Sect 7.2 - Multiplication and Division of Rational Expressions

Concept #1 Multiplication of Rational Expressions

Recall that when we multiply two fractions, we always try to reduce first before we multiply the numerators and denominators. We did this by rewriting the numerators and denominators as a product and then dividing out the common factors:

$$\frac{22}{45} \bullet \frac{5}{11} = \frac{2 \bullet 11}{9 \bullet 5} \bullet \frac{5 \bullet 1}{11 \bullet 1} = \frac{2 \bullet \cancel{11}}{9 \bullet \cancel{5}} \bullet \frac{\cancel{5} \bullet 1}{\cancel{11} \bullet 1} = \frac{2}{9} \bullet \frac{1}{1} = \frac{2}{9}$$

With multiplying rational expressions, we want to do the same thing:

1. First, factor the all the numerators and denominators.
2. Divide out all the common factors.
3. Multiply the numerators and multiply the denominators to get the answer.

Multiplication of Rational Expressions
Let p, q, r and s be polynomials such that q ≠ 0 and s ≠ 0, then

$$\frac{p}{q} \bullet \frac{r}{s} = \frac{pr}{qs}$$

Simplify the following:

Ex. 1 $\frac{m^2+4}{m+8} \bullet \frac{m^2+9}{m^4+13m^2+36}$

Solution:
First, factor the numerators and denominators:
Numerator of the 1st fraction: $m^2 + 4$ is prime
Denominator of the 1st fraction: $m + 8$ is prime
Numerator of the 2nd fraction: $m^2 + 9$ is prime
Denominator of the 2nd fraction: $m^4 + 13m^2 + 36$
$= (m^2 + 9)(m^2 + 4)$, but $m^2 + 9$ and $m^2 + 4$ are prime.

Thus, $\frac{m^2+4}{m+8} \bullet \frac{m^2+9}{m^4+13m^2+36} = \frac{1(m^2+4)}{m+8} \bullet \frac{1(m^2+9)}{1(m^2+9)(m^2+4)}$

$= \frac{1\cancel{(m^2+4)}}{m+8} \bullet \frac{1\cancel{(m^2+9)}}{1\cancel{(m^2+9)}\cancel{(m^2+4)}} = \frac{1}{m+8} \bullet \frac{1}{1} = \frac{1}{m+8}$

Ex. 2 $\frac{5v+5}{v-2} \bullet \frac{v^2-4v+4}{v^2-1}$

Solution:
First, factor the numerators and denominators:
Numerator of the 1st fraction: $5v + 5 = 5(v + 1)$
Denominator of the 1st fraction: $v - 2$ is prime
Numerator of the 2nd fraction: $v^2 - 4v + 4 = (v - 2)(v - 2)$
Denominator of the 2nd fraction: $v^2 - 1 = (v - 1)(v + 1)$

Thus, $\frac{5v+5}{v-2} \bullet \frac{v^2-4v+4}{v^2-1} = \frac{5(v+1)}{1(v-2)} \bullet \frac{(v-2)(v-2)}{(v-1)(v+1)}$

$$= \frac{5\cancel{(v+1)}}{1\cancel{(v-2)}} \bullet \frac{\cancel{(v-2)}(v-2)}{(v-1)\cancel{(v+1)}} = \frac{5}{1} \bullet \frac{(v-2)}{(v-1)} = \frac{5(v-2)}{v-1}$$

Ex. 3 $\frac{x^3-27}{x^2-9} \bullet \frac{x+3}{x^2+3x+9}$

Solution:
First, factor the numerators and denominators:
Numerator of the 1st fraction: $x^3 - 27 = (x - 3)(x^2 + 3x + 9)$
Denominator of the 1st fraction: $x^2 - 9 = (x - 3)(x + 3)$
Numerator of the 2nd fraction: $x + 3$ is prime
Denominator of the 2nd fraction: $x^2 + 3x + 9$ is prime

Thus , $\frac{x^3-27}{x^2-9} \bullet \frac{x+3}{x^2+3x+9} = \frac{1(x-3)(x^2+3x+9)}{(x-3)(x+3)} \bullet \frac{1(x+3)}{1(x^2+3x+9)}$

$$= \frac{1\cancel{(x-3)}\cancel{(x^2+3x+9)}}{1\cancel{(x-3)}\cancel{(x+3)}} \bullet \frac{1\cancel{(x+3)}}{1\cancel{(x^2+3x+9)}} = \frac{1}{1} \bullet \frac{1}{1} = 1$$

Concept #2 Division of Rational Expressions

Recall that when we divide two fractions, we change the operation of division to the operation of multiplication and flip the fraction to the right. More precisely, we are multiplying the first fraction with the reciprocal of the second fraction. Then, we follow the steps for multiplication:

$$\frac{6}{5} \div \frac{4}{7} = \frac{6}{5} \bullet \frac{7}{4} = \frac{2 \cdot 3}{5} \bullet \frac{7}{2 \cdot 2} = \frac{\cancel{2} \cdot 3}{5} \bullet \frac{7}{\cancel{2} \cdot 2} = \frac{3}{5} \bullet \frac{7}{2} = \frac{21}{10}$$

Division of Rational Expressions

Let p, q, r and s be polynomials such that $q \neq 0$, $r \neq 0$, and $s \neq 0$, then

$$\frac{p}{q} \div \frac{r}{s} = \frac{p}{q} \bullet \frac{s}{r} = \frac{ps}{qr}$$

Simplify the following:

Ex. 4 $\frac{a}{a-b} \div \frac{b}{a-b}$

Solution:

First, invert the second fraction and multiply:

$\frac{a}{a-b} \div \frac{b}{a-b} = \frac{a}{a-b} \bullet \frac{a-b}{b}$

There is nothing to factor, so we proceed with the next step:

$\frac{a}{a-b} \bullet \frac{a-b}{b} = \frac{a}{1(a-b)} \bullet \frac{1(a-b)}{b} = \frac{a}{1} \bullet \frac{1}{b} = \frac{a}{b}$

Ex. 5 $\frac{4x-12}{4} \div \frac{6-2x}{6}$

Solution:

First, invert the second fraction and multiply:

$\frac{4x-12}{4} \div \frac{6-2x}{6} = \frac{4x-12}{4} \bullet \frac{6}{6-2x}$

Next, factor the numerators and denominators:

Numerator of the 1st fraction: 4x – 12 = 4(x – 3)

Denominator of the 1st fraction: 4 is prime

Numerator of the 2nd fraction: 6 is prime

Denominator of the 2nd fraction: 6 – 2x = – 2x + 6 = – 2(x – 3)

$\frac{4x-12}{4} \bullet \frac{6}{6-2x} = \frac{4(x-3)}{4} \bullet \frac{6}{-2(x-3)} = \frac{4}{4} \bullet \frac{6}{-2} = \frac{1}{1} \bullet \frac{3}{-1} = -3$

Ex. 6 $\frac{36x^2-48x+16}{3x^2+13x-10} \div \frac{4x^2-12x+9}{2x^2+7x-15}$

Solution:

First, invert the second fraction and multiply:

$\frac{36x^2-48x+16}{3x^2+13x-10} \div \frac{4x^2-12x+9}{2x^2+7x-15} = \frac{36x^2-48x+16}{3x^2+13x-10} \bullet \frac{2x^2+7x-15}{4x^2-12x+9}$

Next, factor the numerators and denominators:

Numerator of the 1st fraction: $36x^2 - 48x + 16 = 4(9x^2 - 12x + 4)$

$= 4(3x - 2)(3x - 2)$

Denominator of the 1st fraction: $3x^2 + 13x - 10 = (3x - 2)(x + 5)$

Numerator of the 2nd fraction: $2x^2 + 7x - 15 = (2x - 3)(x + 5)$

Denominator of the 2nd fraction: $4x^2 - 12x + 9 = (2x - 3)(2x - 3)$

$= \frac{36x^2-48x+16}{3x^2+13x-10} \bullet \frac{2x^2+7x-15}{4x^2-12x+9}$

$= \frac{4(3x-2)(3x-2)}{(3x-2)(x+5)} \bullet \frac{(2x-3)(x+5)}{(2x-3)(2x-3)} = \frac{4(3x-2)}{1(x+5)} \bullet \frac{1(x+5)}{(2x-3)} = \frac{4(3x-2)}{1} \bullet \frac{1}{(2x-3)}$

$= \frac{4(3x-2)}{2x-3}$.

Ex. 7 $\frac{x^3-2x^2-16x+32}{4x^2-14x+12} \div \frac{x^4-4x^3+64x-256}{2x-3}$

Solution:

First, invert the second fraction and multiply:

$\frac{x^3-2x^2-16x+32}{4x^2-14x+12} \div \frac{x^4-4x^3+64x-256}{2x-3} = \frac{x^3-2x^2-16x+32}{4x^2-14x+12} \bullet \frac{2x-3}{x^4-4x^3+64x-256}$

Next, factor the numerators and denominators:

Numerator of the 1st fraction: $x^3 - 2x^2 - 16x + 32$
$= (x^3 - 2x^2) + (-16x + 32)$
$= x^2(x - 2) - 16(x - 2)$
$= (x - 2)\mathbf{(x^2 - 16)}$
$= (x - 2)\mathbf{(x - 4)(x + 4)}$

Denominator of the 1st fraction: $4x^2 - 14x + 12 = 2(2x^2 - 7x + 6)$
$= 2(2x - 3)(x - 2)$

Numerator of the 2nd fraction: $(2x - 3)$ is prime

Denominator of the 2nd fraction: $x^4 - 4x^3 + 64x - 256$
$= (x^4 - 4x^3) + (64x - 256)$
$= x^3(x - 4) + 64(x - 4)$
$= (x - 4)\mathbf{(x^3 + 64)}$
$= (x - 4)\mathbf{(x + 4)(x^2 - 4x + 16)}$

$= \frac{x^3-2x^2-16x+32}{4x^2-14x+12} \bullet \frac{2x-3}{x^4-4x^3+64x-256}$

$= \frac{(x-2)(x-4)(x+4)}{2(2x-3)(x-2)} \bullet \frac{1(2x-3)}{(x-4)(x+4)(x^2-4x+16)}$

$= \frac{(x-4)(x+4)}{2(2x-3)} \bullet \frac{1(2x-3)}{(x-4)(x+4)(x^2-4x+16)}$

$= \frac{1(x-4)(x+4)}{2} \bullet \frac{1}{(x-4)(x+4)(x^2-4x+16)}$

$= \frac{1}{2} \bullet \frac{1}{(x^2-4x+16)}$

$= \frac{1}{2(x^2-4x+16)}$