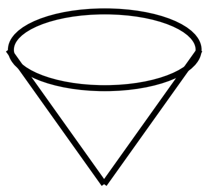
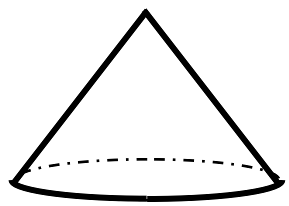
Sect 9.4 – Cones and Frustums of Cones

Objective 1: Understanding Cones, Volume, and Surface Area.

A **Cone** is a pyramid-like solid figure with a circular base. The radius of the cone is the radius of the circle. The circumference of the base is $2\pi r$ and the area of the base is πr^2, the height of the cone is perpendicular height from the apex to the base. The slant height is the distance from the apex to the edge of the base. We can use the same formulas for a pyramid for a cone.

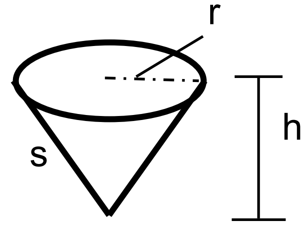




We can use the same formulas we used for a pyramid for a cone.

Properties of Right Cones:

Let r = the radius of the base
B = the area of the base
h = the height of the cone
s = the slant height

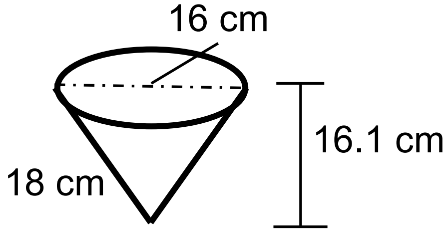


Then

1) Lateral surface area: $L = \frac{1}{2}Cs = \pi rs$
2) Total surface area: $SA = L + B = \pi rs + \pi r^2$
3) Volume: $V = \frac{1}{3}Bh = \frac{1}{3}\pi r^2 h$

Find a) the lateral surface area, b) the surface area, and c) the volume of the following. Round your answers to three significant digits:

Ex. 1



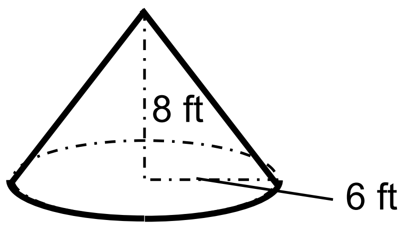
<u>Solution:</u>
The radius is half the diameter: r = 16/2 = 8 cm
The area of base is $B = \pi r^2 = \pi(8)^2 = 64\pi = 201.06\ldots\ cm^2$

a) $L = \pi rs = \pi(8)(18) = 144\pi = 452.3\ldots \approx 452\ cm^2$

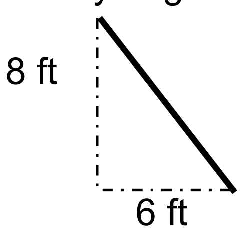
b) $SA = B + L = 201.0\ldots + 452.3\ldots = 653.4\ldots \approx 653\ cm^2$

c) $V = \frac{1}{3}Bh = \frac{1}{3}(201.06\ldots)(16.1) = 1079.0\ldots \approx 1080\ cm^3$

Ex. 2



<u>Solution:</u>
Notice that the radius and the height form a right triangle. We can use the Pythagorean Theorem to find the slant height.



$s^2 = a^2 + b^2$
$s^2 = 8^2 + 6^2$
$s^2 = 64 + 36$
$s^2 = 100$
$s = \pm\sqrt{100} = \pm 10$

So, the slant height s = 10 ft
$B = \pi r^2 = \pi(6)^2 = 36\pi = 113.09\ldots\ ft^2$

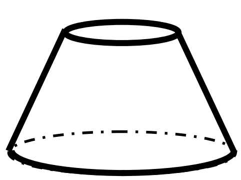
a) $L = \pi rs = \pi(6)(10) = 60\pi = 188.4\ldots \approx 188\ ft^2$

b) $SA = B + L = 113.09\ldots + 188.4\ldots = 301.5\ldots \approx 302\ ft^2$

c) $V = \frac{1}{3}Bh = \frac{1}{3}(113.09\ldots)(8) = 301.5\ldots \approx 302\ ft^3$

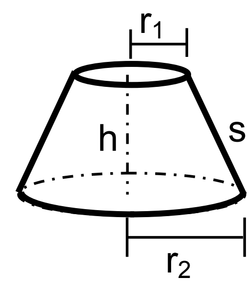
Objective 2: Understanding Frustums, Volumes, and Surface Area.

The **frustum of a cone** is the remaining part of the cone after cutting off the top portion parallel to the base. A frustum of cone has two parallel circular bases which are similar (the same shape, but are not the same size).



We can use the same formulas we used for a frustum of a pyramid for a frustum of a cone. Replacing p_1 by $2\pi r_1$, p_2 by $2\pi r_2$, B_1 by πr_1, and B_2 by πr_2 in the formulas, we get the following properties

Properties of Frustums of Right Cones:



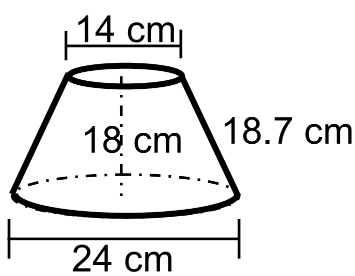
Let r_1 = the radius of the base on the top
r_2 = the radius of the base on the bottom
h = the height of the Frustum
s = the slant height

Then

1) Lateral surface area: $L = \pi(r_1 + r_2)s$
2) Total surface area: $SA = L + \pi r_1^2 + \pi r_2^2$
3) Volume: $V = \frac{1}{3}\pi(r_1^2 + r_2^2 + r_1r_2)h$

Find a) the lateral surface area, b) the surface area, and c) the volume of the following. Round your answers to three significant digits:

Ex. 3



Solution:

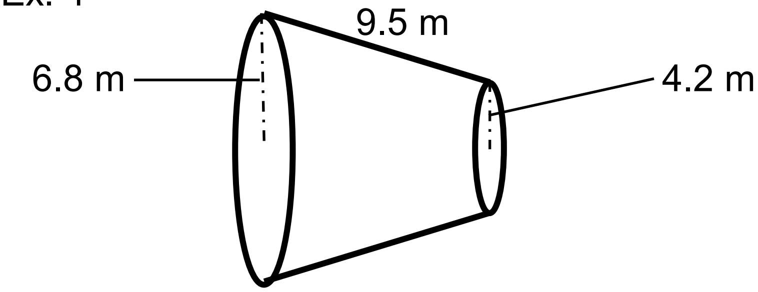
$r_1 = 14/2 = 7$ cm and $r_2 = 24/2 = 12$ cm

a) $L = \pi(r_1 + r_2)s = \pi(7 + 12)(18.7) = \pi(19)(18.7) = 355.3\pi$
$= 1116.2\ldots \approx 1120 \text{ cm}^2$

b) $SA = L + \pi r_1^2 + \pi r_2^2 = 1116.2\ldots + \pi(7)^2 + \pi(12)^2$
$= 1116.2\ldots + 49\pi + 144\pi = 1116.2\ldots + 153.9\ldots + 452.3\ldots$
$= 1722.5\ldots \approx 1720 \text{ ft}^2$

c) $V = \frac{1}{3}\pi(r_1^2 + r_2^2 + r_1r_2)h = \frac{1}{3}\pi((7)^2 + (12)^2 + (7)(12))(18)$
$= \frac{1}{3}\pi(49 + 144 + 84)(18) = \frac{1}{3}\pi(277)(18) = 1662\pi$
$= 5221.3\ldots \approx 5220 \text{ ft}^3$

Ex. 4

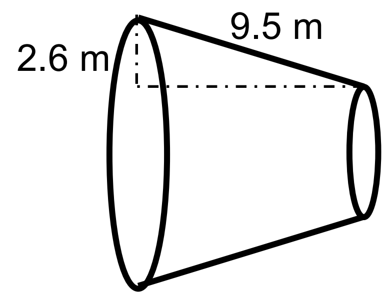


Solution:

a) $L = \pi(r_1 + r_2)s = \pi(6.8 + 4.2)(9.5) = \pi(11)(9.5) = 104.5\pi$
$= 328.2\ldots \approx 328 \text{ m}^2$

b) $SA = L + \pi r_1^2 + \pi r_2^2 = 328.2\ldots + \pi(6.8)^2 + \pi(4.2)^2$
$= 328.2\ldots + 46.24\pi + 17.64\pi = 328.2\ldots + 145.2\ldots + 55.4\ldots$
$= 528.9\ldots \approx 529 \text{ m}^2$

c) To find the height, we can use an embedded right triangle in the figure. The hypotenuse is 9.5 m and the length of the leg is 6.8 – 4.2 = 2.6 m. Using the Pythagorean Theorem, we get



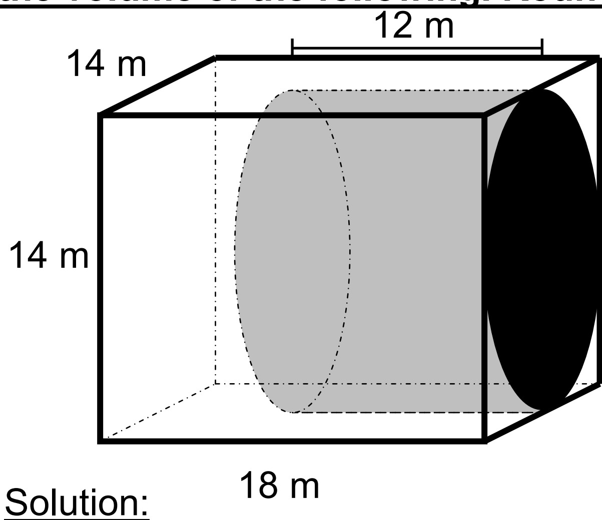
$(9.5)^2 = (2.6)^2 + h^2$
$90.25 = 6.76 + h^2$ (subtract 6.76 from both sides)
$83.49 = h^2$
$h = \pm\sqrt{83.49} = \pm 9.13\ldots$
So, the height is 9.13… m
$V = \frac{1}{3}\pi(r_1^2 + r_2^2 + r_1r_2)h = \frac{1}{3}\pi((6.8)^2 + (4.2)^2 + (6.8)(4.2))(9.13\ldots)$
$= \frac{1}{3}\pi(46.24 + 17.64 + 28.56)(9.13\ldots) = \frac{1}{3}\pi(92.44)(9.13\ldots)$
$= (281.5\ldots)\pi = 884.5\ldots \approx 885 \text{ m}^3$

Objective #3: Finding the volume of composite figures.

Keep in mind that when we are talking about the volume of any figure, we want to find the total amount of space the object occupies; i.e., how much soda a soda can holds or how much concrete needs to be poured to make a driveway. The key to solving a composite figure problem is to split it apart into a series of basic figures. Let's look at some examples.

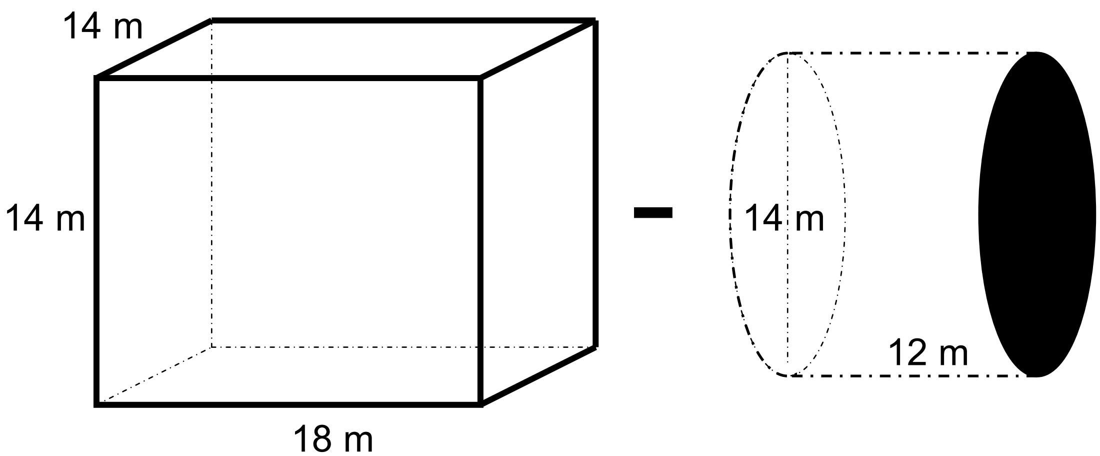
Find the volume of the following. Round to the nearest hundred.

Ex. 5



Solution:

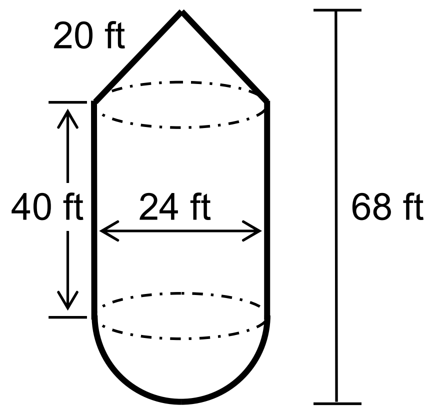
We can split this figure into a rectangular prism and a cylinder. Here, the cylinder has been "tunneled out" of the prism. So, to find the volume of the composite figure, we will subtract the volume of the cylinder from the volume of the prism:



Since the diameter of the cylinder is equal to the height of the rectangular prism (which is 14 m), the radius of the cylinder is 14 m ÷ 2 = 7 m. The volume of the composite figure is the difference between the volume of the rectangular prism and the volume of the cylinder:

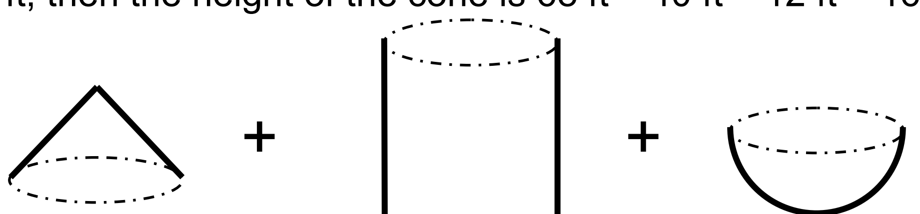
$$V = Lwh - \pi r^2 h = (18)(14)(14) - \pi(7)^2(12) = 3528 - 588\pi$$
$$= 3528 - 1847.25\ldots = 1680.7\ldots \approx 1700 \text{ m}^3$$

Ex. 6 How many gallons of water can the water tank pictured below hold? Round to the nearest hundred. How many gallons of weatherproofing paint are need for two coats of paint if each gallon covers 250 ft^2. Round to the nearest ten gallon.



Solution:

a) To find the volume, we need to split this figure into three parts: The diameter of the hemisphere is 24 ft, so its radius 12 ft. The radius of the hemisphere is also the height of the hemisphere and the radius of the cone and cylinder. Since the total height is 68 ft, then the height of the cone is 68 ft – 40 ft – 12 ft = 16 ft.



r = 12 ft
h = 16 ft

r = 12 ft
h = 40 ft

r = 12 ft
Only a hemisphere, so we have half of the volume.

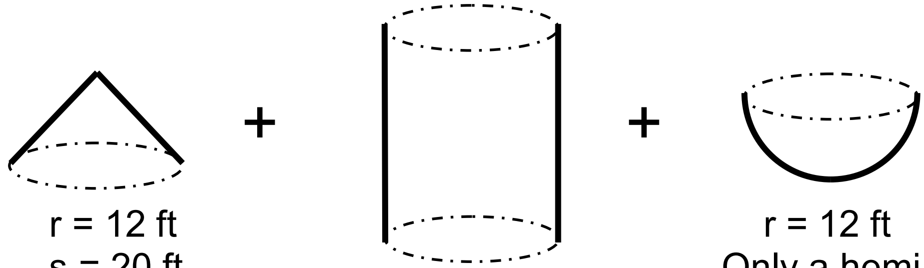
$$V = \frac{1}{3}\pi r^2 h \quad + \quad \pi r^2 h \quad + \quad \frac{4}{3}\pi r^3 \div 2$$
$$= \frac{1}{3}\pi(12)^2(16) \quad + \quad \pi(12)^2(40) \quad + \quad \frac{4}{3}\pi(12)^3 \div 2$$
$$= \frac{1}{3}\pi(144)(16) + \pi(144)(40) + \frac{4}{3}\pi(1728) \div 2$$
$$= 768\pi + 5760\pi + 1152\pi = 7680\pi = 24127.4\ldots \text{ ft}^3$$

Now, convert to gallons:

$$\frac{24127.4\ldots \text{ ft}^3}{1} \bullet \frac{7.48052 \text{ gal}}{1 \text{ ft}^3} = 180485.7\ldots. \approx 180500 \text{ gal}$$

The tank can hold 180,500 gallons of water.

b) The surface area of the tank is equal to the lateral surface area of the cone plus the lateral surface of the cylinder plus the surface area of the hemisphere.



$$SA = \pi rs + 2\pi rh + 4\pi r^2/2$$
$$= \pi(12)(20) + 2\pi(12)(40) + 4\pi(12)^2/2$$
$$= 240\pi + 960\pi + 288\pi = 1488\pi = 4674.6\ldots \text{ ft}^2$$

For two coats of paint, the total area to be covered is $2(4674.6\ldots) = 9349.3\ldots \text{ ft}^2$

Since each gallon covers 250 ft^2, then the total amount the paint needed is:

$$(9349.3\ldots \text{ ft}^2)(1 \text{ gal}/250 \text{ ft}^2) = 37.39\ldots \approx 40 \text{ gallons.}$$

Forty gallons of paint will be needed to paint the water tank.