energy distribution of the photoemitted electrons may be not completely under control. The space-charge effect and the reverse current can be possible sources of error mostly when the applied voltage is near to zero. Finally, the quality of the results may depend on the physical conditions of the photocell potassium layer. Notwithstanding these remarks, we believe that the quality of the measurements is fully satisfactory for a student laboratory experiment and this experiment can be profitably used to introduce undergraduate students to modern physics.

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Scattering by a finite periodic potential

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The problem of scattering in one dimension by a potential which consists of N identical cells is solved in a transparent manner. The N-cell transmission and reflection amplitudes are expressed in terms of the single-cell amplitudes and the Bloch phase. As examples the results are applied to a row of delta-function potentials, and to a row of square wells, and it is shown that these expressions provide an immediate understanding of the results of detailed calculations.

I. INTRODUCTION

There is a recurring interest in one-dimensional scattering problems, particularly cases where the potential consists of a finite number of identical "cells." Kiang[1] for example, showed that in the case of N delta-function potentials, even a small N of order 5 is sufficient to give rise to rather spectacular interference effects on the transmission probability. More recently, Griffiths and Taussig[2] have discussed the same example from a transfer matrix point of view. In a more general case, of arbitrary potential cells, Kalotas and Lee[3] have given a general transfer matrix formulation and applied it to the square well and parabolic potential cases. A few years earlier, Vezzetti and Cahay[4] formulated a similar transfer matrix approach and derived a very compact expression for the N-cell transmission probability in terms of the Bloch phase.

In this paper, we draw together these various developments in one general formulation, and derive some compact formulas from which one can understand at a glance how the N-cell transmission should look. We consider scattering by a potential which consists of N identical cells. We show how the solution for a single cell can be easily extended to the finitely periodic system. In this step, the Bloch phase angle, which is familiar from the infinite periodic system, plays a crucial role. By letting $N\to\infty$, one recovers the Bloch zones of allowed and excluded wave numbers. We feel that the simplicity and generality of the approach developed in this paper should be known to those teaching quantum mechanics. It may also be useful as a complement to the traditional presentation of the infinite periodic system, as found for example in Merzbacher.[5] The strong interference effects which are manifest in our solutions illustrate the importance of interference in wave propagation.

In Sec. II we introduce the transfer matrix which connects the solutions at two ends of one potential cell. In Sec. III we adopt a more convenient form of the transfer matrix which leads in Sec. IV to an immediate solution for the transmission and reflection coefficients of the scattering problem for N cells. We then apply this in Sec. V to the case of a row of delta-function potentials. This leads to a detailed understanding of the results which Kiang[1] and others have found by numerical evaluation of their solutions. Most of the interesting structure in the transmission is due to the periodicity. In Sec. VI we apply our method to a series of square barriers, one of the examples of Kalotas and Lee.[3] The conclusions are given in Sec. VII.

II. THE TRANSFER MATRIX AS A MAPPING

We consider an arbitrary shape potential $U(x)$, defined on $0<x<d$. The potential $U_N(x)$ is defined to be the same potential repeated N times, on the interval $0<x<Nd$. This is what we mean by a finite periodic potential. What we will show is that the solution for the problem $U_N(x)$ follows immediately from the solution for the single cell case. In this section we follow closely the presentation in Ref. 4.

A solution of the Schrödinger equation, $u(x)$, is specified by the boundary conditions at an arbitrary point x. We will define a vector

$$\mathbf{v}(x)=\begin{pmatrix} u(x) \\ u'(x) \end{pmatrix}, \tag{1}$$

which conveys this information. Two particular solutions can be selected corresponding to the conditions $\mathbf{v}_1(0)=(1,0)^\dagger$ and $\mathbf{v}_2(0)=(0,1)^\dagger$ at $x=0$. Integrating from 0 to d, we arrive at the corresponding boundary values $\mathbf{v}_1(d)=[u_1(d),u_1'(d)]^\dagger$ and $\mathbf{v}_2(d)=[u_2(d),u_2'(d)]^\dagger$. We define the transfer matrix $\mathbf{W}$ to be the matrix with these two vectors as its columns.

$$\mathbf{W}(x)=\begin{pmatrix} u_1(x) & u_2(x) \\ u_1'(x) & u_2'(x) \end{pmatrix}. \tag{2}$$

This matrix has the property that if $\mathbf{v}(0)$ characterizes an arbitrary solution of the Schrödinger equation at $x=0$, then $\mathbf{v}(d)=\mathbf{W}(d)\mathbf{v}(0)$.

Note that det $\mathbf{W}$ is the Wronskian of two independent solutions, u_1, u_2, and this is a constant for the Schrödinger equation. Computing its value at the origin gives det $\mathbf{W}=1$. The eigenvalues of $\mathbf{W}$ are of the form [see Eq. (4) below]

$$\lambda_1=1/\lambda_2=\mu+\sqrt{\mu^2-1}=e^{i\phi}, \tag{3}$$

where $2\mu=\mathrm{Tr}\mathbf{W}\equiv 2\cos\phi$. When $|\mu|\leqslant 1$, ϕ will be a real angle; otherwise we take $\phi=i\theta$ or $\theta=\pi+i\theta$ according to the sign of μ. Then the $\cos\phi$ becomes $\pm\cosh\theta$. The eigenstates of $\mathbf{W}$ are particular solutions for which the solution at $x=d$ differs from that at $x=0$ by a phase, as if periodic boundary conditions were imposed. In fact, ϕ is the Bloch phase associated with the infinitely periodic potential whose unit is cell $U(x)$. When ϕ is imaginary, the eigenstates are those solutions for which both u and u' differ by the same real factor between the two ends of the interval.

For the problem U_N, we need only apply the transfer matrix N times to map an arbitrary solution from $x=0$ to Nd. The Cayley–Hamilton theorem (see for example, Ref. 6) tells us that any matrix satisfies its own eigenvalue equation. Hence (and one can easily check it),

$$\mathbf{W}^2=2\mathbf{W}\cos\phi-1. \tag{4}$$

This can be used iteratively to write any power of $\mathbf{W}$ in terms of $\mathbf{W}$ and the unit matrix. By induction one can show that (see the Appendix)

$$\mathbf{W}^N=\frac{1}{\sin\phi}[\mathbf{W}\sin N\phi-\mathbf{1}\sin(N-1)\phi]. \tag{5}$$

This new matrix has determinant unity and $\mathrm{Tr}\,\mathbf{W}^N=2\cos(N\phi)$, since its eigenvalues are simply the Nth powers of those of $\mathbf{W}$.

III. AN ALTERNATIVE MAPPING

What is important in Sec. I is the existence and properties of the transfer matrix, not the way we obtained it. Before we proceed to solve the transmission problem, it is convenient to introduce an alternative form of the transfer matrix. This is based upon a division of the wave function into right and left-moving components. We define

$$u(x)=f(+k,x)e^{+ikx}+f(-k,x)e^{-ikx},$$
$$u'(x)=ik[f(+k,x)e^{+ikx}-f(-k,x)e^{-ikx}]. \tag{6}$$

It may seem strange that a single function can be replaced by a pair, but the division is unique because we have imposed a condition on the derivatives of the two amplitude functions $f(\pm k,x)$, namely that $f'(+k,x)e^{+ikx}+f'(-k,x)e^{-ikx}=0$. Knowing these two functions is equivalent to knowing $u(x)$ and its derivative $u'(x)$, and vice versa. A particular solution of the Schrödinger equation may therefore be characterized by a vector

$$\mathbf{c}(x)\equiv\begin{pmatrix} f(+k,x)e^{ikx} \\ f(-k,x)e^{-ikx} \end{pmatrix}, \tag{7}$$

which is related to $\mathbf{v}$ by

$$\mathbf{v}(x)=\mathbf{L}\mathbf{c}(x), \tag{8}$$

where

$$\mathbf{L}=\begin{pmatrix} 1 & 1 \\ ik & -ik \end{pmatrix}. \tag{9}$$

Correspondingly, the transfer matrix in this representation, which translates from $x=0$ to d is

$$\mathbf{M}^{-1}=\mathbf{L}^{-1}\mathbf{W}\mathbf{L}, \qquad \mathbf{c}(0)=\mathbf{M}\mathbf{c}(d). \tag{10}$$

$\mathbf{M}^{-1}$ is seen to retain the property of having determinant unity. Explicitly one has

$$\mathbf{M}^{-1}=\frac{1}{2}\begin{pmatrix} \left(\mathrm{Tr}\,W+\frac{W_{21}-k^2W_{12}}{ik}\right) & \left(W_{11}-W_{22}+\frac{W_{21}+k^2W_{12}}{ik}\right) \\ \left(W_{11}-W_{22}-\frac{W_{21}+k^2W_{12}}{ik}\right) & \left(\mathrm{Tr}\,W-\frac{W_{21}-k^2W_{12}}{ik}\right) \end{pmatrix}. \tag{11}$$

Since M^{-1} is unimodular, M can be written down trivially. It is also seen that the trace is preserved by this similarity transformation, and so are the eigenvalues. The methods applied to the matrix **W** apply equally to **M**; in particular Eq. (5) holds for **M**.

IV. IMMEDIATE SOLUTION OF THE TRANSMISSION PROBLEM

To describe scattering with an incident wave from the left, we impose the boundary conditions:

$$\psi(x)=\begin{cases} e^{+ikx}+R_N e^{-ikx}, & x<0,\\ T_N e^{+ik(x-Nd)}, & x>Nd. \end{cases} \tag{12}$$

From this we can read off the values of the vector $\mathbf{c}(x)$:

$$\mathbf{c}(0)=\begin{pmatrix}1\\R_N\end{pmatrix},\quad \mathbf{c}(Nd)=\begin{pmatrix}T_N\\0\end{pmatrix}. \tag{13}$$

According to Eq. (10) these are related by

$$1=(M^N)_{11}T_N,\quad R_N=(M^N)_{21}T_N. \tag{14}$$

Very useful relations can be obtained when these are combined with Eq. (5):

$$\mathbf{M}^N=\frac{1}{\sin\phi}[\mathbf{M}\sin N\phi-\mathbf{1}\sin(N-1)\phi]. \tag{15}$$

First we note that $M_{21}=R_1/T_1$, while $(M^N)_{21}=R_N/T_N$. However, these *off-diagonal* elements differ only by a factor $\sin N\phi/\sin\phi$, giving

$$\frac{R_N}{T_N}=\frac{\sin N\phi}{\sin\phi}\frac{R_1}{T_1}. \tag{16}$$

By taking the absolute square of this relation we reproduce the results of the paper of Vezzetti and Cahay,[4] but since Eq. (16) relates the amplitudes directly it is much more powerful. Their relation can be put in the form

$$\frac{1}{|T_N|^2}=1+\frac{\sin^2 N\phi}{\sin^2\phi}\left(\frac{1}{|T_1|^2}-1\right). \tag{17}$$

From this it is evident that perfect transmission occurs whenever $T_1=1$, but in addition there are $N-1$ new possibilities

$$N\phi=m\pi,\quad m=1,2,...(N-1) \tag{18}$$

in each allowed band, where ϕ increases by π.

With a little more work we can solve explicitly for the amplitudes T_N and R_N from Eq. (14).

$$\frac{1}{T_N}=(M^N)_{11}=\frac{1}{\sin\phi}[M_{11}\sin N\phi-\sin(N-1)\phi]$$
$$=\frac{1}{\sin\phi}\left(\frac{1}{T_1}\sin N\phi-\sin(N-1)\phi\right), \tag{19}$$

where M without a superscript is the case $N=1$, and use has been made of Eq. (15). Again, one sees that when $\sin N\phi$ vanishes, $T_N=(-)^m$. On the other hand, when $\sin(N-1)\phi$ vanishes, $T_N=\pm T_1$. [This of course says that if $(N-1)$ barriers are transparent, it is only the last one which causes reflection.] For large N, these two points will be very close together, showing that T_N is a rapidly fluctuating function of wave number k or of the Bloch phase $\phi(k)$ when it is real.

When $N\phi$ is an odd multiple of $\pi/2$, the $\sin^2 N\phi$ on the right of Eq. (17) takes its maximum value of one. For large N, this will be the most rapidly varying term, hence one will have a minimum $|T_N|^2$. Even for moderate N, the minimum occurs very close to this point as we will see below.

To complete the derivation of R_N, on the right hand side of Eq. (16) we replace T_1 from Eq. (19), giving

$$\frac{R_N}{T_N}=R_1\left(\frac{1}{T_N}+\frac{\sin(N-1)\phi}{\sin\phi}\right). \tag{20}$$

Then

$$R_N=R_1\left(1+\frac{\sin(N-1)\phi}{\sin\phi}T_N\right)$$
$$=R_1\left(1-\frac{\sin(N-1)\phi}{\sin N\phi}T_1\right)^{-1}. \tag{21}$$

Here, we have used T_N from Eq. (19) and have cleared a common factor of $\sin\phi$ in the second line.

To conclude this section, we will note that in Eq. (11), **W** is a real matrix. This, together with Eq. (14) allows us to write in general that

$$\mathbf{M}=\begin{pmatrix}\dfrac{1}{T_1} & \dfrac{R_1^*}{T_1^*}\\ \dfrac{R_1}{T_1} & \dfrac{1}{T_1^*}\end{pmatrix}. \tag{22}$$

This is the easiest way to construct the transfer matrix, after one has solved the single-cell transmission problem. Taking the trace shows that

$$\cos\phi=\mathrm{Re}M_{11}=\mathrm{Re}(1/T_1). \tag{23}$$

V. CASE OF A ROW OF DELTA FUNCTIONS

As an application of our result, we consider scattering by a row of N delta-function potentials of strength C_0 and spacing d. We choose this example because the solution is simple and it nicely illustrates the previous results. We can compare with the calculations of Kiang[1] and obtain a better understanding of his result.

The unit cell $U(x)$ may be taken to have the delta function at its center; any other placement simply alters the phase of the reflection amplitude. The Schrödinger equation for the single cell is

$$\psi''(x)+k^2\psi(x)=2\Omega\delta(x-d/2)\psi(x), \tag{24}$$

where

$$k^2=\frac{2m}{\hbar^2}E\quad\text{and}\quad 2\Omega=\frac{2m}{\hbar^2}C_0.$$

One finds that

$$T_1=\frac{k}{k+i\Omega}e^{ikd},\quad R_1=\frac{-i\Omega}{k+i\Omega}e^{ikd}. \tag{25}$$

The transmission probability

$$|T_1|^2=\frac{k^2}{k^2+\Omega^2} \tag{26}$$

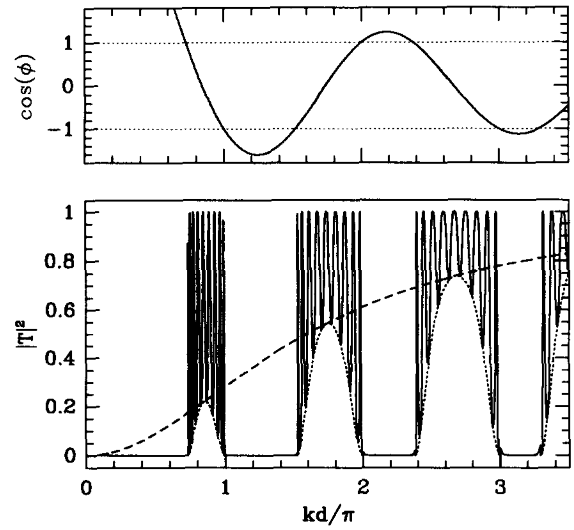

Fig. 1. Transmission probability through an array of $N=10$ delta functions, with $\Omega/d=5$, plotted against kd/π. The zone boundaries and the envelope of the minima [the dotted line, Eq. (30)] are independent of N. In the upper part of the figure, Eq. (29) for the Bloch phase is plotted.

is a very structureless function, so all the structure which Kiang[1] found for the N delta case must arise from the Bloch phase.

To compute this Bloch phase, it is convenient to define

$$\tan\beta=\Omega/k. \tag{27}$$

(This $\beta=-\delta_{\text{even}}$, the even-parity phase shift for a single delta, if we follow Lipkin's[7] treatment of one-dimensional scattering.) Look at Eq. (22), we see that

$$M_{11}=T_1^{-1}=\frac{e^{-i(kd-\beta)}}{\cos\beta}, \tag{28}$$

where Eqs. (25) and (27) have been used. Using Eq. (23), one has

$$\cos\phi=\frac{\cos(kd-\beta)}{\cos\beta}=\cos kd+\frac{\Omega}{k}\sin kd. \tag{29}$$

Supposing that $\Omega>0$, β begins at zero energy at $+\pi/2$, and decreases slowly. In order to have real ϕ, kd must increase until $2\beta<kd<\pi$, and in this interval ϕ rises from zero to π. There is then an excluded region of width 2β, in which $\cos\phi<-1$. This is illustrated in the upper part of Fig. 1. Following this ϕ is real until $kd=2\pi$, and so on. In the excluded region, ϕ becomes complex, as explained earlier, and the cosine becomes a cosh. In the second allowed region, ϕ starts again at π and increases to 2π, etc. Of course, while kd is rising, β is falling from $\pi/2$ towards zero, but the latter is a slow variation. As a result, the widths of the excluded regions become progressively narrower as the energy (and, hence, k) increases.

When N is at all large, in the "excluded zones" of complex ϕ, the factor $\sin N\phi$ in Eq. (16) will be huge. The only way to satisfy this equation is for $|T_N|$ to approach zero, giving essentially zero transmission. In each of the "allowed" zones, there will be exactly $N-1$ places where the $\sin N\phi$ vanishes, giving unit transmission. These are the points corresponding to Eq. (18). In the case of the single delta-function potential, there are no points with perfect transmission, as seen from the dashed line of Fig. 1; otherwise these would show as additional resonances. This explains Kiang's figure 2 perfectly.[1] The slow decline of β explains why the allowed zones become slowly wider as kd increases.

In Fig. 1 we show calculated results for $|T_N(k)|^2$ for $N=10$, and $\Omega/d=5$, the values chosen by Kiang. All the features discussed above are evident. In addition we have drawn the locus of the transition minima as a dotted line in the allowed zones. These minima occur very close to the points where $\sin^2 N\phi=1$. Hence, by setting $\sin^2 N\phi=1$ in Eq. (17) we obtain the envelope of these minima, which is a curve independent of N:

$$\frac{1}{|T_N|^2_{\min}}=1+\frac{1}{\sin^2\phi}\left(\frac{1}{|T_1|^2}-1\right). \tag{30}$$

The line is drawn as a dotted line in Figs. 1 and 3 and agrees well with the calculations. Notice that at the center of an allowed band, where $\phi=\pi/2$, the $\sin\phi$ in the denominator will also be unity, and hence for odd N this minimum value of $|T_N|$ will be equal to $|T_1|$. Indeed the envelope of the minima just touches the curve for $|T_1|^2$ (the dashed line) at these points in Fig. 1.

For very large N, and real ϕ, the $\sin^2 N\phi$ in Eq. (17) will oscillate rapidly around the value $\frac{1}{2}$. The transmission probability will therefore on average approach the curve obtained by setting this value into Eq. (17):

$$\frac{1}{|T_N|^2_{\text{av}}}=1+\frac{1}{2\sin^2\phi}\left(\frac{1}{|T_1|^2}-1\right). \tag{31}$$

So although the regions of real ϕ correspond in the limit of infinite N to the allowed bands of the infinite periodic system, the transmission probability is not unity, whereas in the forbidden zones it *is* zero. However in the case of an infinite periodic lattice one is not interested in measuring the transmission and reflection coefficients, but simply in the fact that in the allowed zones one has solutions which have constant amplitude over the whole lattice. To put it another way, so long as N is finite, there is an outside and an inside to the region of the potential. One can then apply boundary conditions as in Eq. (12) and derive the transmission and reflection coefficients. At a given energy or momentum k, ϕ may be a rational or an irrational multiple of π. In the rational case, as N varies, the transmission will be very often either unity or the minimum value, while in the irrational case it will always take intermediate values. Since the irrational numbers are much more numerous than the rationals, this is the most common situation. If one fixes N at some large value and averages over a narrow k interval, the average value of $\sin^2 N\phi$ in an allowed band will also be one-half.

It is surprising that the limit $N\to\infty$ of the scattering states are not the Bloch waves; rather the limit is a linear combination of the two Bloch states. By considering the scattering state with incident wave from the right we would get an orthogonal linear combination. It is the interference between the two Bloch states which gives rise to the nonperfect transmission in the limit. By combining the left-incident and right-incident scattering states of the form Eq. (12) one can form the individual Bloch states which do proceed without reflection through the infinite crystal. One could also consider that it is the end of the finite periodic system which generates the reflected wave.

Griffiths and Taussig[2] have recently discussed this same example. Since their paper solves only the repeated delta potential, they did not produce a general method such as ours. They drew several very nice pictures of the transmission probability, for $N=1$, 2, 3, 5, 9, and 101, and discussed at length the factors which influence the transmission minima, especially in the forbidden zones. Lacking our Eq. (30), they did not realize the connection to $|T_1|^2$ in the allowed zones. In discussing the limit $N\to\infty$, they give an incorrect explanation. They assumed that points giving $|T_N|^2=1$ would be dense in the allowed band, in the limit. However, as pointed out above, these points correspond to rational values of ϕ/π, and these are a set of zero measure as compared to the irrational numbers for which $|T_N|^2<1$. Blundell[8] in a comment on that paper has made the same point concerning the limit of infinite N.

VI. SQUARE BARRIERS

Although this example has a long history, going back to the Kronig–Penney model,[9] it actually has some topical interest. Ulloa *et al.*[10] have considered electron transport in a mesoscopic device or quantum wire formed by a linear array of potential wells separated by finite height square potential barriers, which can be created on a GaAs chip. Their device is shown schematically in Fig. 2. Because the width a (of the order of 100 nm) is uniform throughout, the problem is strictly one dimensional, as we showed in a recent paper.[11] For the sake of discussion, we will sometimes express energies in terms of the threshold for the first propagating channel which is at $E_0=(\hbar^2/2m)(\pi/a)^2$, even though a does not enter into the one-dimensional problem which we treat.

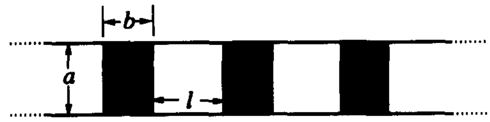

Fig. 2. The structure of a quantum dot superlattice. The electrons are confined between the two horizontal lines and experience a periodic blocking potential $V=V_0$ in the regions marked in black. The potential elsewhere is zero. The entrance/exit lead at each end is considered to be infinitely long.

Since the square well barrier is easily solved, we skip the details of the single barrier case and present the results. The barriers are of height V_0, width b, and separated by l, so the cell is of width $b+l$. The transfer matrix given in Ref. 11 can be put into the present representation Eq. (22) as follows:

$$M=\begin{bmatrix} e^{-i\alpha l}\left[\cos(\gamma b)-\frac{i}{2}\left(\frac{\alpha}{\gamma}+\frac{\gamma}{\alpha}\right)\sin(\gamma b)\right] & +\frac{i}{2}e^{+i\alpha l}\left(\frac{\alpha}{\gamma}-\frac{\gamma}{\alpha}\right)\sin(\gamma b) \\ -\frac{i}{2}e^{-i\alpha l}\left(\frac{\alpha}{\gamma}-\frac{\gamma}{\alpha}\right)\sin(\gamma b) & e^{+i\alpha l}\left[\cos(\gamma b)+\frac{i}{2}\left(\frac{\alpha}{\gamma}+\frac{\gamma}{\alpha}\right)\sin(\gamma b)\right]\end{bmatrix}. \tag{32}$$

Here, $\alpha^2=2mE/\hbar^2$ and $\gamma^2=2m(E-V_0)/\hbar^2$. When the energy E is below the barrier height V_0, $\gamma\to i\bar{\gamma}$, and the above matrix remains valid with this substitution.

In view of Eq. (23), the Bloch phase may be obtained from $\mathrm{Re}\, M_{11}$ giving

$$\cos\phi(\alpha)=\cos(\alpha l)\cos(\gamma b)-\frac{1}{2}\left(\frac{\alpha}{\gamma}+\frac{\gamma}{\alpha}\right)\sin(\alpha l)\sin(\gamma b), \tag{33}$$

which remains real when $\gamma\to i\bar{\gamma}$.

In Fig. 3 we show the cosine of the Bloch phase and the single-cell transmission probability, and the transmission probabilities for the cases $N=6$ and 51. The steep rise in transmission to a plateau is seen to be associated with the second allowed band beginning at $1.5E_0$, where as before $E_0=(\hbar^2/2m)(\pi/a)^2$. The sharp resonances which appear in the vicinity of the barrier height occur precisely in the allowed band between $E=0.46E_0$ and $1.18E_0$ as expected. In a few cases, the peaks fail to reach unity due to the finite resolution of the drawing. In the $N=51$ case, the depression near $E=4.5E_0$ is seen to be due to a very weak forbidden band just at this energy; it requires a large N for the sinh $N\theta$ to become appreciable.

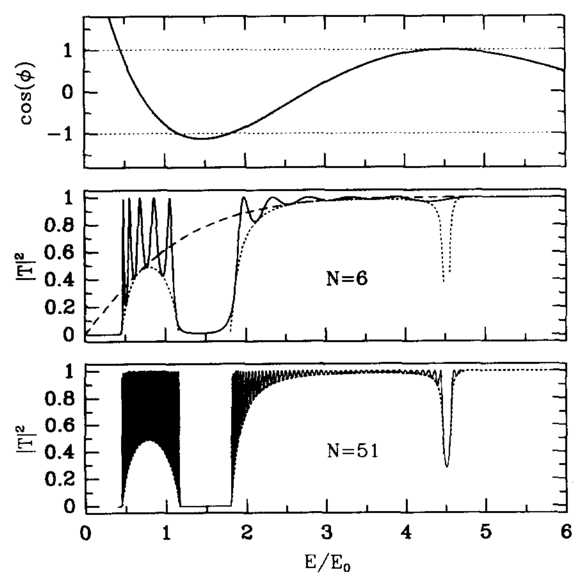

Fig. 3. Transmission for device with $V_0=E_0$, $l=0.5a$, and $b=0.5a$. Upper part: cosine of Bloch phase, Eq. (23). Middle part: $N=6$ cells. Lower part: $N=51$. The single-cell transmission probability is shown as a dashed line, and the envelope of minima, Eq. (30), as a dotted line.

In Fig. 4 we show a second case in which the barrier height has doubled. Here the first allowed band is narrower, and the corresponding sub-barrier resonances more compressed. The forbidden band near $E=5.0E_0$ is much stronger in this case and is seen in the $N=6$ case already. The envelope of the transmission minima is lower in this figure because the single-cell transmission (the dashed line) is lower than in Fig. 3. As a result, the oscillations in the plateau region are more pronounced. Once again this shows the power of the present formulation to explain results that are otherwise obscure.

After we completed this work, we became aware of the paper by Kalotas and Lee[3] in which a similar treatment of the finite periodic potential was expounded. The matrix Π of their Eq. (14) is our M. Their T_{KL} is twice our $\cos\phi$, but the relation of T_{KL} to the Bloch phase was not used. They deduced an algebraic recurrence relation to construct Π^N, equivalent to our Eq. (5), whose coefficients they called $P_m(T_{KL})$. Clearly, the simple trigonometric relations underlying our Eq. (5) are much more familiar and less intimidating. They were able to work out the condition for perfect transmission in a form equivalent to our Eq. (18), and they illustrated it graphically. Perhaps, if they had recognized that their $P_m(T_{KL})$ were the Chebysheff polynomials of the second kind (as pointed out by Ref. 2), they might have made the connection with the Bloch phase. They recognized that the "forbidden bands" are associated with $|T_{KL}|>2$, but they did not discuss how the closely spaced resonances in the "allowed bands" go over to the nondecaying states of the Kronig–Penney model.

VII. CONCLUSION

We have shown that the problem of transmission through N identical potential cells, can be best understood in terms of the Bloch phase and the single-cell transmission amplitude. Essentially we have extended a theorem of Vezzetti and Cahay to apply the amplitude rather than the probability of transmission. By going over to a transfer matrix based on the representation in terms of left- and right-moving waves, the derivation becomes trivial.

Our main results are as follows: Once the single-cell transmission problem has been solved, the transfer matrix can be constructed as in Eq. (22), and the Bloch phase found as in Eq. (23). The N-cell transmission and reflection amplitudes are then given by Eqs. (19) and (21), respectively. The transmission probability is given by Eq. (17), from which one deduces the conditions for transparency. The N-cell problem is transparent whenever the single cell is transparent, and in addition at $N-1$ points in each allowed band, in which the Bloch phase ϕ is real; see Eq. (18). In between each maximum of $|T_N|$ there is a minimum, and the envelope of the minima is given by Eq. (30). At the midpoint of each band, where $\phi=\pi/2 (\bmod\ \pi)$, the envelope equals the single-cell transmission probability so this sets the vertical scale for variation of $|T_N|^2$ in each allowed band. Knowing this one can sketch out the N-cell transmission probability qualitatively without detailed calculation, working from T_1 and $\cos\phi$.

When the Bloch phase is complex, the $\sin^2 N\phi$ in Eq. (17) becomes a $\sinh^2 N\theta$, and for large N this becomes very large, which forces $|T_N|^2$ to become very small. This situation corresponds to the forbidden Bloch zones. In Fig. 4 we have an example where θ happens to be very small, so the forbidden zone is only weakly expressed, even for $N=51$.

The formalism was applied to the case of a finite number of delta functions and square barriers. This allows us to understand the features of this system which were noted by Kiang. Equations (17), (19), and (30) are the essential tools.

Scattering by a finite series of barriers is much richer than for a single barrier. We hope that this formulation will make it more accessible to students.

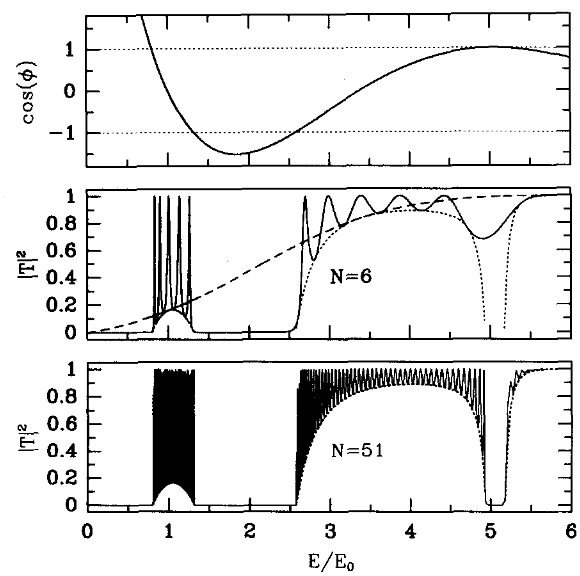

Fig. 4. Same as Fig. 3, but for $V_0=2E_0$.

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APPENDIX A: RECURRENCE FOR EQ. (5)

Write Eq. (5) as

$$\mathbf{W}^N=[\mathbf{W}U_{N-1}(\mu)-\mathbf{1}U_{N-2}(\mu)], \tag{A1}$$

where $\mu=\cos\phi$ and from Eq. (4)

$$U_1(\mu)=\frac{\sin(2\phi)}{\sin\phi}, \quad U_0(\mu)=1. \tag{A2}$$

Multiply by $\mathbf{W}$, linearize using Eq. (4), and one will have the correct form for $\mathbf{W}^{N+1}$ providing that

$$U_N(\mu)=2\mu U_{N-1}(\mu)-U_{N-2}(\mu). \tag{A3}$$

The solution of this recurrence is

$$U_{N-1}(\mu)=\sin N\phi/\sin\phi. \tag{A4}$$

According to Arfken,[12] this is a definition of the Chebyshef polynomials of the second kind. If we compare with Kalotas and Lee,[3] we see that $P_N(2\mu)=U_N(\mu)$.

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CAPA—An integrated computer-assisted personalized assignment system

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A new integrated computer-assisted personalized assignment (CAPA) system that creates individual assignments for each student has been developed and found to be a powerful motivator. The CAPA system allows students to enter their answers to personalized assignments directly via networked terminals, gives immediate feedback and hints (allowing challenging questions), while providing the instructor with on-line performance information. The students are encouraged to study together which is known to be an effective learning strategy, but each must still obtain his/her own correct answers. Students are allowed to re-enter solutions to the problems before the due date without penalty, thus providing students with different skills levels the opportunity and incentive to understand the material without being judged during the learning process. The features and operation of the system are described, observations on its use in an introductory general physics class are reported, and some of the highly favorable student reactions are included.

I. INTRODUCTION

Universities continue to search for more effective ways to provide students a quality education. Scientific literacy, an important component of a modern education, is an elusive goal requiring the study and understanding of unfamiliar concepts and the ability to solve numerical problems. Physics education relies heavily on numerical treatment of problems which appear daunting to many beginning students.

In all of their studies, students are challenged to achieve certain levels of understanding and skills.[1] Meeting the goals is a cooperative task, and diligent instruction does not insure success without a commensurate effort by students. In the traditional format, the instructor sets the level, explains the material and assigns practice problems for the student. The instructor can then guide and help, but the student learns and achieves the goals[2] by serious work. The instructor can "explain" or "clarify" effectively only after the student has tried to solve example problems. Usually, the student works on the problems independently, or in peer groups, and does not know if the solutions are correct at the time they are worked out. The student needs timely feedback on the correctness of his or her problem solution. For effective learning, each student should do some independent work. When students work together to solve similar but not identical problems, both collaboration and independence can be fostered.

We have developed a computer based system to create personalized, that is, individual numerical and multiple-

