

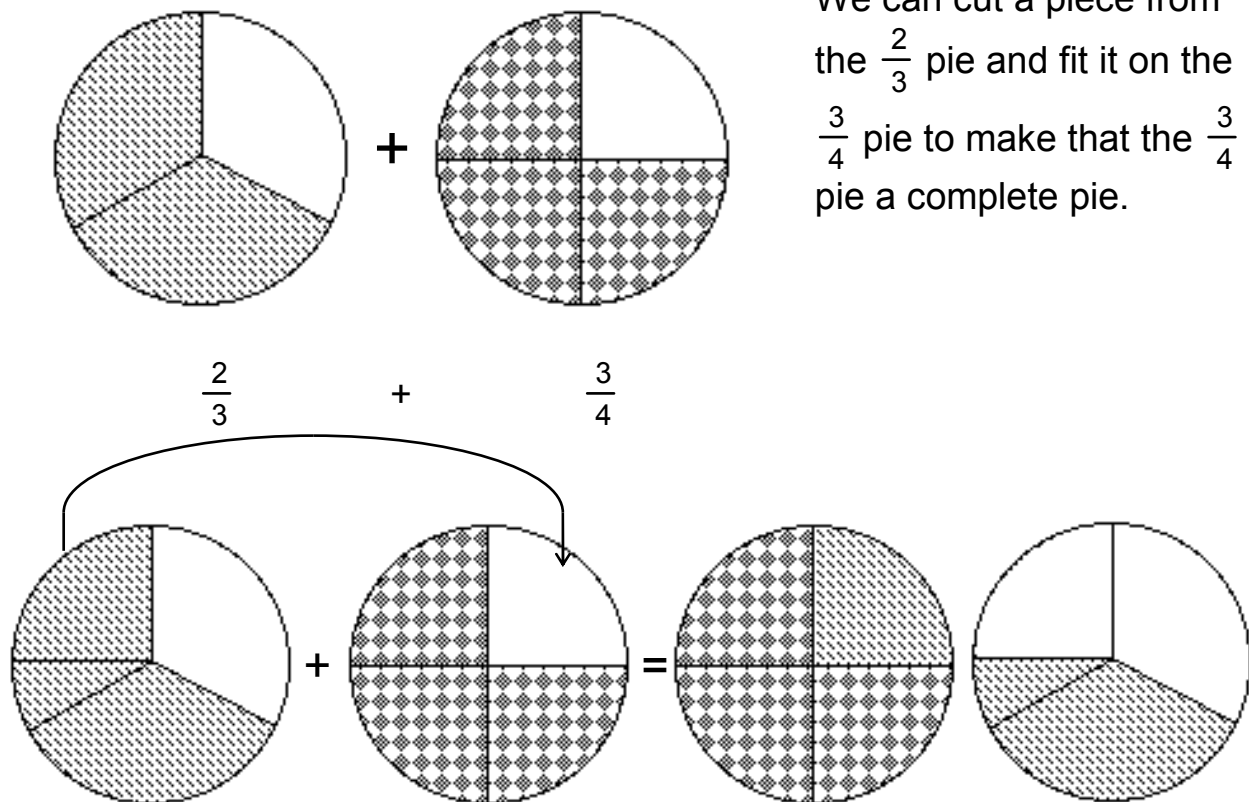
Sect 3.2 - Least Common Multiple

Let's start with an example:

Ex. 1 Suppose a family has two different pies. If they have $\frac{2}{3}$ of one type of pie and $\frac{3}{4}$ of another pie, is it possible to combine the pies into one pie tin? How much pie do they have altogether?

Solution:

Let's draw a picture:

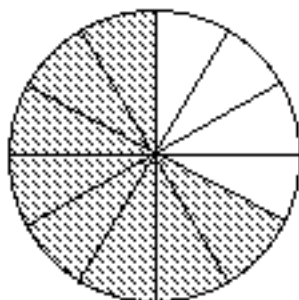


We can cut a piece from the $\frac{2}{3}$ pie and fit it on the $\frac{3}{4}$ pie to make that the $\frac{3}{4}$ pie a complete pie.

Clearly, our result is larger than one. Thus, if we try to add the numerators and add the denominators to get $\frac{5}{7}$, the answer does not make sense since $\frac{5}{7}$ is smaller than 1.

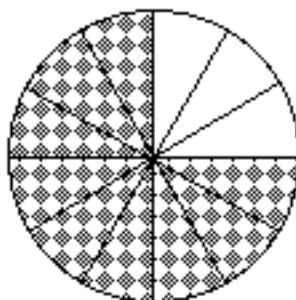
This is where having little kids around can help. With little kids, it is important to be sure that your slices of pie are the same size. Otherwise, you will not hear the end of the griping. So, that is what we are going to do; we will divide the slices so the resulting

pieces will be the same size. In the $\frac{2}{3}$ pie, we will divide each piece into four equal parts and in the $\frac{3}{4}$ pie, we will divide each piece into three equal parts:

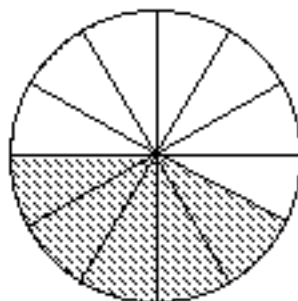
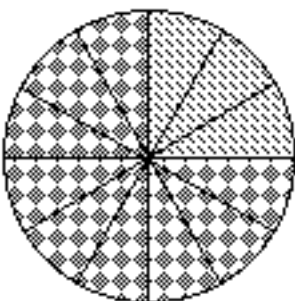


$$= \frac{8}{12}$$

+



$$\frac{9}{12} =$$



Notice that we now have 8 out of 12 slices of the first pie or $\frac{8}{12}$ of a pie and 9 out of 12 slices of the second pie or $\frac{9}{12}$ of a pie. Note that

$$\frac{8}{12} = \frac{2}{3}$$

$$\text{and } \frac{9}{12} = \frac{3}{4}.$$

Now, add and we get $\frac{17}{12}$ or $1\frac{5}{12}$.

$$\begin{array}{r} 1 \\ 12 \overline{) 17} \\ \underline{- 12} \\ 5 \end{array}$$

Notice that our result does match our picture. The important thing to get out of this example is that when we add or subtract fractions with different denominators, we must first transform the fractions so that they have the same denominator.

Objective a: Understanding the Least Common Multiple.

A **Multiple** of a number is a value that is divisible by the number.

The multiples of 3 are 3, 6, 9, 12, 15, 18, 21, 24, 27, 30, 33, 36, 39, ...

The multiples of 4 are 4, 8, 12, 16, 20, 24, 28, 32, 36, 40, ...

The **Common Multiples** of are multiples that a set of numbers have in common. So, the common multiples of 3 and 4 are 12, 24, 36, 48, ... The **Least Common Multiple** (abbreviated L.C.M.) is the smallest multiple the numbers have in common. Thus, the L.C.M. of 3 and 4 is 12. The **Least Common Denominator** (L.C.D.) is the L.C.M. of the denominators of a set of fractions.

Be careful not confuse multiples and factors. A multiple of a number is a value that is divisible by the number, but a factor of a number divides the number evenly with no remainder. For example, the multiples of 15 are 15, 30, 45, 60, 75, ... whereas the factors of 15 are 1, 3, 5, and 15.

Objective b: Finding the LCM by Using Prime Factors

In general, to find the L.C.M. of a number, we will begin by finding the prime factorization of each number. Next, we will write down the product of each factor that appears in the prime factorizations and choose the highest power of the factor in the factorizations. This result is our L.C.M.

Find the L.C.M. of:

Ex. 2 6, 12, and 5

Solution:

Write down the prime factorization of 6, 12, and 5:

$$6 = 2 \cdot 3$$

$$12 = 4 \cdot 3 = 2 \cdot 2 \cdot 3 = 2^2 \cdot 3$$

$$5 = 5$$

Now, write the product of each factor that appears in the prime factorizations and choose the highest power of the factor in the prime factorizations.

So, that would be $2^2 \cdot 3 \cdot 5 = 4 \cdot 3 \cdot 5 = 12 \cdot 5 = 60$. So, the L.C.M. = 60.

Ex. 3 49, 7, and 21

Solution:

Write down the prime factorization of 49, 7, and 21:

$$49 = 7 \cdot 7 = 7^2$$

$$7 = 7$$

$$21 = 3 \cdot 7$$

Now, write the product of each factor that appears in the prime factorizations and choose the highest power of the factor in the prime factorizations.

So, that would be $3 \cdot 7^2 = 3 \cdot 49 = 147$. So, the L.C.M. = 147.

Ex. 4 99, 110, and 22

Solution:

Write down the prime factorization of 99, 110, and 22:

$$99 = 9 \cdot 11 = 3^2 \cdot 11$$

$$110 = 10 \cdot 11 = 2 \cdot 5 \cdot 11$$

$$22 = 2 \cdot 11$$

Now, write the product of each factor that appears in the prime factorizations and choose the highest power of the factor in the prime factorizations.

So, that would be $2 \cdot 3^2 \cdot 5 \cdot 11 = 2 \cdot 9 \cdot 5 \cdot 11 = 18 \cdot 5 \cdot 11 = 90 \cdot 11 = 990$.

So, the L.C.M. = 990.

Ex. 5 24, 36, 22, and 63

Solution:

Write down the prime factorization of 24, 36, 22, and 63:

$$24 = 3 \cdot 8 = 3 \cdot 2 \cdot 4 = 3 \cdot 2 \cdot 2 \cdot 2 = 2^3 \cdot 3 \qquad 36 = 6 \cdot 6 = 2 \cdot 3 \cdot 2 \cdot 3 = 2^2 \cdot 3^2$$

$$22 = 2 \cdot 11$$

$$63 = 9 \cdot 7 = 3^2 \cdot 7$$

Now, write the product of each factor that appears in the prime factorizations and choose the highest power of the factor in the prime factorizations.

So, that would be $2^3 \cdot 3^2 \cdot 7 \cdot 11 = 8 \cdot 9 \cdot 7 \cdot 11 = 72 \cdot 7 \cdot 11 = 504 \cdot 11 = 5544$.

So, the L.C.M. = 5544.

Objective c: Finding the L.C.M. by Examining the Multiplies of the Largest Denominator.

You could write the multiples of every number until you find the match.

Recall when we listed the multiples of 3 and 4:

The multiples of 3 are 3, 6, 9, 12, 15, 18, 21, 24, 27, 30, 33, 36, 39, ...

The multiples of 4 are 4, 8, 12, 16, 20, 24, 28, 32, 36, 40, ...

With the number 3, we had to go to the fourth multiple to get to the L.C.M. whereas with 4, we had to go to the third multiple to get to the L.C.M. This means we should take the largest number and list its multiples until we find a number that is divisible by all the denominators.

Find the L.C.M. of:

Ex. 6 6, 12, and 5

Solution:

We will write down the multiples of 12:

12, 24, 36, 48, 60,

12, 24, 36, and 48 are not divisible by 5, but 60 is and 6 also divides into 60 evenly as well. So, the L.C.M. = 60.

Ex. 7 28, 7, and 21

Solution:

We will write down the multiples of 28:

28, 56, 84

28 and 56 are not divisible by 21, but 84 is and 7 also divides into 84 evenly as well. So, the L.C.M. = 84.

Ex. 8 33, 110, 44

Solution:

We will write down the multiples of 110:

110, 220, 330, 440, 550, 660

110, 330, 550 are not divisible by 44 and 220, 440 are not divisible by 33, but 660 is divisible by both 33 and 44 so the L.C.M. = 660.

Ex. 9 8, 10, and 35

Solution:

We will write down the multiples of 35:

35, 70, 105, 140, 175, 210, 245, 280

Even though 70, 140, and 210 are divisible by 10 there are not divisible by 8. We have to go to 280 before we can find a number that is divisible by both 8 and 10. So, the L.C.M. = 280.

Objective d: Application of the L.C.M.

Ex. 10 In a timed lock on a safe, one gear completes a full rotation in 6 hours while the other two gears complete a full rotation in 8 hours and 10 hours respectively. The safe will only open when the three gears are aligned. If the gears are initially aligned, how many hours must pass before the safe can be opened again?

Solution:

The shortest interval of time until the gears are aligned is the L.C.M. of 6, 8, and 10. Write down the prime factorization of 6, 8, and 10:

$$6 = 2 \cdot 3$$

$$8 = 4 \cdot 2 = 2 \cdot 2 \cdot 2 = 2^3$$

$$10 = 2 \cdot 5$$

Now, write the product of each factor that appears in the prime factorizations and choose the highest power of the factor in the prime factorizations.

So, that would be $2^3 \cdot 3 \cdot 5 = 8 \cdot 3 \cdot 5 = 24 \cdot 5 = 120$.

The safe can be opened in 120 hours.

Objective e: Equivalent Fractions and Comparing Fractions.

When we reduce fractions to lowest terms, we need to divide out the same number in numerator and denominator. Similarly, if we want to go backwards, we need to multiply the numerator and denominator by the same number. This is referred to as building fractions.

Write the following as an equivalent fraction with the given denominator:

Ex. 11 $\frac{2}{3} = \frac{\quad}{66}$

Solution:

We take the old denominator and divide it into the new denominator. Whatever our answer is, we multiply the top and the bottom of the old fraction to get the equivalent new fraction. Since $66 \div 3 = 22$, then we multiply the top and the bottom by 22:

$$\frac{2 \cdot 22}{3 \cdot 22} = \frac{44}{66}.$$

Ex. 12 $5\frac{7}{8} = 5\frac{\quad}{96}$

Solution:

We take the old denominator and divide it into the new denominator. Whatever our answers is, we multiply the top and the bottom of the old fraction to get the equivalent new fraction. Since $96 \div 8 = 12$, then we multiply the top and the bottom by 12

$$5\frac{7 \cdot 12}{8 \cdot 12} = 5\frac{84}{96}.$$

Ex. 13 $\frac{2}{3} = \frac{\quad}{12}$ and $\frac{3}{4} = \frac{\quad}{12}$

Solution:

Since $12 \div 3 = 4$, multiply the top and bottom of the first fraction by 4 and since $12 \div 4 = 3$, multiply the top and bottom of the second fraction by 3:

$$\frac{2 \cdot 4}{3 \cdot 4} = \frac{8}{12} \text{ and } \frac{3 \cdot 3}{4 \cdot 3} = \frac{9}{12}$$

Ex. 14 $6 = \frac{\quad}{14}$

Solution:

First, write 6 over 1. Since $14 \div 1 = 14$, multiply the top and bottom by fourteen.

$$\frac{6 \cdot 14}{1 \cdot 14} = \frac{84}{14}.$$

In order to compare fractions, they need to have the same denominator. So, we will need to find the L.C.M. of the denominators. We will refer to the L.C.M. of the denominators as the Least Common Denominator (L.C.D.). After finding the L.C.D, we will build the fractions so that the denominators are equal to the L.C.D. Finally, we can compare the numerators to see which one is bigger.

Compare using <, >, or =:

Ex. 15 $\frac{7}{9}$ $\frac{11}{15}$

Solution:

The L.C.D. = 45, so

$$\frac{7}{9} = \frac{7 \cdot 5}{9 \cdot 5} = \frac{35}{45} \text{ and } \frac{11}{15} = \frac{11 \cdot 3}{15 \cdot 3} = \frac{33}{45}. \text{ Since } \frac{35}{45} > \frac{33}{45},$$

$$\text{then } \frac{7}{9} > \frac{11}{15}.$$

Ex. 16 $\frac{9}{34}$ $\frac{14}{51}$

Solution:

The L.C.D. = 102, so

$$\frac{9}{34} = \frac{9 \cdot 3}{34 \cdot 3} = \frac{27}{102} \text{ and } \frac{14}{51} = \frac{14 \cdot 2}{51 \cdot 2} = \frac{28}{102}. \text{ Since } \frac{27}{102} < \frac{28}{102},$$

$$\text{then } \frac{9}{34} < \frac{14}{51}.$$

Put in order from smallest to largest:

Ex. 17 $\frac{5}{6}, \frac{7}{12}, \frac{11}{18}, \frac{5}{9}$

Solution:

First, find the L.C.D.:

$$6 = 2 \cdot 3$$

$$18 = 6 \cdot 3 = 2 \cdot 3 \cdot 3 = 2 \cdot 3^2$$

$$\text{L.C.D.} = 2^2 \cdot 3^2 = 4 \cdot 9 = 36$$

$$\frac{5}{6} = \frac{5 \cdot 6}{6 \cdot 6} = \frac{30}{36}$$

$$\frac{11}{18} = \frac{11 \cdot 2}{18 \cdot 2} = \frac{22}{36}$$

$$12 = 4 \cdot 3 = 2 \cdot 2 \cdot 3 = 2^2 \cdot 3$$

$$9 = 3 \cdot 3 = 3^2$$

$$\frac{7}{12} = \frac{7 \cdot 3}{12 \cdot 3} = \frac{21}{36}$$

$$\frac{5}{9} = \frac{5 \cdot 4}{9 \cdot 4} = \frac{20}{36}$$

So, $\frac{20}{36}$ is the smallest, then $\frac{21}{36}$, then $\frac{22}{36}$, and finally $\frac{30}{36}$.

Thus, from smallest to largest: $\frac{5}{9}, \frac{7}{12}, \frac{11}{18}, \frac{5}{6}$.