Sect 7.3 - Least Common Denominator

Concept #1 Writing Equivalent Rational Expressions

Two fractions are equivalent if they are equal. In other words, they are equivalent if they both reduce to the same fraction. For instance, the fractions $\frac{30}{21}$ and $\frac{50}{35}$ are equivalent because they both reduce to $\frac{10}{7}$. We can also build fractions so that they have a given denominator by multiplying the top and bottom by the same non-zero number. If we want to write $\frac{10}{7}$ so that it has a denominator of 35, we would multiply top and bottom by 5 since 7 goes into 35 five times:

$$\frac{10}{7} = \frac{10}{7}(1) = \frac{10}{7} \bullet \frac{5}{5} = \frac{50}{35}.$$

We can do the same thing with rational expressions.

Write the following as an equivalent rational expression with the indicated denominator:

Ex. 1 $\frac{3}{xy} = \frac{\quad}{5x^2yz}$

Solution:
Take the new denominator divide by the old denominator:
$5x^2yz \div xy = 5xz$.
So, we need to multiply top and bottom by 5xz:

$$\frac{3}{xy} = \frac{3}{xy}(1) = \frac{3}{xy} \bullet \frac{5xz}{5xz} = \frac{15xz}{5x^2yz}$$

Ex. 2 $\frac{7x}{3y} = \frac{\quad}{21y^4}$

Solution:
Since $21y^4 \div 3y = 7y^3$, we need to multiply top and bottom by $7y^3$:

$$\frac{7x}{3y} = \frac{7x}{3y}(1) = \frac{7x}{3y} \bullet \frac{7y^3}{7y^3} = \frac{49xy^3}{21y^4}$$

Ex. 3 $\frac{2}{2w-5} = \frac{\quad}{10w^2-31w+15}$

Solution:
Since $10w^2 - 31w + 15 = (2w - 5)(5w - 3)$ and

(2w – 5)(5w – 3) ÷ (2w – 5) = (5w – 3), then we need to multiply top and bottom by (5w – 3):

$$\frac{2}{2w-5}\bullet\frac{(5w-3)}{(5w-3)} = \frac{2(5w-3)}{(2w-5)(5w-3)} = \frac{10w-6}{(2w-5)(5w-3)}$$

Notice that we left the denominator in factored form. In general, we usually leave the denominator of a rational expression in factored form.

Ex. 4 $\frac{8}{9x^2-16} = \frac{}{81x^4+108x^3-192x-256}$

Solution:

First, factor the denominators:

$9x^2 - 16 = (3x - 4)(3x + 4)$

$81x^4 + 108x^3 - 192x - 256$

$= (81x^4 + 108x^3) + (-192x - 256)$

$= 27x^3(3x + 4) - 64(3x + 4)$

$= (3x + 4)\mathbf{(27x^3 - 64)}$

$= (3x + 4)\mathbf{(3x - 4)(9x^2 + 12x + 16)}$

Since $(3x + 4)(3x - 4)(9x^2 + 12x + 16) \div (3x - 4)(3x + 4)$

$= (9x^2 + 12x + 16)$, then we need to multiply top and bottom by $(9x^2 + 12x + 16)$:

$$\frac{8}{9x^2-16} = \frac{8}{(3x-4)(3x+4)}\bullet\frac{(9x^2+12x+16)}{(9x^2+12x+16)} = \frac{8(9x^2+12x+16)}{(3x-4)(3x+4)(9x^2+12x+16)}$$

$$= \frac{72x^2+96x+128}{(3x-4)(3x+4)(9x^2+12x+16)}$$

Concept #2 Least Common Denominator

When adding two fractions with different denominators, we had to first find the least common denominator. To find the LCD of two fractions, we first find the prime factorization of each denominator. Next, we write down the product of each factor that appears in the prime factorizations. Finally, we choose the highest power of the factor in the prime factorizations. The resulting product is our L.C.D. For instance, if our two denominators are 18 and 60, the prime factorization of each is:

$18 = 2\bullet9 = 2\bullet3\bullet3 = 2\bullet3^2$

$60 = 4\bullet15 = 2\bullet2\bullet3\bullet5 = 2^2\bullet3\bullet5$

The factors 2, 3, and 5 appear in the prime factorizations so we write their product: $2\bullet3\bullet5$

But the highest power of the factors of 2 is 2 and the highest power of the factors of 3 is 2, so we write:

$2^2 \bullet 3^2 \bullet 5$

The principle is the same for finding the L.C.D. of rational expressions:

To Find the L.C.D. of Two or More Rational Expressions:

1. Factor the denominator of each rational expression.

2. Write down as a product each factor that appears in at least one of the factored denominators.

3. Raise each factor in step #2 to the highest power of that factor that appears in the factored denominators.

<u>Find the L.C.D. of the following rational expression:</u>

Ex. 5 $\frac{1}{2a^3b}, \frac{1}{8ab^2c}$

<u>Solution:</u>

We do not need to factor the denominators since they each have one term.

The least common multiple of 2 and 8 is 8. For the variable part, we write the product of each factor that appears:

8abc

The highest power of a is 3, of b is 2, and c is 1.

Thus, L.C.D. is $8a^3b^2c$

Ex. 6 $\frac{5x-7}{4x^2-4}, \frac{7x}{18x^2+12x-6}$

<u>Solution:</u>

First, factor the denominators:

$4x^2 - 4 = 4(x^2 - 1) = 4(x - 1)(x + 1)$

$18x^2 + 12x - 6 = 6(3x^2 + 2x - 1) = 6(3x - 1)(x + 1)$

The least common multiple of 4 and 6 is 12. For the variable part, we write the product of each factor that appears:

$12(x - 1)(x + 1)(3x - 1)$

The highest power of each factor is one.

Thus, L.C.D. = $12(x - 1)(x + 1)(3x - 1)$

Ex. 7 $\frac{7}{12a^3+48a^2+48a}, \frac{5}{4a^2+20a+24}$

Solution:
First, factor the denominators:
$12a^3 + 48a^2 + 48a = 12a(a^2 + 4a + 4) = 12a(a + 2)^2$
$4a^2 + 20a + 24 = 4(a^2 + 5a + 6) = 4(a + 3)(a + 2)$
The least common multiple of 4 and 12 is 12. For the variable part, we write the product of each factor that appears:

$12a(a + 2)(a + 3)$

The highest power of each factor is one except of the factor of $(a + 2)$; its highest power is 2
Thus, L.C.D. = $12a(a + 2)^2(a + 3)$

Ex. 8 $\frac{7y}{1000y^3 + 27}, \frac{41y}{1000y^3 - 300y^2 - 90y + 27}$

Solution:
First, factor the denominators:
$1000y^3 + 27 = (10y)^3 + (3)^3 = (10y + 3)(100y^2 - 30y + 9)$
$1000y^3 - 300y^2 - 90y + 27 = (1000y^3 - 300y^2) + (-90y + 27)$
$= 100y^2(10y - 3) - 9(10y - 3)$
$= (10y - 3)\mathbf{(100y^2 - 9)}$
$= (10y - 3)\mathbf{(10y - 3)(10y + 3)}$
$= (10y - 3)^2(10y + 3)$
We write the product of each factor that appears:

$(10y + 3)(10y - 3)(100y^2 - 30y + 9)$

The highest power of each factor is one except of the factor of $(10y - 3)$; its highest power is 2
Thus, L.C.D. = $(10y + 3)(10y - 3)^2(100y^2 - 30y + 9)$

Concept #3 Writing Rational Expressions with the L.C.D.

Now, let's put concepts #1 and #2 together.

Find the L.C.D. of each pair of rational expressions and then write each one as an equivalent rational expression with the L.C.D. for the denominator:

Ex. 9 $\frac{5}{14a^2b}, \frac{11}{21ab^3}$

Solution:
The L.C.D. of $\frac{5}{14a^2b}, \frac{11}{21ab^3}$ is $42a^2b^3$.
Since $42a^2b^3 \div 14a^2b = 3b^2$ and $42a^2b^3 \div 21ab^3 = 2a$, multiply the top and bottom of the first fraction by $3b^2$ and the top and bottom of

the second by 2a:

$$\frac{5}{14a^2b}\bullet\frac{3b^2}{3b^2} = \frac{15b^2}{42a^2b^3}$$

$$\frac{11}{21ab^3}\bullet\frac{2a}{2a} = \frac{22a}{42a^2b^3}$$

So, our fractions are $\frac{15b^2}{42a^2b^3}$ and $\frac{22a}{42a^2b^3}$.

Ex. 10 $\frac{7}{64x^3-1}, \frac{5}{1-16x^2}$

<u>Solution:</u>

First, factor the denominators:

$64x^3 – 1 = (4x)^3 – (1)^3 = (4x – 1)(16x^2 + 4x + 1)$

$1 – 16x^2 = – 16x^2 + 1 = – 1(16x^2 – 1) = – 1(4x – 1)(4x + 1)$

The question becomes what to do with the negative one? But, the following are equivalent:

$$\frac{5}{1-16x^2} = \frac{5}{-1(4x-1)(4x+1)} = \frac{-5}{(4x-1)(4x+1)}$$

So, if we get a – 1 in the denominator as part of the product, we can get rid of it by changing the sign of the numerator:

So, our denominators are $(4x – 1)(16x^2 + 4x + 1)$ & $(4x – 1)(4x + 1)$.

We write the product of each factor that appears:

$(4x – 1)(4x + 1)(16x^2 + 4x + 1)$

The highest power of each factor is one.

Thus, L.C.D. = $(4x – 1)(4x + 1)(16x^2 + 4x + 1)$

Since $(4x – 1)(4x + 1)(16x^2 + 4x + 1) \div (4x – 1)(16x^2 + 4y + 1)$ $= (4x + 1)$ and $(4x – 1)(4x + 1)(16x^2 + 4x + 1) \div (4x – 1)(4x + 1) =$ $(16x^2 + 4x + 1)$, multiply the top and bottom of the first fraction by $(4x + 1)$ and the top and bottom of the second by $(16x^2 + 4x + 1)$:

$$\frac{7}{64x^3-1} = \frac{7}{(4x-1)(16x^2+4x+1)} = \frac{7}{(4x-1)(16x^2+4x+1)}\bullet\frac{(4x+1)}{(4x+1)}$$

$$= \frac{7(4x+1)}{(4x-1)(4x+1)(16x^2+4x+1)} = \frac{28x+7}{(4x-1)(4x+1)(16x^2+4x+1)}$$

$$\frac{5}{1-16x^2} = \frac{-5}{(4x-1)(4x+1)} = \frac{-5}{(4x-1)(4x+1)}\bullet\frac{(16x^2+4x+1)}{(16x^2+4x+1)}$$

$$= \frac{-5(16x^2+4x+1)}{(4x-1)(4x+1)(16x^2+4x+1)} = \frac{-80x^2-20x-5}{(4x-1)(4x+1)(16x^2+4x+1)}$$

So, our fractions are $\frac{28x+7}{(4x-1)(4x+1)(16x^2+4x+1)}$ & $\frac{-80x^2-20x-5}{(4x-1)(4x+1)(16x^2+4x+1)}$.

Ex. 11 $\frac{2x}{6x^2+11x-35}$, $\frac{5}{8x^2+10x-63}$

Solution:

First, factor the denominators:

$6x^2 + 11x – 35 = (3x – 5)(2x + 7)$

$8x^2 + 10x – 63 = (4x – 9)(2x + 7)$

We write the product of each factor that appears:

$(3x – 5)(2x + 7)(4x – 9)$

The highest power of each factor is one.

Thus, L.C.D. = $(3x – 5)(2x + 7)(4x – 9)$

Since $(3x – 5)(2x + 7)(4x – 9) \div (3x – 5)(2x + 7) = (4x – 9)$ and $(3x – 5)(2x + 7)(4x – 9) \div (4x – 9)(2x + 7) = (3x – 5)$, multiply the top and bottom of the first fraction by $(4x – 9)$ and the top and bottom of the second by $(3x – 5)$:

$$\frac{2x}{6x^2+11x-35} = \frac{2x}{(3x-5)(2x+7)} = \frac{2x}{(3x-5)(2x+7)} \bullet \frac{(4x-9)}{(4x-9)}$$

$$= \frac{2x(4x-9)}{(3x-5)(2x+7)(4x-9)} = \frac{8x^2-18x}{(3x-5)(2x+7)(4x-9)}$$

$$\frac{5}{8x^2+10x-63} = \frac{5}{(4x-9)(2x+7)} = \frac{5}{(4x-9)(2x+7)} \bullet \frac{(3x-5)}{(3x-5)}$$

$$= \frac{5(3x-5)}{(3x-5)(2x+7)(4x-9)} = \frac{15x-25}{(3x-5)(2x+7)(4x-9)}$$

So, our fractions are $\frac{8x^2-18x}{(3x-5)(2x+7)(4x-9)}$ and $\frac{15x-25}{(3x-5)(2x+7)(4x-9)}$.

Ex. 12 $\frac{5}{9-7x}$, $\frac{11}{7x-9}$

Solution:

Since $\frac{5}{9-7x} = \frac{5}{-7x+9} = \frac{5}{-(7x-9)} = \frac{-5}{7x-9}$, we do not need to find the L.C.D. Our fractions are $\frac{-5}{7x-9}$ and $\frac{11}{7x-9}$.