - 4 = 0$

Let $u = (m^{1/3})$. So, the equation becomes:

$u^2 - 2u - 4 = 0$ (use the quadratic formula)

$a = 1$, $b = -2$, and $c = -4$

$$u = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(-4)}}{2(1)} = \frac{2 \pm \sqrt{4+16}}{2} = \frac{2 \pm \sqrt{20}}{2}$$

$$= \frac{2 \pm \sqrt{2^2 \bullet 5}}{2} = \frac{2 \pm 2\sqrt{5}}{2} = 1 \pm \sqrt{5}$$

So, $u = 1 - \sqrt{5}$ or $u = 1 + \sqrt{5}$

Now, replace $u = m^{1/3}$:

$m^{1/3} = 1 - \sqrt{5}$ or $m^{1/3} = 1 + \sqrt{5}$ (cube both sides)

$(m^{1/3})^3 = (1 - \sqrt{5})^3$ or $(m^{1/3})^3 = (1 + \sqrt{5})^3$ (simplify)

$m = (1 - \sqrt{5})^2(1 - \sqrt{5})$ or $m = (1 + \sqrt{5})^2(1 + \sqrt{5})$

$m = (1 - 2\sqrt{5} + (\sqrt{5})^2)(1 - \sqrt{5})$ or

$m = (1 + 2\sqrt{5} + (\sqrt{5})^2)(1 + \sqrt{5})$

$m = (6 - 2\sqrt{5})(1 - \sqrt{5})$ or $m = (6 + 2\sqrt{5})(1 + \sqrt{5})$

$m = 6 - 6\sqrt{5} - 2\sqrt{5} + 2(\sqrt{5})^2$ or $m = 6 + 6\sqrt{5} + 2\sqrt{5} + 2(\sqrt{5})^2$

$m = 6 - 6\sqrt{5} - 2\sqrt{5} + 10$ or $m = 6 + 6\sqrt{5} + 2\sqrt{5} + 10$

$m = 16 - 8\sqrt{5}$ or $m = 16 + 8\sqrt{5}$

Check: $m = 16 - 8\sqrt{5} \approx -1.889$ $\qquad$ $m = 16 + 8\sqrt{5} \approx 33.889$

$(-1.889)^{2/3} - 2(-1.889)^{1/3} \approx 4$ $\qquad$ $(33.889)^{2/3} - 2(33.889)^{1/3} \approx 4$

$1.528 - (-2.472) \approx 4$ $\qquad$ $10.472 - 6.472 \approx 4$

$4 \approx 4$ True $\qquad$ $4 \approx 4$ True

The solutions are $\{16 - 8\sqrt{5}, 16 + 8\sqrt{5}\}$.

Ex. 8 $\quad x^4 - 15 = 2(2x^2 + 15)$

Solution:

$x^4 - 15 = 2(2x^2 + 15)$ (simplify)

$x^4 - 15 = 4x^2 + 30$ (get zero on one side)

$x^4 - 4x^2 - 45 = 0$ $(x^4 = (x^2)^2)$

$(x^2)^2 - 4(x^2) - 45 = 0$ (let $u = x^2$)

$u^2 - 4u - 45 = 0$ (factor)

$(u - 9)(u + 5) = 0$ (solve)

$u - 9 = 0$ or $u + 5 = 0$

$u = 9$ or $u = -5$ (replace u by x^2)

$x^2 = 9$ or $x^2 = -5$ (square root property)

$x = \pm\sqrt{9}$ $\qquad$ $x = \pm\sqrt{-5}$

$x = \pm 3$ $\qquad$ $x = \pm i\sqrt{5}$

The solutions are $\{-i\sqrt{5}, i\sqrt{5}, -3, 3\}$.

Ex. 9 $\quad 2(3 - \sqrt{x})^2 + 5(3 - \sqrt{x}) = 42$

Solution:

$2(3 - \sqrt{x})^2 + 5(3 - \sqrt{x}) = 42$ (get zero on one side)

$2(3 - \sqrt{x})^2 + 5(3 - \sqrt{x}) - 42 = 0$ (let $u = 3 - \sqrt{x}$)

$2u^2 + 5u - 42 = 0$ (factor)

$(2u - 7)(u + 6) = 0$ (solve)

$2u - 7 = 0$ or $u + 6 = 0$

$u = 3.5$ or $u = -6$ (replace u by $3 - \sqrt{x}$)

$3 - \sqrt{x} = 3.5$ or $3 - \sqrt{x} = -6$

$-\sqrt{x} = 0.5$ or $-\sqrt{x} = -9$ (square both sides)

$(-\sqrt{x})^2 = (0.5)^2$ or $(-\sqrt{x})^2 = (-9)^2$ (simplify)

$x = 0.25$ or $x = 81$

Check: x = 0.25

$2(3 - \sqrt{0.25})^2 + 5(3 - \sqrt{0.25}) = 42$?

$2(3 - 0.5)^2 + 5(3 - 0.5) = 42$?

$2(2.5)^2 + 5(2.5) = 42$?

$12.5 + 12.5 = 42$?

25 = 42 False

Check: x = 81

$2(3 - \sqrt{81})^2 + 5(3 - \sqrt{81}) = 42$?

$2(3 - 9)^2 + 5(3 - 9) = 42$?

$2(-6)^2 + 5(-6) = 42$?

$72 - 30 = 42$?

42 = 42 True

The solution is {81}.