Sect 7.5 - Complex Fractions

Concept #1 Simplifying Complex Fractions (Method #1 - Division)

Definition

A **Complex Fraction** is a fraction that contains at least one rational expression in either the numerator or denominator.

Recall the following example from arithmetic:

Simplify the following:

Ex. 1 $\dfrac{6\frac{3}{4}}{7\frac{1}{2}}$

Solution:

This is a division problem: $\frac{a}{b}$ = a ÷ b so the problem becomes:

$$\frac{6\frac{3}{4}}{7\frac{1}{2}} = 6\frac{3}{4} \div 7\frac{1}{2} = \frac{27}{4} \div \frac{15}{2} = \frac{27}{4} \bullet \frac{2}{15} = \frac{\cancel{3}\bullet 9}{\cancel{2}\bullet 2} \bullet \frac{\cancel{2}\bullet 1}{\cancel{3}\bullet 5} = \frac{9}{10}.$$

This example illustrates the first method of simplifying complex fractions:

Method #1 (Division)

1. Simplify the numerator to form single fraction in the numerator and simplify the denominator to form single fraction in the denominator.

2. Rewrite the complex fraction as the numerator divided by the denominator.

3. Invert the second fraction and multiply.

Simplify:

Ex. 2 $\dfrac{\frac{10x+12y}{7x+14y}}{\frac{15x+18y}{4x+8y}}$

Solution:

Since there is already a single fraction in the numerator and a single fraction in the denominator, we can rewrite this as a division problem:

$$\frac{\frac{10x+12y}{7x+14y}}{\frac{15x+18y}{4x+8y}} = \frac{10x+12y}{7x+14y} \div \frac{15x+18y}{4x+8y} = \frac{10x+12y}{7x+14y} \bullet \frac{4x+8y}{15x+18y}$$

Next, factor the numerators and denominators:
Numerator of the 1st fraction: $10x + 12y = 2(5x + 6y)$
Denominator of the 1st fraction: $7x + 14y = 7(x + 2y)$
Numerator of the 2nd fraction: $4x + 8y = 4(x + 2y)$
Denominator of the 2nd fraction: $15x + 18y = 3(5x + 6y)$
Thus,

$$\frac{10x+12y}{7x+14y} \bullet \frac{4x+8y}{15x+18y} = \frac{2(5x+6y)}{7(x+2y)} \bullet \frac{4(x+2y)}{3(5x+6y)} = \frac{2}{7} \bullet \frac{4}{3} = \frac{8}{21}$$

Ex. 3 $\dfrac{\frac{3}{m^2-16} - \frac{3}{m+4}}{\frac{4}{m-4} - \frac{36}{m^2-16}}$

Solution:
First, simplify the numerator to a single fraction and the denominator to a single fraction:

$$\frac{3}{m^2-16} - \frac{3}{m+4} = \frac{3}{(m-4)(m+4)} - \frac{3}{m+4} = \frac{3}{(m-4)(m+4)} - \frac{3}{(m+4)} \bullet \frac{(m-4)}{(m-4)}$$

$$= \frac{3-3(m-4)}{(m-4)(m+4)} = \frac{3-3m+12}{(m-4)(m+4)} = \frac{-3m+15}{(m-4)(m+4)} = \frac{-3(m-5)}{(m-4)(m+4)}$$

$$\frac{4}{m-4} - \frac{36}{m^2-16} = \frac{4}{m-4} - \frac{36}{(m-4)(m+4)} = \frac{4}{m-4} \bullet \frac{(m+4)}{(m+4)} - \frac{36}{(m-4)(m+4)}$$

$$= \frac{4(m+4)-36}{(m-4)(m+4)} = \frac{4m+16-36}{(m-4)(m+4)} = \frac{4m-20}{(m-4)(m+4)} = \frac{4(m-5)}{(m-4)(m+4)}$$

Thus, $\dfrac{\frac{3}{m^2-16} - \frac{3}{m+4}}{\frac{4}{m-4} - \frac{36}{m^2-16}} = \dfrac{\frac{-3(m-5)}{(m-4)(m+4)}}{\frac{4(m-5)}{(m-4)(m+4)}}$. Now, rewrite as a division problem:

$$\frac{\frac{-3(m-5)}{(m-4)(m+4)}}{\frac{4(m-5)}{(m-4)(m+4)}} = \frac{-3(m-5)}{(m-4)(m+4)} \div \frac{4(m-5)}{(m-4)(m+4)} = \frac{-3(m-5)}{1(m-4)(m+4)} \bullet \frac{1(m-4)(m+4)}{4(m-5)}$$

$$= \frac{-3}{1} \bullet \frac{1}{4} = -\frac{3}{4}$$

Ex. 4 $\dfrac{\frac{2}{a+5} + \frac{2}{a-3}}{a+5 - \frac{a^2-6a+9}{a+5}}$

Solution:
First, simplify the numerator to a single fraction and the denominator to a single fraction:

$$\frac{2}{a+5}+\frac{2}{a-3}=\frac{2}{(a+5)}\bullet\frac{(a-3)}{(a-3)}+\frac{2}{(a-3)}\bullet\frac{(a+5)}{(a+5)}=\frac{2(a-3)+2(a+5)}{(a+5)(a-3)}$$

$$=\frac{2a-6+2a+10}{(a+5)(a-3)}=\frac{4a+4}{(a+5)(a-3)}=\frac{4(a+1)}{(a+5)(a-3)}$$

$$a+5-\frac{a^2-6a+9}{a+5}=\frac{(a+5)}{1}\bullet\frac{(a+5)}{(a+5)}-\frac{a^2-6a+9}{a+5}=\frac{(a+5)(a+5)-(a^2-6a+9)}{(a+5)}$$

$$=\frac{a^2+10a+25-a^2+6a-9}{(a+5)}=\frac{16a+16}{(a+5)}=\frac{16(a+1)}{(a+5)}$$

Hence, $\dfrac{\frac{2}{a+5}+\frac{2}{a-3}}{a+5-\frac{a^2-6a+9}{a+5}}=\dfrac{\frac{4(a+1)}{(a+5)(a-3)}}{\frac{16(a+1)}{a+5}}=\dfrac{4(a+1)}{(a+5)(a-3)}\div\dfrac{16(a+1)}{(a+5)}$

$$=\frac{4(a+1)}{(a+5)(a-3)}\bullet\frac{1(a+5)}{16(a+1)}=\frac{4}{(a-3)}\bullet\frac{1}{16}=\frac{1}{4(a-3)}$$

Concept #2 Simplifying Complex Fractions (Method #2 - L.C.D.)

An alternative way to simplifying rational expressions is by clearing the fractions out of the numerator and denominator first. This is done by multiplying top and bottom of the complex fraction by the L.C.D. of all the rational expressions in the numerator and denominator.

Method #2 (L.C.D.)

1. Factor all the denominators of the rational expressions in the numerator and denominator of the complex fraction.
2. Find the L.C.D. of all the factored denominators and multiply top and bottom of the complex fraction by the L.C.D.
3. Distribute and simplify.

Simplify:

Ex. 5 $\dfrac{\frac{3}{4p^2}+\frac{7}{6p}}{\frac{5}{12p}+\frac{11}{3p^2}}$

Solution:
The L.C.D. = $12p^2$
Multiply the top and the bottom of the complex fraction by $12p^2$.

$$\frac{\frac{3}{4p^2}+\frac{7}{6p}}{\frac{5}{12p}+\frac{11}{3p^2}} = \left(\frac{\frac{3}{4p^2}+\frac{7}{6p}}{\frac{5}{12p}+\frac{11}{3p^2}}\right) \bullet \frac{\frac{12p^2}{1}}{\frac{12p^2}{1}} = \frac{\frac{3}{4p^2}\bullet\frac{12p^2}{1}+\frac{7}{6p}\bullet\frac{12p^2}{1}}{\frac{5}{12p}\bullet\frac{12p^2}{1}+\frac{11}{3p^2}\bullet\frac{12p^2}{1}} = \frac{\frac{3}{1}\bullet\frac{3}{1}+\frac{7}{1}\bullet\frac{2p}{1}}{\frac{5}{1}\bullet\frac{p}{1}+\frac{11}{1}\bullet\frac{4}{1}}$$

$$= \frac{3\bullet 3+7\bullet 2p}{5\bullet p+11\bullet 4} = \frac{9+14p}{5p+44} = \frac{14p+9}{5p+44}.$$

Ex. 6 $\quad \dfrac{\frac{2}{4m+12}-\frac{2}{2m^2+12m+18}}{\frac{3}{2}-\frac{3}{m+3}}$

Solution:

First, factor all the denominators:

$4m + 12 = 4(m + 3)$

$2m^2 + 12m + 18 = 2(m^2 + 6m + 9) = 2(m + 3)^2$

2 and $(m + 3)$ are prime.

The L.C.D. = $4(m + 3)^2$. Multiply the top and the bottom of the complex fraction by $4(m + 3)^2$.

$$\frac{\frac{2}{4m+12}-\frac{2}{2m^2+12m+18}}{\frac{3}{2}-\frac{3}{m+3}} = \left(\frac{\frac{2}{4(m+3)}-\frac{2}{2(m+3)^2}}{\frac{3}{2}-\frac{3}{(m+3)}}\right)\bullet\frac{\frac{4(m+3)^2}{1}}{\frac{4(m+3)^2}{1}}$$

$$= \frac{\frac{2}{4(m+3)}\bullet\frac{4(m+3)^2}{1}-\frac{1}{2(m+3)^2}\bullet\frac{4(m+3)^2}{1}}{\frac{3}{2}\bullet\frac{4(m+3)^2}{1}-\frac{3}{(m+3)}\bullet\frac{4(m+3)^2}{1}} = \frac{\frac{2}{1}\bullet\frac{(m+3)}{1}-\frac{2}{1}\bullet\frac{2}{1}}{\frac{3}{1}\bullet\frac{2(m+3)^2}{1}-\frac{3}{1}\bullet\frac{4(m+3)}{1}}$$

$$= \frac{2(m+3)-4}{6(m+3)^2-12(m+3)} = \frac{2m+6-4}{6(m^2+6m+9)-12m-36} = \frac{2m+2}{6m^2+36m+54-12m-36}$$

$$= \frac{2m+2}{6m^2+24m+18} = \frac{2(m+1)}{6(m^2+4m+3)} = \frac{2(m+1)}{6(m+1)(m+3)} = \frac{2}{6(m+3)} = \frac{1}{3(m+3)}.$$

Ex. 7 $\quad \dfrac{\frac{6}{a^2}+\frac{13}{a}+6}{10+\frac{1}{a}-\frac{21}{a^2}}$

Solution:

The L.C.D. = a^2. Multiply the top and the bottom of the complex fraction by a^2.

$$\frac{\frac{6}{a^2}+\frac{13}{a}+6}{10+\frac{1}{a}-\frac{21}{a^2}} = \left(\frac{\frac{6}{a^2}+\frac{13}{a}+6}{10+\frac{1}{a}-\frac{21}{a^2}}\right)\bullet\frac{\frac{a^2}{1}}{\frac{a^2}{1}} =$$

$$\frac{\frac{6}{a^2}\bullet\frac{a^2}{1}+\frac{13}{a}\bullet\frac{a^2}{1}+6\bullet a^2}{10\bullet a^2+\frac{1}{a}\bullet\frac{a^2}{1}-\frac{21}{a^2}\bullet\frac{a^2}{1}} = \frac{\frac{6}{1}\bullet\frac{1}{1}+\frac{13}{1}\bullet\frac{a}{1}+6a^2}{10a^2+\frac{1}{1}\bullet\frac{a}{1}-\frac{21}{1}\bullet\frac{1}{1}} = \frac{6+13a+6a^2}{10a^2+a-21}$$

$$= \frac{6a^2+13a+6}{10a^2+a-21} = \frac{(3a+2)(2a+3)}{(5a-7)(2a+3)} = \frac{3a+2}{5a-7}.$$

Ex. 8 $\dfrac{\frac{x+10}{x^3-8}}{\frac{3}{x-2}+\frac{21}{x^2+2x+4}}$

Solution:

First, factor all the denominators:

$x^3 - 8 = (x - 2)(x^2 + 2x + 4)$

$(x - 2)$ and $(x^2 + 2x + 4)$ are prime.

The L.C.D. = $(x - 2)(x^2 + 2x + 4)$. Multiply the top and the bottom of the complex fraction by $(x - 2)(x^2 + 2x + 4)$:

$$\frac{\frac{x+10}{x^3-8}}{\frac{3}{x-2}+\frac{21}{x^2+2x+4}} = \left(\frac{\frac{(x+10)}{(x-2)(x^2+2x+4)}}{\frac{3}{(x-2)}+\frac{21}{(x^2+2x+4)}}\right)\bullet\frac{\frac{(x-2)(x^2+2x+4)}{1}}{\frac{(x-2)(x^2+2x+4)}{1}}$$

$$= \frac{\frac{(x+10)}{(x-2)(x^2+2x+4)}\bullet\frac{(x-2)(x^2+2x+4)}{1}}{\frac{3}{(x-2)}\bullet\frac{(x-2)(x^2+2x+4)}{1}+\frac{21}{(x^2+2x+4)}\bullet\frac{(x-2)(x^2+2x+4)}{1}}$$

$$= \frac{\frac{(x+10)}{1}\bullet\frac{1}{1}}{\frac{3}{1}\bullet\frac{(x^2+2x+4)}{1}+\frac{21}{1}\bullet\frac{(x-2)}{1}} = \frac{(x+10)}{3(x^2+2x+4)+21(x-2)}$$

$$= \frac{(x+10)}{3x^2+6x+12+21x-42} = \frac{(x+10)}{3x^2+27x-30}$$

$$= \frac{(x+10)}{3(x^2+9x-10)} = \frac{1(x+10)}{3(x+10)(x-1)} = \frac{1}{3(x-1)}.$$