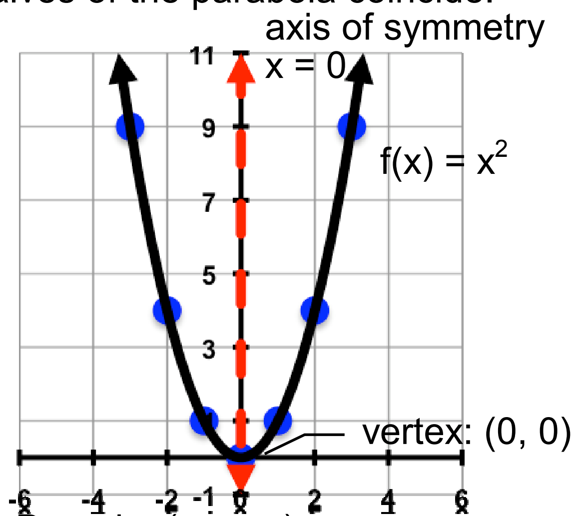
Sect 12.4 - Graphs of Quadratic Functions

In section 8.3, we graphed the function $f(x) = x^2$: Its graph is called a **parabola**. The **vertex** is the lowest point on the graph (0, 0) and the vertical line through the vertex is called the **axis of symmetry**. It is where we can fold the graph paper so that both halves of the parabola coincide.

x	$f(x) = x^2$	Points
0	$(0)^2 = 0$	(0, 0)
1	$(1)^2 = 1$	(1, 1)
2	$(2)^2 = 4$	(2, 4)
3	$(3)^2 = 9$	(3, 9)
– 1	$(-1)^2 = 1$	(– 1, 1)
– 2	$(-2)^2 = 4$	(– 2, 4)
– 3	$(-3)^2 = 9$	(– 3, 9)



Domain: $(-\infty, \infty)$
Range: $[0, \infty)$

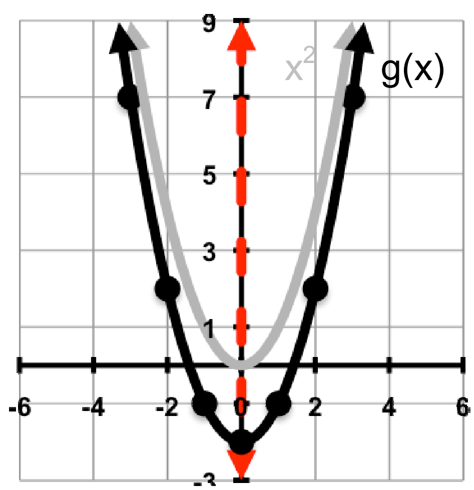
Concept #1 Functions in the Form $g(x) = f(x) + k$

Graph the following and compare it to $y = x^2$:

Ex. 1 $g(x) = x^2 - 2$

Solution:

x	$f(x) = x^2$	$g(x) = x^2 - 2$
0	$(0)^2 = 0$	– 2
1	$(1)^2 = 1$	– 1
2	$(2)^2 = 4$	2
3	$(3)^2 = 9$	7
– 1	$(-1)^2 = 1$	– 1
– 2	$(-2)^2 = 4$	2
– 3	$(-3)^2 = 9$	7



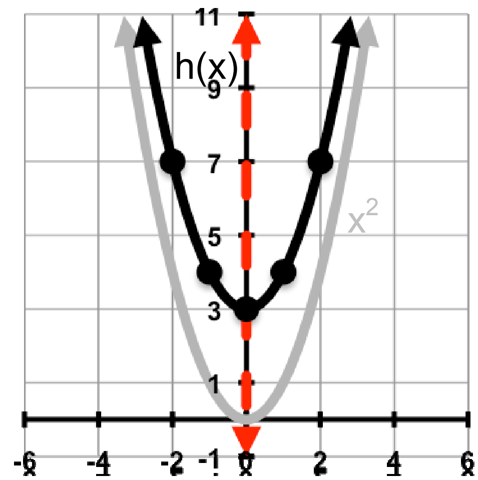
Vertex: (0, – 2)
Axis of Symmetry: x = 0

Notice that g is the graph of x^2 shifted down two units.

Ex. 2 $h(x) = x^2 + 3$

Solution:

x	$f(x) = x^2$	$h(x) = x^2 + 3$
0	$(0)^2 = 0$	3
1	$(1)^2 = 1$	4
2	$(2)^2 = 4$	7
3	$(3)^2 = 9$	12
– 1	$(-1)^2 = 1$	4
– 2	$(-2)^2 = 4$	7
– 3	$(-3)^2 = 9$	12



Vertex: (0, 3)
Axis of Symmetry: x = 0

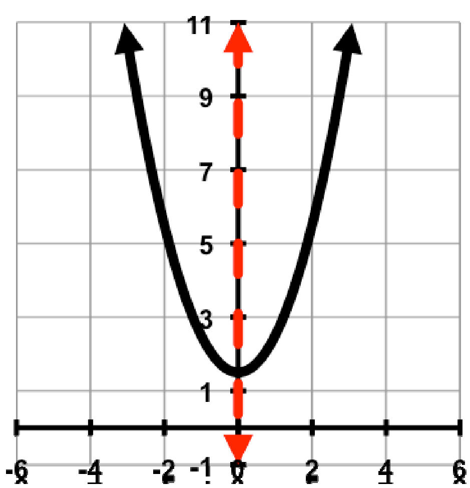
Notice that h is the graph of x^2 shifted up three units.

Vertical Shift
The graph $g(x) = f(x) + k$ is the graph of $f(x)$ shifted vertically by k units.

Graph the following:

Ex. 3a $g(x) = x^2 + 1.5$

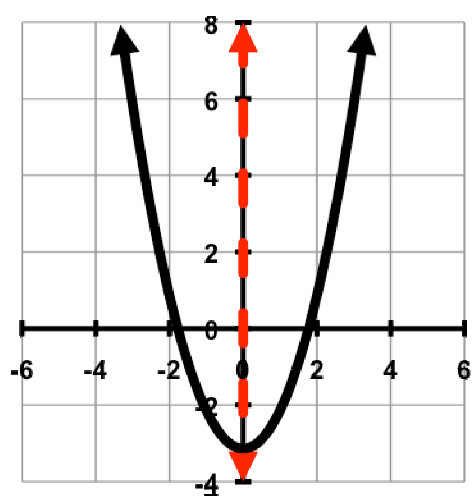
Solution:
Since k = 1.5, then g is the graph of x^2 shifted up 1.5 units.



Vertex: (0, 1.5)
Axis of Symmetry: x = 0

Ex. 3b $h(x) = x^2 - \pi$

Solution:
Since $k = -\pi$, then h is the graph of x^2 shifted down π units.



Vertex: $(0, -\pi)$
Axis of Symmetry: x = 0

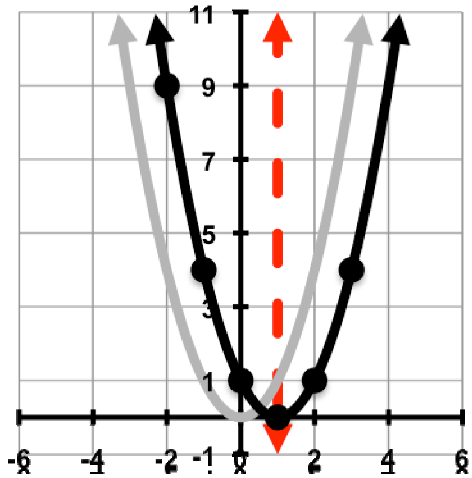
Concept #2 Functions in the Form g(x) = f(x – h)

Graph the following and compare it to y = x^2:

Ex. 4 h(x) = $(x – 1)^2$

Solution:

x	x – 1	h(x) = $(x – 1)^2$
0	0 – 1 = – 1	$(– 1)^2 = 1$
1	1 – 1 = 0	$(0)^2 = 0$
2	2 – 1 = 1	$(1)^2 = 1$
3	3 – 1 = 2	$(2)^2 = 4$
– 1	– 1 – 1 = – 2	$(– 2)^2 = 4$
– 2	– 2 – 1 = – 3	$(– 3)^2 = 9$
– 3	– 3 – 1 = – 4	$(– 4)^2 = 16$



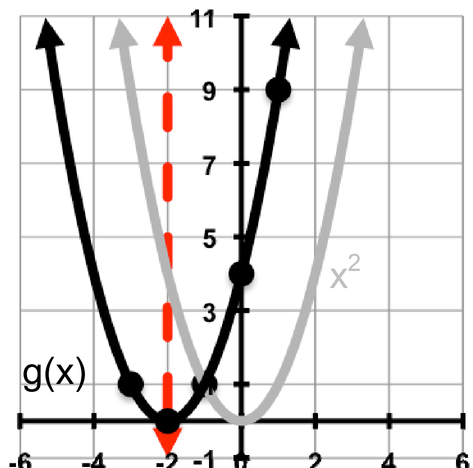
Vertex: (1, 0)
Axis of Symmetry: x = 1

Notice that h is the graph of x^2 shifted horizontally to the *right* by 1 unit. This shift is in the opposite direction to the sign in front of 1.

Ex. 5 g(x) = $(x + 2)^2$

Solution:

x	x + 2	g(x) = $(x + 2)^2$
0	0 + 2 = 2	$(2)^2 = 4$
1	1 + 2 = 3	$(3)^2 = 9$
2	2 + 2 = 4	$(4)^2 = 16$
3	3 + 2 = 5	$(5)^2 = 25$
– 1	– 1 + 2 = 1	$(1)^2 = 1$
– 2	– 2 + 2 = 0	$(0)^2 = 0$
– 3	– 3 + 2 = – 1	$(– 1)^2 = 1$



Vertex: (– 2, 0)
Axis of Symmetry: x = – 2

Notice that g is the graph of x^2 shifted horizontally to the *left* by 2 unit. This shift is in the opposite direction to the sign in front of 2.

Horizontal Shift

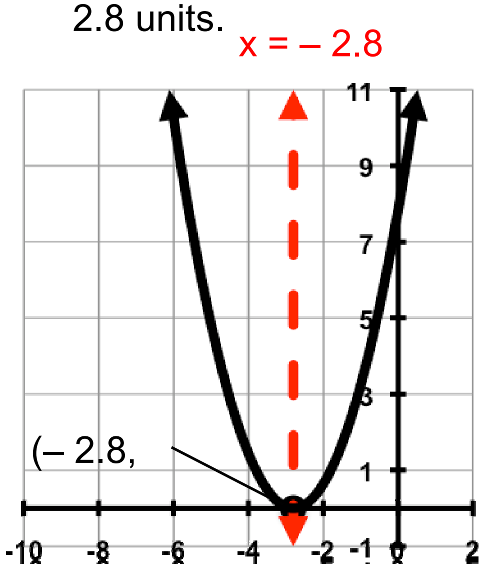
The graph g(x) = f(x – h) is the graph of f(x) shifted horizontally by h units.

<u>Graph the following:</u>

Ex. 6a $g(x) = (x + 2.8)^2$

<u>Solution:</u>

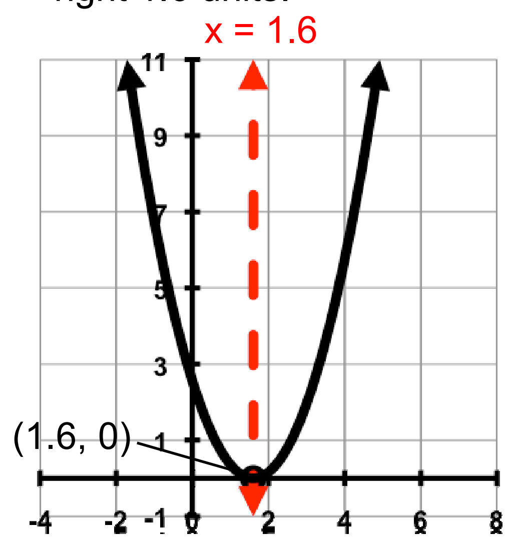
Since h = – 2.8, then g is the graph of x^2 shifted to the left 2.8 units.



Ex. 6b $r(x) = (x – 1.6)^2$

<u>Solution:</u>

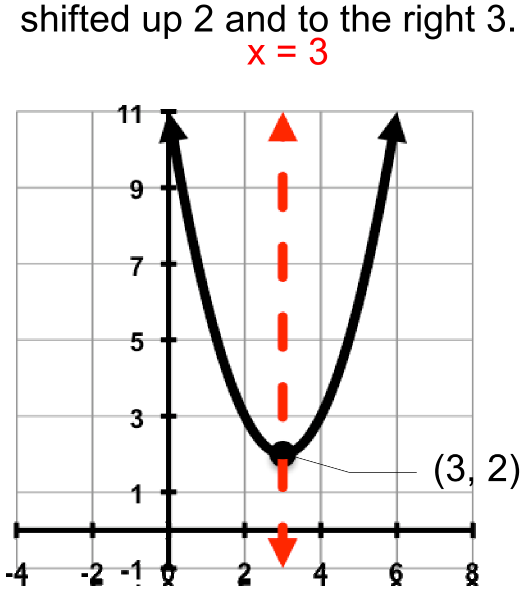
Since h = 1.6, then r is the graph of x^2 shifted to the right 1.6 units.



Ex. 7a $g(x) = (x – 3)^2 + 2$

<u>Solution:</u>

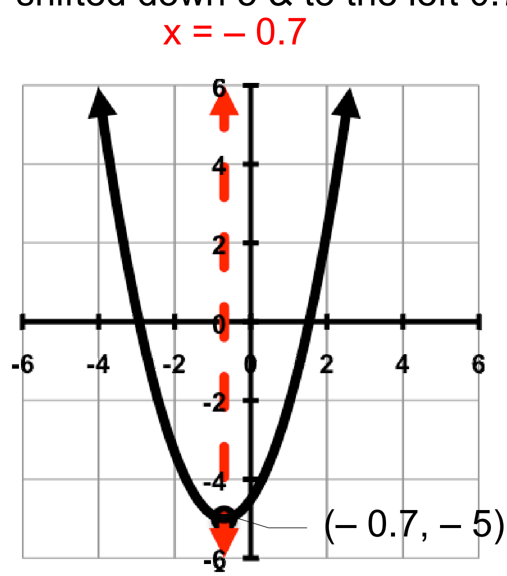
This graph is the graph of x^2 shifted up 2 and to the right 3.



Ex. 7b $q(x) = (x + 0.7)^2 – 5$

<u>Solution:</u>

This graph is the graph of x^2 shifted down 5 & to the left 0.7.



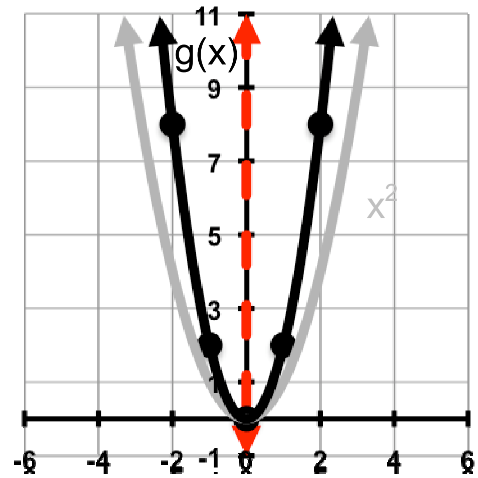
Concept #3 Functions in the Form g(x) = a•f(x)

Graph the following and compare it to $y = x^2$:

Ex. 8 $g(x) = 2x^2$

Solution:

x	$f(x) = x^2$	$g(x) = 2x^2$
0	$(0)^2 = 0$	0
1	$(1)^2 = 1$	2
2	$(2)^2 = 4$	8
3	$(3)^2 = 9$	18
– 1	$(-1)^2 = 1$	2
– 2	$(-2)^2 = 4$	8
– 3	$(-3)^2 = 9$	18



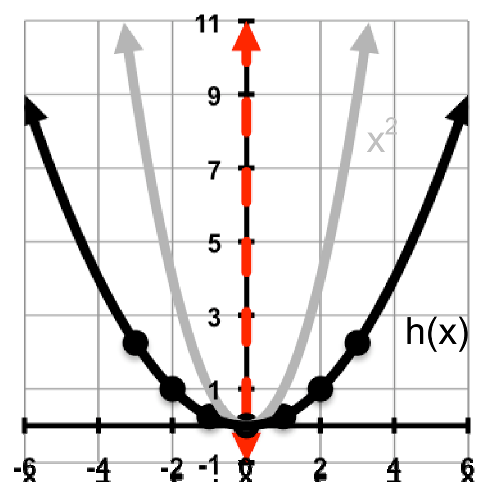
Vertex: (0, 0)
Axis of Symmetry: x = 0

Notice that g is the graph of x^2 stretched by a factor of 2, so the shape is "skinnier" than x^2.

Ex. 9 $h(x) = \frac{1}{4}x^2$

Solution:

x	$f(x) = x^2$	$h(x) = \frac{1}{4}x^2$
0	$(0)^2 = 0$	0
1	$(1)^2 = 1$	0.25
2	$(2)^2 = 4$	1
3	$(3)^2 = 9$	2.25
– 1	$(-1)^2 = 1$	0.25
– 2	$(-2)^2 = 4$	1
– 3	$(-3)^2 = 9$	2.25



Vertex: (0, 0)
Axis of Symmetry: x = 0

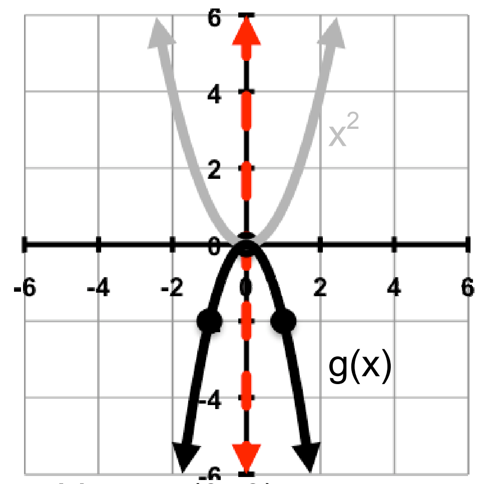
Notice that h is the graph of x^2 shrunk to 1/4 of its size, so the shape is "fatter" than x^2.

Stretch/Shrink (shape)
The graph of g(x) = a•f(x) is the graph of f(x) stretched by a factor of a if | a | > 1 and shrunk by a factor of a if | a | < 1.

Ex. 10 $g(x) = -2x^2$

Solution:

x	$f(x) = x^2$	$g(x) = -2x^2$
0	$(0)^2 = 0$	0
1	$(1)^2 = 1$	– 2
2	$(2)^2 = 4$	– 8
3	$(3)^2 = 9$	– 18
– 1	$(-1)^2 = 1$	– 2
– 2	$(-2)^2 = 4$	– 8
– 3	$(-3)^2 = 9$	– 18



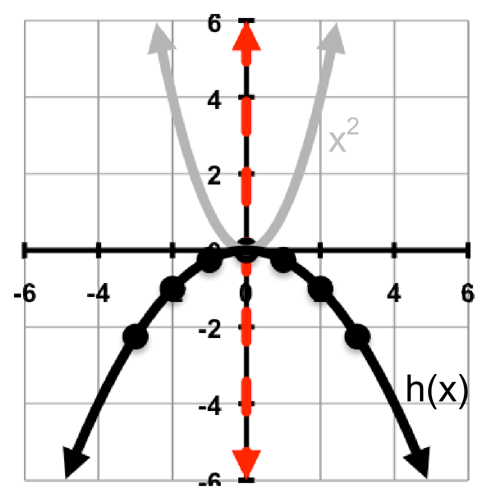
Vertex: (0, 0)
Axis of Symmetry: x = 0

Notice that g is the graph of x^2 stretched by a factor of 2, so the shape is "skinnier" than x^2 and it is reflected across the x-axis.

Ex. 11 $h(x) = -\frac{1}{4}x^2$

Solution:

x	$f(x) = x^2$	$h(x) = -\frac{1}{4}x^2$
0	$(0)^2 = 0$	0
1	$(1)^2 = 1$	– 0.25
2	$(2)^2 = 4$	– 1
3	$(3)^2 = 9$	– 2.25
– 1	$(-1)^2 = 1$	– 0.25
– 2	$(-2)^2 = 4$	– 1
– 3	$(-3)^2 = 9$	– 2.25



Vertex: (0, 0)
Axis of Symmetry: x = 0

Notice that h is the graph of x^2 shrunk to 1/4 of its size, so the shape is "fatter" than x^2 and it is reflected across the x-axis.

Reflection
The graph of g(x) = – f(x) is the graph of f(x) reflected across the x-axis.

Concept #4 Functions in the Form g(x) = a•f(x – h) + k

Let's summarize all the techniques discussed so far and develop a general strategy for graphing functions without point plotting, but based on the graph of y = f(x)

General Strategy for transformations:

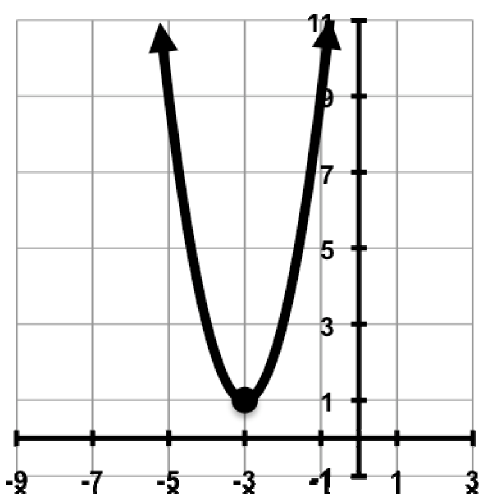
In graphing y = a•f(x – h) + k, we will start with the graph of f(x) and

i) Stretch it by a factor of a if $|a| > 1$
or shrink it by a factor of a if $|a| < 1$.
ii) Reflect it across the x-axis if a is negative.
iii) Shift it horizontally by h units and vertically by k units.

Graph the following:

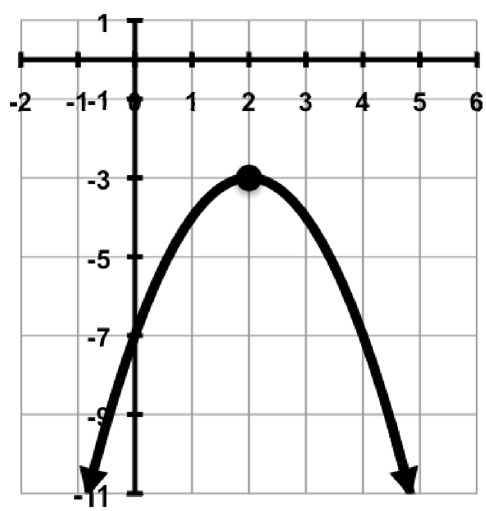
Ex. 12a $h(x) = 2(x + 3)^2 + 1$ Ex. 12b $f(x) = -(x - 2)^2 - 3$

Solution:
Since a = 2, the graph is stretched by a factor of 2. Since k is 1 and h is – 3, the graph is shifted up 1 unit and to the left by 3 units.



Vertex: (– 3, 1)
Axis of Symmetry: x = – 3

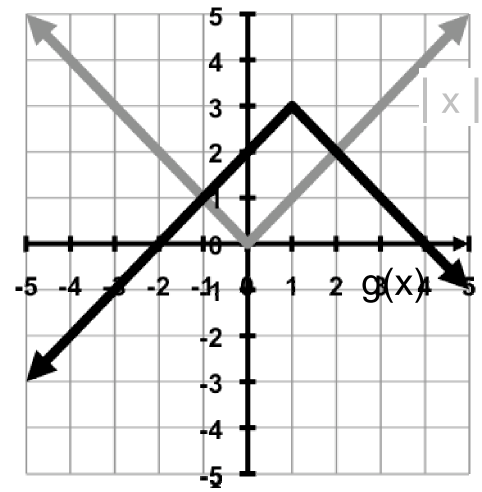
Solution:
Since a = – 1, the graph is reflected across the x - axis. Since k = – 3 and h = 2, the graph is shifted down 3 units & to the right by 2 units.



Vertex: (2, – 3)
Axis of Symmetry: x = 2

These techniques can be applied to graphing many other types of functions aside from quadratic functions such as f(x) = | x | .

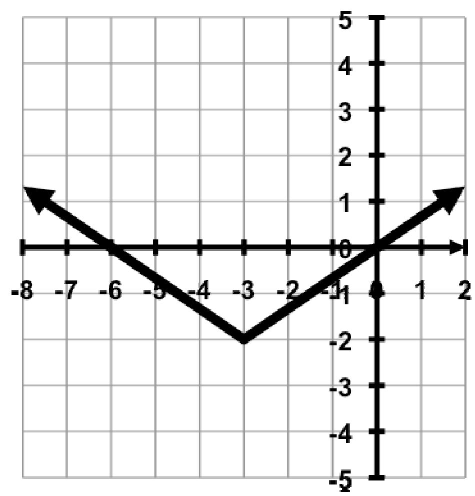
Ex. 13a Graph g(x) = – | x – 1 | + 3.



Solution:
Let's go through the steps of our general strategy :

i) Since | a | = 1, the graph is not stretched or shrunk.
ii) Since a is negative, the graph is reflected across the x-axis.
iii) Since k is 3 and h is 1, the graph is shifted up by 3 units and to the right by 1 unit.

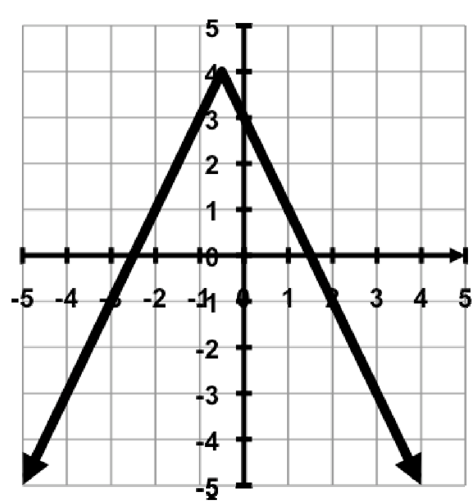
Ex. 13b Graph h(x) = $\frac{2}{3}$ | x + 3 | – 2



Solution:
Let's go through the steps of our general strategy:

i) Since | a | = $\frac{2}{3}$, the graph is shrunk to $\frac{2}{3}$ of its size.
ii) Since a is positive, the graph is not reflected across the x-axis.
iii) Since k is – 2 and h is – 3, the graph is shifted down by 2 units and to the left by 3 units.

Ex. 13c Graph f(x) = – 2 | x + 1/2 | + 4



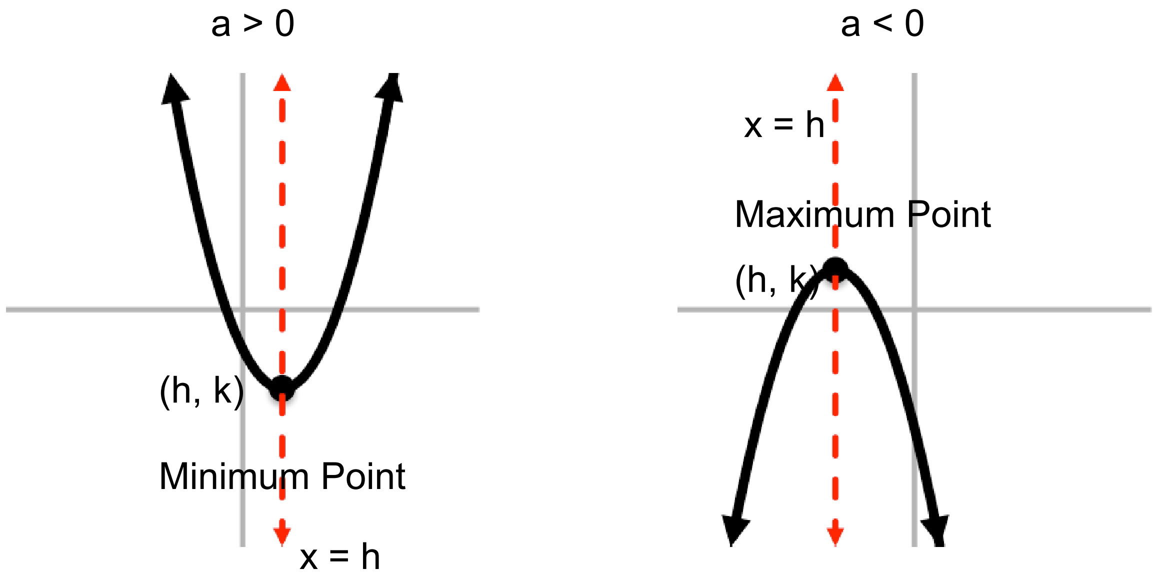
Solution:
Let's go through the steps of our general strategy :

i) Since | a | = 2, the graph is stretched by a factor of 2.
ii) Since a is negative, the graph is reflected across the x-axis.
iii) Since k is 4 and h is – 1/2, the graph is shifted up by 4 units and to the left by 1/2 unit.

Now, let's look at some specifics just for quadratic functions.

Graphs of $f(x) = a(x - h)^2 + k$

1) The vertex is the point (h, k).
2) The axis of symmetry is the line x = h.
3) If a > 0, the graph opens upward (smile), and k is the **minimum value** of the function.
4) If a < 0, the graph opens downward (frown), and k is the **maximum value** of the function.



<u>Without graphing, a) find the vertex, b) the axis of symmetry, c) and find the maximum or minimum value of the function:</u>

Ex. 14 $g(x) = -2(x - 4)^2 + 3$

<u>Solution:</u>

a) Since h = 4 and k = 3, the vertex is (4, 3).

b) The axis of symmetry is x = 4.

c) Since a < 0, the function opens downwards and has a maximum value of 3 at x = 4.

Ex. 15 $q(x) = 3(x + 1)^2 - 9$

<u>Solution:</u>

a) Since h = – 1 & k = – 9, the vertex is (– 1, – 9)

b) The axis of symmetry is x = – 1.

c) Since a > 0, the function opens upwards and has a minimum value of – 9 at x = – 1