Sect 7.7 – Multiplying and Dividing Algebraic Expressions

Objective 1: Review of Exponential Notation

In the exponential expression 4^5, 4 is called the base and 5 is called the exponent. This says that there are five factors of four being multiplied together. In expanded form, this is equal to 4•4•4•4•4. In general, if n is a positive integer, then a^n means that there are n factors of a being multiplied together.

Definition

Let a be a real number and n be a positive integer. Then

$$a^n = \underbrace{a \bullet a \bullet a \bullet a \ldots \bullet a}_{n \text{ factors of } a}$$

a is the base & n is the exponent or power.

Note, some books use a natural number instead of a positive integer, but a positive integer is the same as a natural number. In this section, we will expand the use of the exponential notation to include the use of variables and develop some rules for simplifying variable expressions involving exponents.

Write the following in expanded form:

Ex. 1a x^7

Ex. 1b $(r – 8)^3$

Ex. 1c $(– 7t)^4$

Ex. 1d $– 7t^4$

Solution:

a) The base is x, and the power is 7, so x^7 = x•x•x•x•x•x•x.

b) The base is (r – 8) and the power is 3, so $(r – 8)^3$ = (r – 8)•(r – 8)•(r – 8).

c) The base is (– 7t) and the power is 4, so $(– 7t)^4$ = (– 7t)•(– 7t)•(– 7t)•(– 7t).

d) The base is t and the power is 4, so $– 7t^4$ = – 7t•t•t•t.

Objective 2: Multiplying and Dividing Common Bases

To see how multiplication works, let us examine the following example:

Simplify the following:

Ex. 2 $x^5 \bullet x^3$

Solution:

Write x^5 and x^3 in expanded form and simplify:

$x^5 \bullet x^3 = (x \bullet x \bullet x \bullet x \bullet x) \bullet (x \bullet x \bullet x) = x \bullet x \bullet x \bullet x \bullet x \bullet x \bullet x \bullet x = x^8$.

Notice that if we add the exponents, we get the same result:

$x^5 \bullet x^3 = x^{5 + 3} = x^8$. This introduces our first rule for exponents.

Multiplication of Like Bases (The Product Rule for Exponents)

If m and n are positive integers and a is a non-zero real number, then $\mathbf{a^m \bullet a^n = a^{m + n}}$.

In words, when multiplying powers of the same base, add the exponents and keep the same base.

Simplify the following:

Ex. 3a $y^7 \bullet y^8$

Ex. 3b $(-5y)^{11} \bullet (-5y)^{15}(-5y)$

Ex. 3c $7^{14} \bullet 7^{12}$

Ex. 3d $a^2 \bullet a^5 \bullet a \bullet b^3 \bullet b \bullet b^{11}$

Solution:

a) $y^7 \bullet y^8 = y^{7 + 8} = y^{15}$.

b) $(-5y)^{11} \bullet (-5y)^{15}(-5y) = (-5y)^{11 + 15 + 1} = (-5y)^{27}$.

c) $7^{14} \bullet 7^{12} = 7^{14 + 12} = 7^{26}$.

d) $a^2 \bullet a^5 \bullet a \bullet b^3 \bullet b \bullet b^{11} = a^{2 + 5 + 1}b^{3 + 1 + 11} = a^8b^{15}$

Notice that we cannot simplify a^8b^{15} since the bases are not the same. Keep in mind that property #1 only works for multiplication involving one term. It does not work for addition and subtraction.

Ex. 4a $3x^4 \bullet 5x^4$

Ex. 4b $3x^4 + 5x^4$

Ex. 4c $x^3 - x^4$

Ex. 4d $x^3(-x^4)$

Ex. 4e $(-4x^3)(-7y^2)$

Solution:

a) $3x^4 \bullet 5x^4 = 3 \bullet 5x^4 \bullet x^4 = 15x^8$.

b) Caution, the operation is addition. Combine like terms: $3x^4 + 5x^4 = 8x^4$.

c) There are no like terms to combine, so our answer is $x^3 - x^4$.

d) $x^3(-x^4) = -x^{3 + 4} = -x^7$

e) $(-4x^3)(-7y^2) = 28x^3y^2$. We cannot simplify the x^3y^2 since the bases are not the same.

To see how division works, let us examine the following example:

Ex. 5 $\frac{y^8}{y^5}$

Solution:
Write both the numerator and denominator in expanded form and divide out the common factors of y:

$$\frac{y^8}{y^5} = \frac{y \bullet y \bullet y \bullet y \bullet y \bullet y \bullet y \bullet y}{y \bullet y \bullet y \bullet y \bullet y} = \frac{y \bullet y \bullet y \bullet \cancel{y} \bullet \cancel{y} \bullet \cancel{y} \bullet \cancel{y} \bullet \cancel{y}}{\cancel{y} \bullet \cancel{y} \bullet \cancel{y} \bullet \cancel{y} \bullet \cancel{y}} = \frac{y \bullet y \bullet y \bullet 1 \bullet 1 \bullet 1 \bullet 1 \bullet 1}{1 \bullet 1 \bullet 1 \bullet 1 \bullet 1} = y^3.$$

This example illustrates our second property:

Division of Like Bases (The Quotient Rule for Exponents)

If m and n are positive integers where m ≥ n and a ≠ 0, then $\mathbf{\frac{a^m}{a^n} = a^{m-n}}$. In words, when dividing powers of the same base, take the exponent in the numerator minus the exponent in the denominator and keep the same base.

Simplify the following:

Ex. 6a $\frac{x^{24}}{x^6}$

Ex. 6b $\frac{8^{93}}{8^{31}}$

Ex. 6c $\frac{y^8}{y}$

Ex. 6d $\frac{x^6y^3}{xy^2}$

Solution:

a) $\frac{x^{24}}{x^6} = x^{24-6} = x^{18}$.

b) $\frac{8^{93}}{8^{31}} = 8^{93-31} = 8^{62}$

c) $\frac{y^8}{y} = y^{8-1} = y^7$

d) $\frac{x^6y^3}{xy^2} = x^{6-1}y^{3-2} = x^5y^1 = x^5y$.

Ex. 7a $(-4a^3b^4c)(7a^7bc^3)$

Ex. 7b $\frac{-28x^6y^7z^5}{21x^2yz^4}$

Ex. 7c $\frac{(-6a^2b^5c)(-5ab^4c^2)}{(-10ab)(ab^2)}$

Solution:

a) $(-4a^3b^4c)(7a^7bc^3)$ (use the commutative and associative properties of multiplication to group common bases together)

$= -4 \bullet 7a^3 \bullet a^7 \bullet b^4 \bullet b \bullet c \bullet c^3$

$= -28a^{3+7}b^{4+1}c^{1+3}$

$= -28a^{10}b^5c^4$.

b) Be careful with $\frac{-28}{21}$ since they are not exponents. We will need to reduce that part: $\frac{-28x^6y^7z^5}{21x^2yz^4} = \frac{-28}{21}x^{6-2}y^{7-1}z^{5-4}$

$= -\frac{4}{3}x^4y^6z$.

c) First, simplify the numerator and denominator:

$\frac{(-6a^2b^5c)(-5ab^4c^2)}{(-10ab)(ab^2)} = \frac{-6 \bullet -5a^{2+1}b^{5+4}c^{1+2}}{-10a^{1+1}b^{1+2}} = \frac{30a^3b^9c^3}{-10a^2b^3}$

Now, perform the division:

$\frac{30a^3b^9c^3}{-10a^2b^3} = \frac{30}{-10}a^{3-2}b^{9-3}c^3 = -3ab^6c^3$.

Objective 3: Understanding Zero and Negative Exponents

Let's examine the quotient rule when the powers are equal.

Simplify:

Ex. 8 $\frac{2^5}{2^5}$

Solution:

There are two ways to view this problem. First, any non-zero number divided by itself is 1, so, $\frac{2^5}{2^5} = 1$. But, using the quotient rule, $\frac{2^5}{2^5} = 2^{5-5} = 2^0$. This says that $2^0 = 1$. We can do this same trick with any base except for zero.

Zero Exponents

If a is any non-zero real number, then $a^0 = 1$.

Simplify:

Ex. 9a	3^0	Ex. 9b	$(-3)^0$
Ex. 9c	-3^0	Ex. 9d	$-3x^0$
Ex. 9e	$-3x^2y^0$	Ex. 9f	$-3(x^2y)^0$
Ex. 9g	$-(3x^2y)^0$	Ex. 9h	$(-3x^2y)^0$

Solution:

a) $3^0 = 1$

b) $(-3)^0 = 1$

c) $-3^0 = -(1) = -1$ (the 0 exponent only applies to 3)

d) $-3x^0 = -3(1) = -3$ (the 0 exponent only applies to x)

e) $-3x^2y^0 = -3x^2(1) = -3x^2$. (the 0 exponent only applies to y)

f) $-3(x^2y)^0 = -3(1) = -3$. (the 0 exponent only applies to x^2y)

g) $-(3x^2y)^0 = -(1) = -1$. (the 0 exponent only applies to $3x^2y$)

h) $(-3x^2y)^0 = (1) = 1$.

Let's examine the quotient rule when the power in the denominator is larger than the power in the numerator.

Simplify:

Ex. 10 $\frac{2^5}{2^8}$

Solution:

There are two ways to view this problem. First, $2^5 = 32$ and $2^8 = 256$, then $\frac{2^5}{2^8} = \frac{32}{256}$ which reduces to $\frac{1}{8}$. But, $\frac{1}{8} = \frac{1}{2^3}$, so, $\frac{2^5}{2^8} = \frac{1}{2^3}$.

But, using the quotient rule, $\frac{2^5}{2^8} = 2^{5-8} = 2^{-3}$. This says that $2^{-3} = \frac{1}{2^3}$. We can do this same trick with any base except for zero.

Also, $\frac{1}{4^{-2}} = 4^2 = 16$ since $\frac{1}{4^{-2}} = 1 \div 4^{-2} = 1 \div \frac{1}{4^2} = 1 \bullet 4^2 = 4^2 = 16$

Negative Exponents

If a and b are any non-zero real numbers, then $a^{-n} = \frac{1}{a^n}$ and $\frac{1}{b^{-n}} = b^n$. Note this also implies that $\left(\frac{a}{b}\right)^{-n} = \frac{a^{-n}}{b^{-n}} = \frac{b^n}{a^n} = \left(\frac{b}{a}\right)^n$.

In words, when raising a quantity to a negative power, take the reciprocal of the base and change the sign of the exponent.

Simplify the following. Write your answer using positive exponents:

Ex. 11a 11^{-2}

Ex. 11b $(-3)^{-4}$

Ex. 11c $\frac{1}{5^{-2}}$

Ex. 11d $\frac{5}{(-2)^{-4}}$

Ex. 11e $\frac{7^{-2}}{6^{-3}}$

Solution:

a) 11^{-2} (apply the definition of a negative exponent)

$= \frac{1}{11^2}$ (simplify)

$= \frac{1}{121}$.

b) $(-3)^{-4}$ (apply the definition of a negative exponent)

$= \frac{1}{(-3)^4}$ (simplify)

$= \frac{1}{81}$.

c) $\frac{1}{5^{-2}}$ (apply the definition of a negative exponent)

$= 5^2$ (simplify)

$= 25$.

d) $\frac{5}{(-2)^{-4}}$ (apply the definition of a negative exponent)

$= 5\bullet(-2)^4$ (exponents)

$= 5\bullet16$ (multiplication)

$= 80$.

e) $\frac{7^{-2}}{6^{-3}}$ (apply the definition of a negative exponent)

$= \frac{6^3}{7^2}$ (simplify)

$= \frac{216}{49}$.

Ex. 12a $\frac{-4x^{-3}y^2}{5a^3b^{-5}}$ Ex. 12b $\frac{(-2)^3a^{-3}b^7c^{-5}}{(-7)^2q^{-4}r^2v^0}$

Ex. 12c $\frac{x^2}{h^{-7}v^{-8}h^9}$ Ex. 12d $3^{-1} + \left(\frac{7}{2}\right)^{-2} - 7^0$

Solution:

a) If the exponents are already positive, do not move the factors. Only move the factors that have negative exponents:

$\frac{-4x^{-3}y^2}{5a^3b^{-5}}$ (negative ÷ positive is negative)

` $= -\frac{4x^{-3}y^2}{5a^3b^{-5}}$ (apply the definition of a negative exponent)

$= -\frac{4b^5y^2}{5a^3x^3}$. (Note – 4 is not an exponent, but a number so we do not move it).

b) If the exponents are already positive, do not move the factors. Only move the factors that have negative exponents:

$\frac{(-2)^3a^{-3}b^7c^{-5}}{(-7)^2q^{-4}r^2v^0}$ (simplify)

$= \frac{-8a^{-3}b^7c^{-5}}{49q^{-4}r^2v^0}$ (negative ÷ positive is negative)

$= -\frac{8a^{-3}b^7c^{-5}}{49q^{-4}r^2v^0}$ (apply the definition of a negative exponent)

$= -\frac{8b^7q^4}{49a^3c^5r^2v^0}$

But $v^0 = 1$, so $-\frac{8b^7q^4}{49a^3c^5r^2v^0} = -\frac{8b^7q^4}{49a^3c^5r^2(1)} = -\frac{8b^7q^4}{49a^3c^5r^2}$

c) $\frac{x^2}{h^{-7}v^{-8}h^9}$ (#1 product rule in the denominator)

$= \frac{x^2}{h^2v^{-8}}$ (apply the definition of a negative exponent)

$= \frac{v^8x^2}{h^2}$.

d) $3^{-1} + \left(\frac{7}{2}\right)^{-2} - 7^0$ (apply the definition of a negative exponent)

$= \frac{1}{3} + \left(\frac{2}{7}\right)^2 - 7^0 = \frac{1}{3} + \frac{4}{49} - 1 = -\frac{86}{147}$

Objective 4: Multiplying and Dividing when one expression has more than one term.

Recall how we used the distributive property to simplify the following:

Simplify:

Ex. 13 $-4(-3x + 6 - 2y)$

Solution:

$\mathbf{-4}(-3x + 6 - 2y) = \mathbf{-4}(-3x) + (\mathbf{-4})\bullet 6 - (\mathbf{-4})\bullet 2y$

$= 12x + (-24) + 8y = 12x - 24 + 8y.$

We can extend this idea to multiplying a polynomial by a monomial. Suppose in example 13, we change the -4 to $-4x^2y$. Let's see what happens:

Ex. 14 $-4x^2y(-3x + 6 - 2y)$

Solution:

$\mathbf{-4x^2y}(-3x + 6 - 2y) = \mathbf{-4x^2y}(-3x) + (\mathbf{-4x^2y})\bullet 6 - (\mathbf{-4x^2y})\bullet 2y$

$= 12x^3y + (-24x^2y) + 8x^2y^2 = 12x^3y - 24x^2y + 8x^2y^2.$

Ex. 15 $-\frac{2}{3}a^2\left(5a^3 - 6a^2 + \frac{1}{4}a^{-1} - \frac{2}{7}\right)$

Solution:

$-\frac{2}{3}\mathbf{a^2}\left(5a^3 - 6a^2 + \frac{1}{4}a^{-1} - \frac{2}{7}\right)$

$= -\frac{2}{3}\mathbf{a^2}\left(\frac{5}{1}a^3\right) - \left(-\frac{2}{3}\mathbf{a^2}\right)\frac{6}{1}a^2 + \left(-\frac{2}{3}\mathbf{a^2}\right)\left(\frac{1}{4}a^{-1}\right) - \left(-\frac{2}{3}\mathbf{a^2}\right)\left(\frac{2}{7}\right)$

$= -\frac{10}{3}a^5 + 4a^4 - \frac{1}{6}a + \frac{4}{21}a^2.$

Ex. 16 $-0.7cd(3c^2 - 4cd + 2d^{-3})$

Solution:

$\mathbf{-0.7cd}(3c^2 - 4cd + 2d^{-3})$

$= \mathbf{-0.7cd}(3c^2) - (\mathbf{-0.7cd})(4cd) + (\mathbf{-0.7cd})(2d^{-3})$

$= -2.1c^3d + 2.8c^2d^2 - 1.4cd^{-2}$

$= -2.1c^3d + 2.8c^2d^2 - \frac{1.4c}{d^2}$

The same basic idea applies in division if we are dividing by an expression with one term:

Ex. 17 $\frac{3x^2-6x+12}{3}$

Solution:

Since $\frac{3x^2-6x+12}{3}$ is the same as $\frac{1}{3}(3x^2 - 6x + 12)$, we can distribute the $\frac{1}{3}$ to each term and simplify:

$$\frac{1}{3}(3x^2) - \frac{1}{3}(6x) + \frac{1}{3}(12) = \frac{3x^2}{3} - \frac{6x}{3} + \frac{12}{3} = x^2 - 2x + 4.$$

Notice that we divided each term by 3 to get the answer.

In general, to divide an expression by one term, divide each term of the expression by that term and simplify.

Simplify:

Ex. 18 $\frac{25t^3-15t^2+30t}{15t}$

Solution:

$$\frac{25t^3-15t^2+30t}{15t} = \frac{25t^3}{15t} - \frac{15t^2}{15t} + \frac{30t}{15t} = \frac{5}{3}t^2 - t + 2$$

Ex. 19 $(18x^6 - 27x^5 + 3x^3) \div (3x^3)$

Solution:

$$(18x^6 - 27x^5 + 3x^3) \div (3x^3) = \frac{18x^6}{3x^3} - \frac{27x^5}{3x^3} + \frac{3x^3}{3x^3} = 6x^3 - 9x^2 + 1$$

Ex. 20 $\frac{8xy^2-16x^2y+24xy}{4x^2y^2}$

Solution:

$$\frac{8xy^2-16x^2y+24xy}{4x^2y^2} = \frac{8xy^2}{4x^2y^2} - \frac{16x^2y}{4x^2y^2} + \frac{24xy}{4x^2y^2} = \frac{2}{x} - \frac{4}{y} + \frac{6}{xy}$$

Ex. 21 $(42a^6b^5 - 3a^5b^7 + 21a^3b^8) \div (7a^5b^7)$

Solution:

$$(42a^6b^5 - 3a^5b^7 + 21a^3b^8) \div (7a^5b^7) = \frac{42a^6b^5}{7a^5b^7} - \frac{3a^5b^7}{7a^5b^7} + \frac{21a^3b^8}{7a^5b^7}$$

$$= \frac{6a}{b^2} - \frac{3}{7} + \frac{3b}{a^2}$$