Sect 5.1 - Exponents: Multiplying and Dividing Common Bases

Concept #1 Review of Exponential Notation

In the exponential expression 4^5, 4 is called the base and 5 is called the exponent. This says that there are five factors of four being multiplied together. In expanded form, this is equal to 4•4•4•4•4. In general, if n is a positive integer, then a^n means that there are n factors of a being multiplied together.

Definition

Let a be a real number and n be a positive integer. Then

$$a^n = \underbrace{a \bullet a \bullet a \bullet a \ldots \bullet a}_{\text{n factors of a}}$$

a is the base & n is the exponent or power.

Note, some books use a natural number instead of a positive integer, but a positive integer is the same as a natural number. In this section, we will expand the use of the exponential notation to include the use of variables and develop some rules for simplifying variable expressions involving exponents.

Write the following in expanded form:

Ex. 1a x^7

Ex. 1b $(r - 8)^3$

Ex. 1c $(-7t)^4$

Ex. 1d $-7t^4$

Solution:

a) The base is x, and the power is 7, so $x^7 = x \bullet x \bullet x \bullet x \bullet x \bullet x \bullet x$.

b) The base is (r – 8) and the power is 3, so $(r - 8)^3 = (r - 8) \bullet (r - 8) \bullet (r - 8)$.

c) The base is (– 7t) and the power is 4, so $(-7t)^4 = (-7t) \bullet (-7t) \bullet (-7t) \bullet (-7t)$.

d) The base is t and the power is 4, so $-7t^4 = -7t \bullet t \bullet t \bullet t$.

Concept #2 Evaluating Expressions with Exponents

Simplify the following:

Ex. 2a $(-4)^2$

Ex. 2b -4^2

Ex. 2c $(-0.5)^3$

Ex. 2d -0.5^3

Solution:

a) $(-4)^2 = (-4)(-4) = 16$

b) $-4^2 = -4 \bullet 4 = -16$

c) $(-0.5)^3 = (-0.5)(-0.5)(-0.5) = -0.125$

d) $-0.5^3 = -0.5 \bullet 0.5 \bullet 0.5 = -0.125$

Evaluate each expression for a = 3 and b = – 4:

Ex. 3a $5a^2$ Ex. 3b $(5a)^2$

Ex. 3c $5ab^2$ Ex. 3d $(b + a)^2$

Solution:

a) $5a^2$ (replace a by (3))
$= 5(3)^2$ (#2-exponents)
$= 5(9)$ (#3-multiply)
$= 45$

b) $(5a)^2$ (replace a by (3))
$= (5(3))^2$ (#1-parenthesis, #3-multiply)
$= (15)^2$ (#2-exponents)
$= 225$

c) $5ab^2$ (replace a by (3) and b by (– 4))
$= 5(3)(-4)^2$ (#2-exponents)
$= 5(3)(16)$ (#3-multiply)
$= 240$

d) $(b + a)^2$ (replace a by (3) and b by (– 4))
$= ((3) + (-4))^2$ (#1-parenthesis, #4-add)
$= (-1)^2$ (#2-exponents)
$= 1$

Concept #3 Multiplying and Dividing Common Bases

Simplify the following:

Ex. 4 $x^5 \bullet x^3$

Solution:

Write x^5 and x^3 in expanded form and simplify:

$x^5 \bullet x^3 = (x \bullet x \bullet x \bullet x \bullet x) \bullet (x \bullet x \bullet x) = x \bullet x \bullet x \bullet x \bullet x \bullet x \bullet x \bullet x = x^8$.

Notice that if we add the exponents, we get the same result:

$x^5 \bullet x^3 = x^{5+3} = x^8$. This introduces our first property for exponents.

Multiplication of Like Bases (The Product Rule for Exponents)

If m and n are positive integers and a is a non-zero real number, then $\mathbf{a^m \bullet a^n = a^{m+n}}$. **Property #1**

In words, when multiplying powers of the same base, add the exponents and keep the same base.

Simplify the following:

Ex. 5a $y^7 \bullet y^8$

Ex. 5b $(-5y)^{11} \bullet (-5y)^{15}(-5y)$

Ex. 5c $7^{14} \bullet 7^{12}$

Ex. 5d $a^2 \bullet a^5 \bullet a \bullet b^3 \bullet b \bullet b^{11}$

Solution:

a) $y^7 \bullet y^8 = y^{7+8} = y^{15}$.

b) $(-5y)^{11} \bullet (-5y)^{15}(-5y) = (-5y)^{11+15+1} = (-5y)^{27}$.

c) $7^{14} \bullet 7^{12} = 7^{14+12} = 7^{26}$.

d) $a^2 \bullet a^5 \bullet a \bullet b^3 \bullet b \bullet b^{11} = a^{2+5+1}b^{3+1+11} = a^8b^{15}$

Notice that we cannot simplify a^8b^{15} since the bases are not the same. Keep in mind that property #1 only works for multiplication involving one term. It does not work for addition and subtraction.

Ex. 6a $3x^4 \bullet 5x^4$

Ex. 6b $3x^4 + 5x^4$

Ex. 6c $x^3 - x^4$

Ex. 6d $x^3(-x^4)$

Ex. 6e $(-4x^3)(-7y^2)$

Solution:

a) $3x^4 \bullet 5x^4 = 3 \bullet 5x^4 \bullet x^4 = 15x^8$.

b) Caution, the operation is addition. Combine like terms: $3x^4 + 5x^4 = 8x^4$.

c) There are no like terms to combine, so our answer is $x^3 - x^4$.

d) $x^3(-x^4) = -x^{3+4} = -x^7$

e) $(-4x^3)(-7y^2) = 28x^3y^2$. We cannot simplify the x^3y^2 since the bases are not the same.

Ex. 7 $\frac{y^8}{y^5}$

Solution:

Write both the numerator and denominator in expanded form and divide out the common factors of y:

$$\frac{y^8}{y^5} = \frac{y \bullet y \bullet y \bullet y \bullet y \bullet y \bullet y \bullet y}{y \bullet y \bullet y \bullet y \bullet y} = \frac{y \bullet y \bullet y \bullet \cancel{y} \bullet \cancel{y} \bullet \cancel{y} \bullet \cancel{y} \bullet \cancel{y}}{\cancel{y} \bullet \cancel{y} \bullet \cancel{y} \bullet \cancel{y} \bullet \cancel{y}} = \frac{y \bullet y \bullet y \bullet 1 \bullet 1 \bullet 1 \bullet 1 \bullet 1}{1 \bullet 1 \bullet 1 \bullet 1 \bullet 1} = y^3.$$

This example illustrates our second property:

Division of Like Bases (The Quotient Rule for Exponents)

If m and n are positive integers where m ≥ n and a ≠ 0, then $\frac{a^m}{a^n} = a^{m-n}$. In words, when dividing powers of the same base, take the exponent in the numerator minus the exponent in the denominator and keep the same base.

Simplify the following:

Ex. 8a $\frac{x^{24}}{x^6}$

Ex. 8b $\frac{8^{93}}{8^{31}}$

Ex. 8c $\frac{y^8}{y}$

Ex. 8d $\frac{x^6y^3}{xy^2}$

Solution:

a) $\frac{x^{24}}{x^6} = x^{24-6} = x^{18}$.

b) $\frac{8^{93}}{8^{31}} = 8^{93-31} = 8^{62}$

c) $\frac{y^8}{y} = y^{8-1} = y^7$

d) $\frac{x^6y^3}{xy^2} = x^{6-1}y^{3-2} = x^5y^1 = x^5y$.

Concept #4 Simplify Expressions with Exponents

Simplify:

Ex. 9a $(-4a^3b^4c)(7a^7bc^3)$

Ex. 8b $\frac{-28x^6y^7z^5}{21x^2yz^4}$

Ex. 9c $\frac{(-6a^2b^5c)(-5ab^4c^2)}{(-10ab)(ab^2)}$

Solution:

a) $(-4a^3b^4c)(7a^7bc^3)$ (use the commutative and associative properties of multiplication to group common bases together)

$= -4 \bullet 7a^3 \bullet a^7 \bullet b^4 \bullet b \bullet c \bullet c^3$

$= -28a^{3+7}b^{4+1}c^{1+3}$

$= -28a^{10}b^5c^4$.

b) Be careful with $\frac{-28}{21}$ since they are not exponents. We will need to reduce that part: $\frac{-28x^6y^7z^5}{21x^2yz^4} = \frac{-28}{21}x^{6-2}y^{7-1}z^{5-4}$ $= -\frac{4}{3}x^4y^6z$.

c) First, simplify the numerator and denominator:

$$\frac{(-6a^2b^5c)(-5ab^4c^2)}{(-10ab)(ab^2)} = \frac{-6 \bullet -5a^{2+1}b^{5+4}c^{1+2}}{-10a^{1+1}b^{1+2}} = \frac{30a^3b^9c^3}{-10a^2b^3}$$

Now, perform the division:

$$\frac{30a^3b^9c^3}{-10a^2b^3} = \frac{30}{-10}a^{3-2}b^{9-3}c^3 = -3ab^6c^3.$$

Concept #5 Applications of Exponents

Recall that when we calculated simple interest, we used the formula i = prt where i was the interest, p was the principal, r was the annual interest rate written as a decimal or fraction, and t was the time in years. This formula depends on the principal being fixed during the lifetime of the loan which is not always the case. Let us consider an example where interest is calculated on the amount in the account every year.

Solve the following:

Ex. 10 Juan invests $1000 in an account paying 7% annual interest every year on the amount in the account. How much money does he have in the account after three years?

Solution:

For the first year, p = $1000, r = 7% = 0.07, and t = 1 year. Thus, the interest is:

i = prt = (1000)(0.07)(1) = $70.

The total in the account is

A = p + i = 1000 + 70 = $1070

For the second year, p = $1070, r = 7% = 0.07, and t = 1 year. Thus, the interest is:

i = prt = (1070)(0.07)(1) = $74.90.

The total in the account is

A = p + i = 1070 + 74.90 = $1144.90

For the third year, p = $1144.90, r = 7% = 0.07, and t = 1 year. Thus, the interest is:

$i = prt = (1144.90)(0.07)(1) \approx \$80.14$.

The total in the account is

$A = p + i = 1144.90 + 80.14 = \$1225.04$

At the end of three years, Juan has \$1225.04 in his account.

This type of interest computation in called **compound interest**. Not only is one earning interest on the principal, but also earning interest on the previous interest earned. To see how this works, let us examine the previous example more closely.

For the first year, Juan's interest was \$70 = 1000(0.07) and the total he had was \$1070 = 1000 + 70 = 1000 + 1000(0.07) = 1000(1 + 0.07).

Hence, **\$1070 = \$1000(1 + 0.07)**.

For the second year, Juan's interest was \$74.90 = 1070(0.07) and the total he had was \$1144.90 = 1070 + 74.90 = 1070 + 1070(0.07) = 1070(1 + 0.07) .

Hence, \$1144.90 = \$1070(1 + 0.07). But, **\$1070 = \$1000(1 + 0.07)**, so we can substitute **\$1000(1 + 0.07)** in for \$1070:

$\$1144.90 = \underline{\$1070}(1 + 0.07)$.

$\$1144.90 = \mathbf{\$1000(1 + 0.07)}(1 + 0.07) = \$1000(1 + 0.07)^2$

Thus, $\underline{\$1144.90 = \$1000(1 + 0.07)^2}$

For the third year, Juan's interest was \$80.14 = 1144.90(0.07) and the total he had was \$1225.04 = 1144.90 + 80.14= 1144.90 + 1144.90(0.07) = 1144.90(1 + 0.07) .

Hence, \$1225.04 = **\$1144.90**(1 + 0.07). But, $\underline{\$1144.90 = \$1000(1 + 0.07)^2}$, so we can substitute $\underline{\$1000(1 + 0.07)^2}$ in for **\$1144.90**:

$\$1225.04 = \mathbf{\$1144.90}(1 + 0.07)$

$\$1225.04 = \underline{\$1000(1 + 0.07)^2}(1 + 0.07) = \$1000(1 + 0.07)^3$

Hence, $\$1225.04 = \$1000(1 + 0.07)^3$.

Notice that \$1225.04 was the amount, \$1000 was the original principal, 0.07 was the rate as a decimal, and the exponent 3 was the time in years. From this, we can write a formula for interest compounded annually:

$\$1225.04 = \$1000(1 + 0.07)^3$.

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$A = P(1 + r)^t$

Let us verify that the formula gives us the correct answer:

$A = 1000(1 + 0.07)^3$	(#1-parenthesis, # 4-add)
$= 1000(1.07)^3$	(#2-exponents)
$= 1000(1.225043)$	(#3-multiply)

$= 1225.043 \approx \$1225.04$ which matches the answer to #10.

Interest Compounded Annually

If the principal P is invested in an account paying r% interest compounded annually, then the total amount A in the account after t years is given by:

$A = P(1 + r)^t$ where r is converted to a decimal or fraction.

Solve the following:

Ex. 11 LaTonya invests \$3300 in an account paying 8.9% annual interest compounded annually. How much money does she have in the account after five years?

Solution:

The principal is \$3300, the rate is 8.9% = 0.089 and the time is 5 years:

$A = P(1 + r)^t$ (plug the values into the formula)

$A = (3300)(1 + (0.089))^{(5)}$ (#1-parenthesis, # 4-add)

$= 3300(1.089)^5$ (#2-exponents)

$= 3300(1.53157898526)$ (#3-multiply)

$= 5054.21065137 \approx 5054.21$

Thus, there is \$5054.21 in the account after five years.