# Sect 7.4 - Addition and Subtraction of Rational Expressions

Concept #1 Addition and Subtraction of Rational Expressions with the Same Denominator.

When adding and subtracting fractions with the same denominator, we add and subtract only the numerators; the denominator remains the same. For instance:

$$\frac{5}{8} + \frac{7}{8} = \frac{5+7}{8} = \frac{12}{8} = \frac{4\bullet3}{4\bullet2} = \frac{3}{2} \text{ and } \frac{11}{12} - \frac{4}{12} = \frac{11-4}{12} = \frac{7}{12}$$

The same applies to rational expressions. After performing the addition and subtraction and combining like terms in the numerator, we will need to see if the rational expression can be reduced. We will do this by factoring the numerator and denominator and dividing out the common factors of the numerator and denominator.

**Addition and Subtraction of Rational Expressions**

Let p, q, and r be polynomials such that q ≠ 0, then

**1.** $\frac{p}{q} + \frac{r}{q} = \frac{p+r}{q}$ and **2.** $\frac{p}{q} - \frac{r}{q} = \frac{p-r}{q}$

**Simplify the following:**

Ex. 1 $\frac{a}{7} + \frac{3a-4}{7}$

Solution:

$$\frac{a}{7} + \frac{3a-4}{7} = \frac{(a)+(3a-4)}{7} = \frac{a+3a-4}{7} = \frac{4a-4}{7} = \frac{4(a-1)}{7}$$

Ex. 2 $\frac{3a+13}{a+4} - \frac{2a+7}{a+4}$

Solution:

$$\frac{3a+13}{a+4} - \frac{2a+7}{a+4} = \frac{(3a+13)-(2a+7)}{a+4} = \frac{3a+13-2a-7}{a+4} = \frac{a+6}{a+4}$$

Ex. 3 $\frac{a^2-1}{a^2-7a+12} - \frac{8}{a^2-7a+12}$

Solution:

$$\frac{a^2-1}{a^2-7a+12} - \frac{8}{a^2-7a+12} = \frac{(a^2-1)-(8)}{a^2-7a+12} = \frac{a^2-1-8}{a^2-7a+12} = \frac{a^2-9}{a^2-7a+12}$$

But $a^2 - 9 = (a - 3)(a + 3)$ and $a^2 - 7a + 12 = (a - 3)(a - 4)$:

$$\frac{a^2-9}{a^2-7a+12} = \frac{(a-3)(a+3)}{(a-3)(a-4)} = \frac{a+3}{a-4}$$

Ex. 4 $\quad \frac{6y+11}{4y^2+12y-7} - \frac{4y+4}{4y^2+12y-7}$

Solution:

$$\frac{6y+11}{4y^2+12y-7} - \frac{4y+4}{4y^2+12y-7} = \frac{(6y+11)-(4y+4)}{4y^2+12y-7} = \frac{6y+11-4y-4}{4y^2+12y-7} = \frac{2y+7}{4y^2+12y-7}$$

But $4y^2 + 12y - 7 = (2y - 1)(2y + 7)$:

$$\frac{2y+7}{4y^2+12y-7} = \frac{1(2y+7)}{(2y-1)(2y+7)} = \frac{1}{2y-1}$$

Concept #2 Addition and Subtraction of Rational Expressions with Different Denominators.

When adding and subtracting fractions with different denominators, we need to first find the L.C.D., then build the fractions so that they have the same denominator, and finally combine the numerators and reduce:

$$\frac{5}{6} + \frac{3}{4} = \frac{5}{6}\bullet\frac{2}{2} + \frac{3}{4}\bullet\frac{3}{3} = \frac{10}{12} + \frac{9}{12} = \frac{19}{12}$$

We do exactly the same thing for rational expressions. We will find the L.C.D. and build our rational expressions the same way we did in the last section.

**Addition and Subtraction of Rational Expressions**

**1.** Factor the denominators completely.
**2.** Find the L.C.D.
**3.** Build each rational expression into an equivalent expression with the denominator equal to the L.C.D.
**4.** Add and subtract the numerators and write the result over the common denominator.
**5.** Simplify the numerator and factor the numerator to see if you can reduce.

**Simplify:**

Ex. 5 $\quad \frac{6}{z+4} - \frac{2}{3z+12}$

Solution:

First, factor the denominators:

z + 4 is prime

3z + 12 = 3(z + 4)

We write the product of each factor that appears:

$3(z + 4)$

The highest power of each factor is one.

Thus, L.C.D. = $3(z + 4)$

Since $3(z + 4) \div (z + 4) = 3$ and $3(z + 4) \div 3(z + 4) = 1$, multiply the top and bottom of the first fraction by 3 and the top and bottom of the second by 1:

$$\frac{6}{z+4} - \frac{2}{3z+12} = \frac{6}{z+4} - \frac{2}{3(z+4)} = \frac{6}{z+4} \bullet \frac{3}{3} - \frac{2}{3(z+4)} \bullet \frac{1}{1}$$

$$= \frac{6(3)-2(1)}{3(z+4)} = \frac{18-2}{3(z+4)} = \frac{16}{3(z+4)}$$

So our answer is $\frac{16}{3(z+4)}$.

Ex. 6 $\quad \frac{x-4}{x^2-9} + \frac{x+2}{x^2+11x+24}$

Solution:

First, factor the denominators:

$x^2 - 9 = (x - 3)(x + 3)$

$x^2 + 11x + 24 = (x + 3)(x + 8)$

We write the product of each factor that appears:

$(x - 3)(x + 3)(x + 8)$

The highest power of each factor is one.

Thus, L.C.D. = $(x - 3)(x + 3)(x + 8)$

Since $(x - 3)(x + 3)(x + 8) \div (x - 3)(x + 3) = (x + 8)$ and $(x - 3)(x + 3)(x + 8) \div (x + 3)(x + 8) = (x - 3)$, multiply the top and bottom of the first fraction by $(x + 8)$ and the top and bottom of the second by $(x - 3)$:

$$\frac{x-4}{x^2-9} + \frac{x+2}{x^2+11x+24} = \frac{x-4}{(x-3)(x+3)} + \frac{x+2}{(x+3)(x+8)}$$

$$= \frac{(x-4)}{(x-3)(x+3)}\frac{(x+8)}{(x+8)} + \frac{(x+2)}{(x+3)(x+8)}\frac{(x-3)}{(x-3)} = \frac{(x-4)(x+8)+(x+2)(x-3)}{(x-3)(x+3)(x+8)}$$

But, $(x - 4)(x + 8) = x^2 + 4x - 32$ and $(x + 2)(x - 3) = x^2 - x - 6$, so

$$\frac{(x-4)(x+8)+(x+2)(x-3)}{(x-3)(x+3)(x+8)} = \frac{x^2+4x-32+x^2-x-6}{(x-3)(x+3)(x+8)} = \frac{2x^2+3x-38}{(x-3)(x+3)(x+8)}$$

But, $2x^2 + 3x - 38$ is prime, so our answer is $\frac{2x^2+3x-38}{(x-3)(x+3)(x+8)}$.

Ex. 7 $\frac{5}{x^2+17x+16} - \frac{3}{x^2+9x+8}$

Solution:

First, factor the denominators:

$x^2 + 17x + 16 = (x + 16)(x + 1)$

$x^2 + 9x + 8 = (x + 1)(x + 8)$

We write the product of each factor that appears:

$(x + 16)(x + 1)(x + 8)$

The highest power of each factor is one.

Thus, L.C.D. = $(x + 16)(x + 1)(x + 8)$

Since $(x + 16)(x + 1)(x + 8) \div (x + 16)(x + 1) = (x + 8)$ and $(x + 16)(x + 1)(x + 8) \div (x + 1)(x + 8) = (x + 16)$, multiply the top and bottom of the first fraction by $(x + 8)$ and the top and bottom of the second by $(x + 16)$:

$$\frac{5}{x^2+17x+16} - \frac{3}{x^2+9x+8} = \frac{5}{(x+16)(x+1)} - \frac{3}{(x+1)(x+8)}$$

$$= \frac{(5)}{(x+16)(x+1)}\frac{(x+8)}{(x+8)} - \frac{(3)}{(x+1)(x+8)}\frac{(x+16)}{(x+16)} = \frac{(5)(x+8)-3(x+16)}{(x+16)(x+1)(x+8)}$$

$$= \frac{5x+40-3x-48}{(x+16)(x+1)(x+8)} = \frac{2x-8}{(x+16)(x+1)(x+8)} = \frac{2(x-4)}{(x+16)(x+1)(x+8)}$$

But nothing reduces, so our answer is $\frac{2(x-4)}{(x+16)(x+1)(x+8)}$.

Ex. 8 $\frac{3}{2c-1} - \frac{1}{c+2} - \frac{5}{2c^2+3c-2}$

Solution:

First, factor the denominators:

$2c – 1$ and $c + 2$ are prime and $2c^2 + 3c – 2 = (2c – 1)(c + 2)$

Thus, L.C.D. = $(2c – 1)(c + 2)$

We need to multiply the top and bottom of the first fraction by $(c + 2)$ and the top and bottom of the second by $(2c – 1)$:

$$\frac{3}{2c-1} - \frac{1}{c+2} - \frac{5}{2c^2+3c-2} = \frac{3}{2c-1} - \frac{1}{c+2} - \frac{5}{(2c-1)(c+2)}$$

$$= \frac{3}{(2c-1)} \bullet \frac{(c+2)}{(c+2)} - \frac{1}{(c+2)} \bullet \frac{(2c-1)}{(2c-1)} - \frac{5}{(2c-1)(c+2)}$$

$$= \frac{3(c+2)-1(2c-1)-5}{(2c-1)(c+2)} = \frac{3c+6-2c+1-5}{(2c-1)(c+2)} = \frac{c+2}{(2c-1)(c+2)} = \frac{1(c+2)}{(2c-1)(c+2)} = \frac{1}{(2c-1)}.$$

But nothing reduces, so our answer is $\frac{1}{(2c-1)}$.

Ex. 9 $\frac{-50x^2-55x+8}{15x^2+x-2} - \frac{25x}{5x+2} + \frac{25x^2+15x}{3x-1}$

Solution:

First, factor the denominators:

$15x^2 + x - 2 = (5x + 2)(3x - 1)$ and $5x + 2$ and $3x - 1$ are prime.

Thus, L.C.D. = $(5x + 2)(3x - 1)$

We need to multiply the top and bottom of the second fraction by $(3x - 1)$ and the top and bottom of the second by $(5x + 2)$:

$$\frac{-50x^2-55x+8}{15x^2+x-2} - \frac{25x}{5x+2} + \frac{25x^2+15x}{3x-1} = \frac{-50x^2-55x+8}{(5x+2)(3x-1)} - \frac{25x}{5x+2} + \frac{25x^2+15x}{3x-1}$$

$$= \frac{-50x^2-55x+8}{(5x+2)(3x-1)} - \frac{25x}{(5x+2)} \bullet \frac{(3x-1)}{(3x-1)} + \frac{(25x^2+15x)}{(3x-1)} \bullet \frac{(5x+2)}{(5x+2)}$$

$$= \frac{(-50x^2-55x+8)-25x(3x-1)+(25x^2+15x)(5x+2)}{(5x+2)(3x-1)}$$

But, $-25x(3x - 1) = -75x^2 + 25x$ and $(25x^2 + 15x)(5x + 2)$ $= 125x^3 + 50x^2 + 75x^2 + 30x = 125x^3 + 125x^2 + 30x$, so

$$\frac{(-50x^2-55x+8)-25x(3x-1)+(25x^2+15x)(5x+2)}{(5x+2)(3x-1)}$$

$$= \frac{-50x^2-55x+8-75x^2+25x+125x^3+125x^2+30x}{(5x+2)(3x-1)}$$

$$= \frac{125x^3-50x^2-75x^2+125x^2-55x+25x+30x+8}{(5x+2)(3x-1)} = \frac{125x^3+8}{(5x+2)(3x-1)}.$$

Since $125x^3 + 8 = (5x + 2)(25x^2 - 10x + 4)$, then

$$\frac{125x^3+8}{(5x+2)(3x-1)} = \frac{(5x+2)(25x^2-10x+4)}{(5x+2)(3x-1)} = \frac{25x^2-10x+4}{3x-1}$$

So, our answer is $\frac{25x^2-10x+4}{3x-1}$. (Note, $25x^2 + 10x + 4$ is prime).

Ex. 10 $\frac{25y^2}{5y-4} + \frac{16}{4-5y}$

Solution:

Notice that the denominators are opposites of one and another.

The following are equivalent:

$$\frac{16}{4-5y} = \frac{16}{-5y+4} = \frac{16}{-1(5y-4)} = -\frac{16}{(5y-4)}.$$

Thus, $\frac{25y^2}{5y-4} + \frac{16}{4-5y} = \frac{25y^2}{5y-4} - \frac{16}{(5y-4)} = \frac{25y^2-16}{(5y-4)}$

$$= \frac{(5y-4)(5y+4)}{1(5y-4)} = \frac{(5y+4)}{1} = 5y + 4$$

Concept #3 Using Rational Expressions in Translation

**Translate the following into an expression and then simplify:**

Ex. 11 Five times the reciprocal of the sum of a number and 3 subtract quotient of 4 and the total of the same number and negative three.

Solution:

Let n = the number

"sum of a number and 3": (n + 3)

"the reciprocal of (n + 3)": $\frac{1}{(n+3)}$

"Five times": $5 \bullet \frac{1}{(n+3)} = \frac{5}{(n+3)}$

"the total of the same number and negative three": (n + – 3)

"quotient of 4 and (n + – 3)": $4 \div (n + -3) = \frac{4}{(n-3)}$

"$\frac{5}{(n+3)}$ subtract $\frac{4}{(n-3)}$": $\frac{5}{(n+3)} - \frac{4}{(n-3)}$

L.C.D. = (n + 3)(n – 3). Multiply the first fraction by (n – 3) and the second fraction by (n + 3):

$$\frac{5}{(n+3)} \bullet \frac{(n-3)}{(n-3)} - \frac{4}{(n-3)} \bullet \frac{(n+3)}{(n+3)} = \frac{5(n-3)-4(n+3)}{(n+3)(n-3)} = \frac{5n-15-4n-12}{(n+3)(n-3)} = \frac{n-27}{(n+3)(n-3)}$$

So, the answer is $\frac{n-27}{(n+3)(n-3)}$.