Sect 3.3 - Slope and Rate of Change

Concept #1 Introduction to Slope

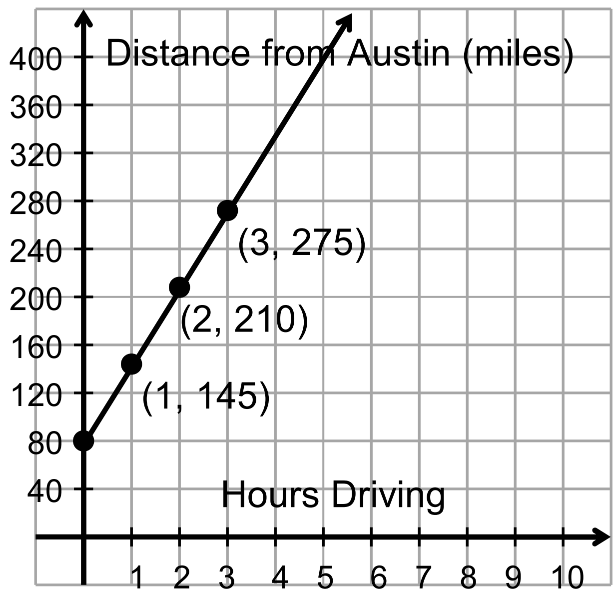
The **slope of a line**, denote by m, is a measure of "steepness" of a line. If (x_1, y_1) and (x_2, y_2) are two distinct points on a line, then

$$m = \frac{\text{"rise"}}{\text{"run"}} = \frac{\text{change in y}}{\text{change in x}} = \frac{y_2 - y_1}{x_2 - x_1}$$

To understand the relationship between slopes and rates of change, let us consider the following example:

Calculate the slope:

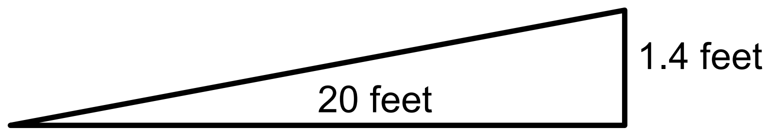
Ex. 1 Leroy started his driving trip 80 miles north of Austin and headed north at 65 mph. After every hour, Leroy plotted the distance he was from Austin versus the time had been on the road. The graph he created is to the right. Calculate the slope and interpret what it represents.



Solution:
We first pick two distinct points on the graph: (2, 210) & (3, 275).
Calculating the slope, we get: $m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{275 - 210}{3 - 2} = \frac{65}{1} = 65$.
Notice that this is the same as his speed. Thus, the slope in this context describes how his distance changes with respect to time or more simply his speed.

Ex. 2 On I-10 outside of Kerrville, a txdot worker determines the road rises at a rate of 1.4 feet for a horizontal distance of 20 feet. Calculate the slope and write it as a percentage.
Solution:

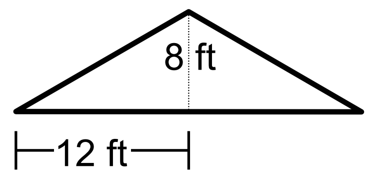
The road rises at a rate of 1.4 feet for a horizontal distance of 20 ft:



Calculating the slope, we get: $m = \frac{\text{"rise"}}{\text{"run"}} = \frac{1.4}{20} = \frac{14}{200} = \frac{7}{100} = 7\%$.

This corresponds to the grade of the road. So, the grade of the road is the slope of the road converted to a percent.

Ex. 3 An architect is designing a house with a roof as illustrated on the right. Find the slope of the roof.



Solution:

Calculating the slope, we get: $m = \frac{\text{"rise"}}{\text{"run"}} = \frac{8}{12} = \frac{2}{3}$. This is referred to as the pitch of the roof. So, the pitch of the roof is $\frac{2}{3}$.

Concept #2 Applying the Slope Formula

Calculate the slope of the line. The sketch the graph:

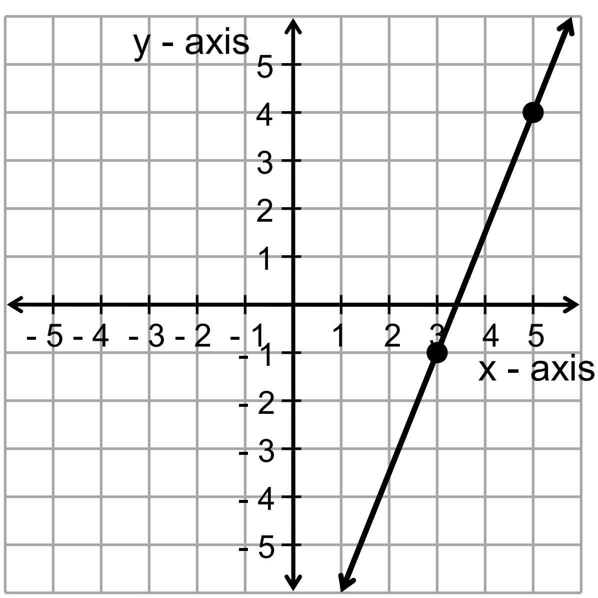
Ex. 4 The line passing through the points (3, – 1) and (5, 4).

Solution:

Calculating the slope, we get:

$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{4-(-1)}{5-3} = \frac{5}{2}$.

Now, sketch the graph:



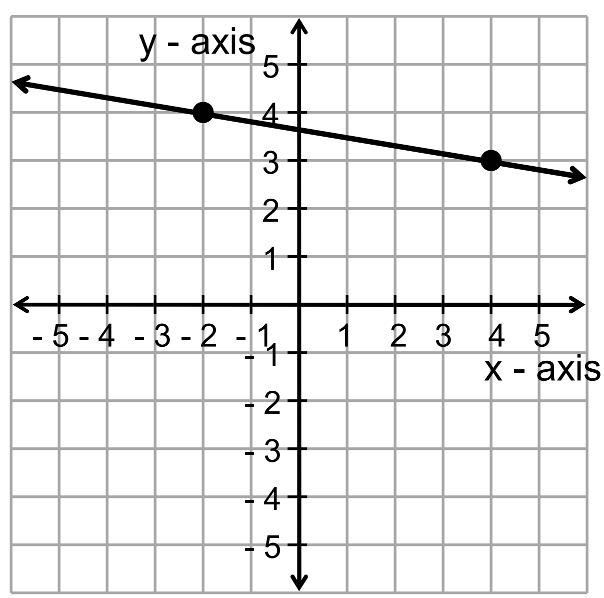
Ex. 5 The line passing through the points (4, 3) and (– 2, 4).

Solution:

Calculating the slope, we get:

$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{4-3}{-2-4} = \frac{1}{-6} = -\frac{1}{6}$.

Now, draw the graph:



Notice as you move from left to right, the graph in example #4 rises (positive slope) and the graph in example #5 falls (negative slope).

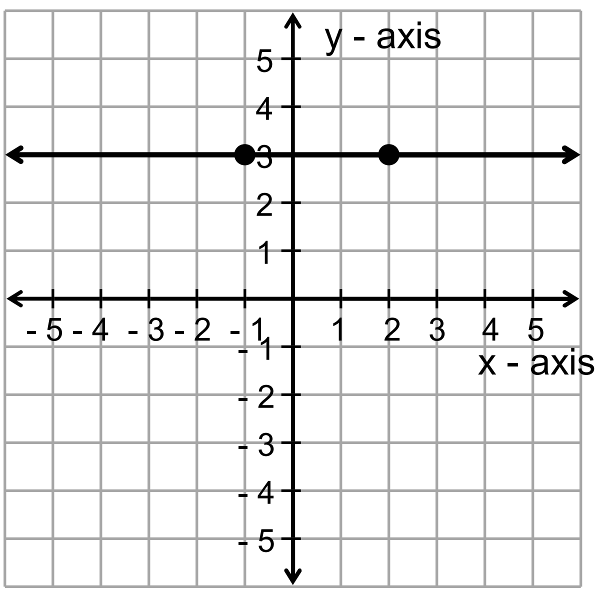
Ex. 6 $y = 3$

Solution:

Since there is no x term, we can make x anything and $y = 3$. So, two points on the line are $(-1, 3)$ and $(2, 3)$.

$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{3-3}{2-(-1)} = \frac{0}{3} = 0.$

Now, draw the graph:



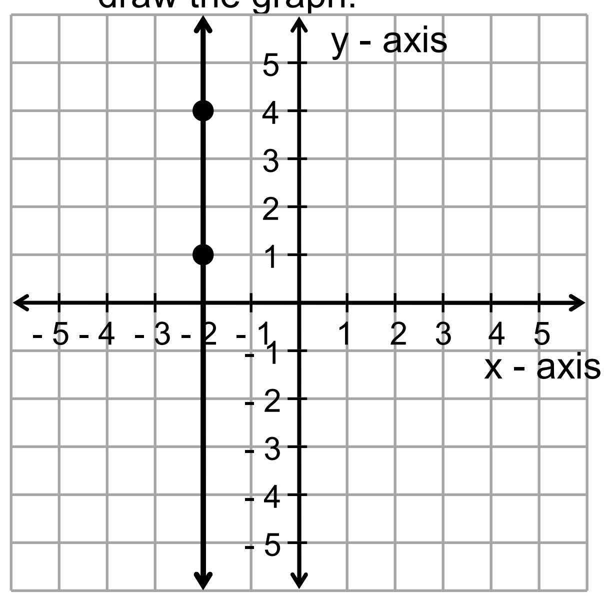
Ex. 7 $x = -2$

Solution:

Since there is no y term, we can make y anything and $x = -2$. So, two points on the line are $(-2, 1)$ & $(-2, 4)$.

$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{4-1}{-2-(-2)} = \frac{3}{0}$

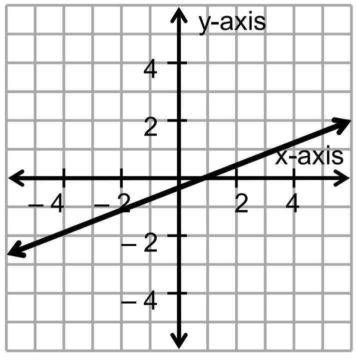
which is undefined. Now, draw the graph:



In general, all horizontal lines have slope of zero. Think of a horizontal as being level ground; it has zero steepness. A vertical line however is so steep that you cannot measure it. It's slope is undefined. In summary:

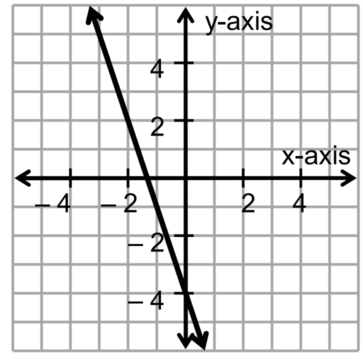
$m > 0$ (positive)

The line rises from left to right.

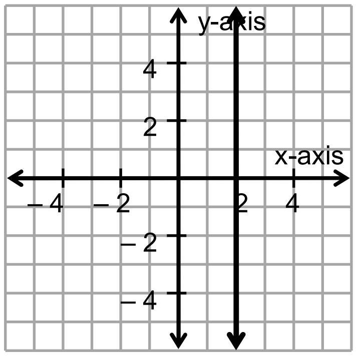


$m < 0$ (negative)

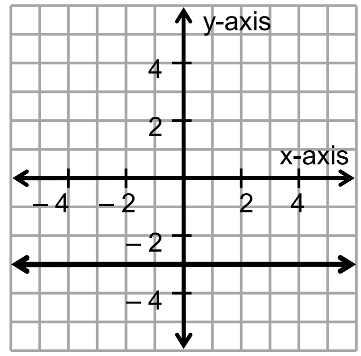
The line falls from left to right.



x = #   (vertical line)
m is undefined
Cannot climb the wall.



y = # (horizontal line)
m = 0
Floor is level.



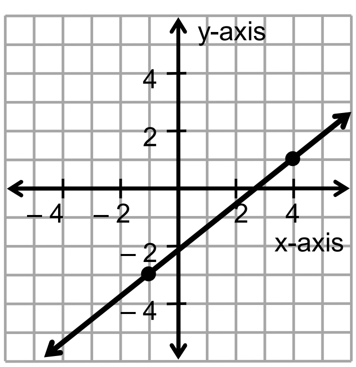
Concept #3   Parallel and Perpendicular Lines

Two distinct lines are **parallel** if and only if they have the same slope. We can denote the slope of a parallel line by $m_{||}$.

Two distinct lines that are neither vertical nor horizontal lines are **perpendicular** if and only if the product of their slopes is – 1. In other words, the slope of one line is the negative reciprocal of the slope of the line perpendicular to it so long as they are not vertical or horizontal lines. Every vertical line is perpendicular to every horizontal line and every horizontal line is perpendicular to every vertical line. We can denote the slope of a perpendicular line by $m_{\perp}$.

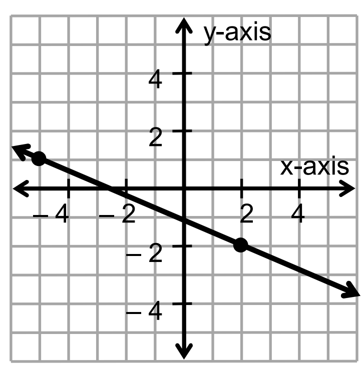
Given the information about the line, find a) the slope of the line parallel to it and b) the slope of the line perpendicular to it.

Ex . 8



The line passes through the points (– 1, – 3) and (4, 1). Thus, the slope of the line is
$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{1-(-3)}{4-(-1)} = \frac{4}{5}$.

Ex. 9



The line passes through the points (– 5, 1) and (2, – 2). Thus, the slope of the line is
$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{1-(-2)}{-5-2} = \frac{3}{-7} = -\frac{3}{7}$.

a) $m_{||} = \frac{4}{5}$.

b) $m_{\perp} = -\frac{5}{4}$.

a) $m_{||} = -\frac{3}{7}$.

b) $m_{\perp} = \frac{7}{3}$.

Ex. 10 The line passes through the points (– 1, 9) and (– 3, 4).

Solution:

Thus, the slope of the line is

$$m = \frac{y_2-y_1}{x_2-x_1} = \frac{4-9}{-3-(-1)} = \frac{-5}{-2} = \frac{5}{2}.$$

a) $m_{||} = \frac{5}{2}$.

b) $m_{\perp} = -\frac{2}{5}$.

Ex. 11 The line passes through the points (6, – 4) and (7, – 4).

Solution:

Thus, the slope of the line is

$$m = \frac{y_2-y_1}{x_2-x_1} = \frac{-4-(-4)}{7-6} = \frac{0}{1} = 0.$$

a) $m_{||} = 0$.

b) $m_{\perp}$ is undefined.

Ex. 12 The line passes through the points (– 2, 5) and (– 2, 11).

Solution:

$$m = \frac{y_2-y_1}{x_2-x_1} = \frac{11-5}{-2-(-2)} = \frac{6}{0}$$

which is undefined.

a) $m_{||}$ is undefined.

b) $m_{\perp} = 0$.

Ex. 13 The line passes through the points $\left(\frac{2}{3}, \frac{7}{6}\right)$ and $\left(\frac{11}{9}, \frac{1}{3}\right)$.

Solution:

$$m = \frac{\frac{1}{3}-\frac{7}{6}}{\frac{11}{9}-\frac{2}{3}} = \left(\frac{1}{3}-\frac{7}{6}\right) \div \left(\frac{11}{9}-\frac{2}{3}\right)$$

$$= \left(\frac{2}{6}-\frac{7}{6}\right) \div \left(\frac{11}{9}-\frac{6}{9}\right) = -\frac{5}{6} \div \frac{5}{9}$$

$$= -\frac{5}{6} \bullet \frac{9}{5} = -\frac{3}{2}.$$

a) $m_{||} = -\frac{3}{2}$.

b) $m_{\perp} = \frac{2}{3}$.

Concept #4 Applications of Slope

Ex. 14 In 1999, there were about 2.8 million high speed lines in the U.S. By 2003, that number had risen to 28.2 million. (source: www.fcc.gov). a) Write two ordered pairs of the form (year, number of high speed lines), b) find the slope of the line through those points and c) interpret what the slope means.

Solution:

a) The ordered pairs are (1999, 2.8) and (2003, 28.2).

b) The slope is $m = \frac{y_2-y_1}{x_2-x_1} = \frac{28.2-2.8}{2003-1999} = \frac{25.4}{4} = 6.35$ million per year.

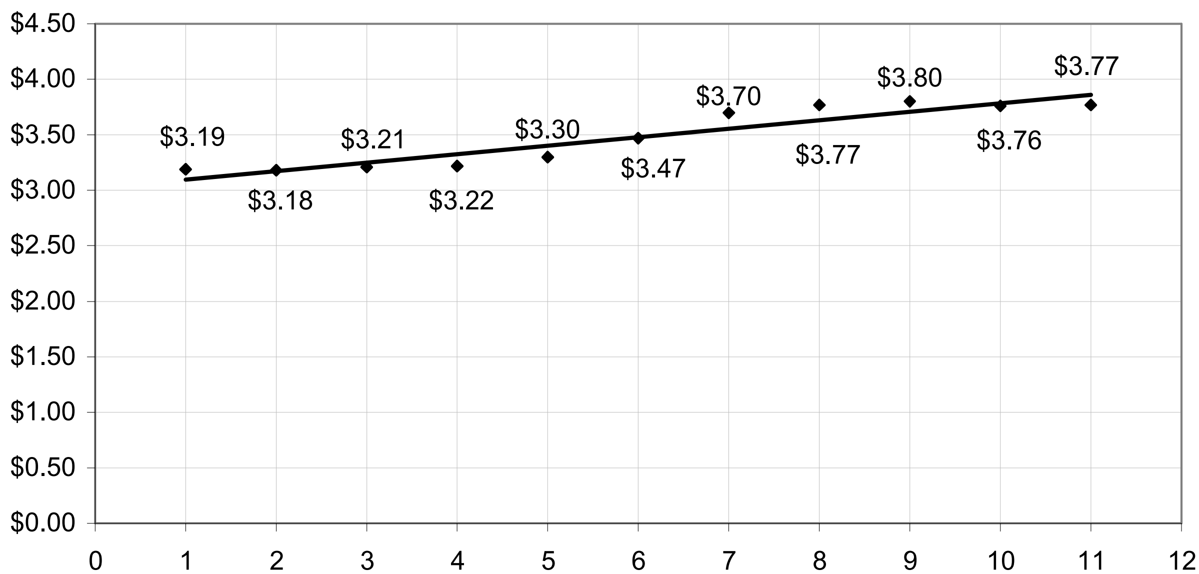
c) The number of high speed lines was increasing by 6.35 million lines per year.

Recall the following example from section 3.1:

Ex. 15 The average price of a gallon of 2% milk from thirty selected cities in the US for the first eleven months of the 2007 is given in the table below:

Month	Price per Gallon of 2% Milk
Jan, 2007	\$3.19
Feb, 2007	\$3.18
Mar, 2007	\$3.21
Apr, 2007	\$3.22
May, 2007	\$3.30
Jun, 2007	\$3.47
Jul, 2007	\$3.70
Aug, 2007	\$3.77
Sep, 2007	\$3.80
Oct, 2007	\$3.76
Nov, 2007	\$3.77

The graph with the trendline is given below:



Calculate the slope of the trendline and interpret what this means.
Solution:
The points (2, \$3.18) and (6, \$3.47) seem to be the closest points to the trend line so we will use them to calculate the slope:

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{3.47 - 3.18}{6 - 2} = \frac{0.29}{4} = \$0.0725 \text{ per month.}$$

This means that the price of a gallon of 2% milk was increasing at a rate of 7.25¢ per month over that eleven month period.