Chapter 2

Graphs of Lines

2.1 The Slope of a Line

In Section 1.2 you learned graphing in Cartesian plane. In this section you will graph linear equations in two variables. The concept of shape, point-slope and slope-intercept forms of linear equations are introduced.

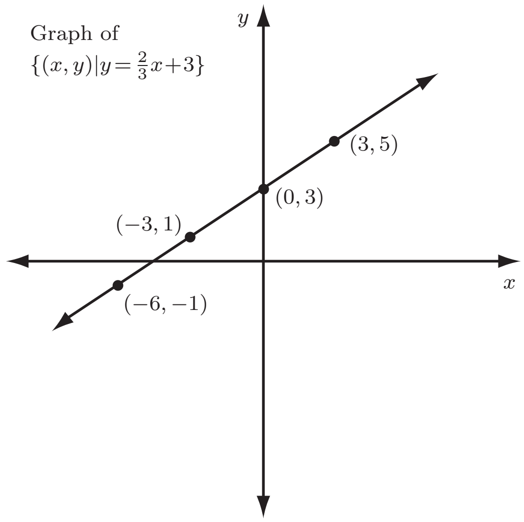
To understand the concept of slope, consider the following example.

Example 1 *Graph the set* $\{(x, y) \mid y = \frac{2}{3}x + 3\}$.

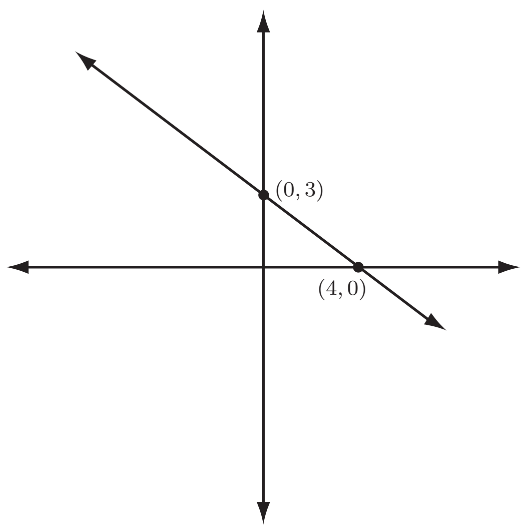
SOLUTION: For a point to be in the set it must satisfy the equation $y = \frac{2}{3}x + 3$. The point $(3, 5)$ is in the set since $5 = \frac{2}{3}(3) + 3$. To get more points in the set, pick a convenient value to substitute for x in the equation and solve for y. Using $x = 0$, -3, and -6 gives $y = \frac{2}{3}(0) + 3 = 3$, $y = \frac{2}{3}(-3) + 3 = 1$, and $y = \frac{2}{3}(-6) + 3 = -1$, respectively. The points $(3, 5)$, $(0, 3)$, $(-3, 1)$, and $(-6, -1)$ are in the set $\{(x, y) \mid y = \frac{2}{3}x + 3\}$. Plotting these points in the Cartesian plane shows that these points lie on the straight line shown.

Example 2 *Graph the set* $\{(x, y) \mid 3x + 4y = 12\}$.

SOLUTION: The points $(0, 3)$ and $(4, 0)$ are easily seen satisfy the equation. these are called the **y-intercept** and the **x-intercept** respectively.



To get the y-intercept, substitute $x = 0$ in the equation and solve for y. To get the x-intercept, substitute $y = 0$ in the equation and solve for x. Using these two points gives the required graph.



The graphs of the lines have some characteristics that require particular attention. The point where the line crosses the y-axis is known as the y-intercept and the point where the line crosses the x-axis is the x-intercept. In general $(a, 0)$ represents the x-intercept, and $(0, b)$ the y-intercept. Another feature of the lines is the fact that both lines are slanting or sloping. The measure of "steepness" of the slant is called the **slope of the line**.

> **Slope:** Geometrically, the slope of a line is the ratio of the vertical change to the horizontal change as you move from one point to the other along the line.

Example 3 *Find the slope of the line given by* $\{(x, y) \mid y = \frac{2}{3}x + 3\}$.

SOLUTION: Suppose you are at $(0, 3)$ and moving along the line till you get to the point $(3, 5)$. The vertical change is $5 - 3 = 2$ and the horizontal change is $3 - 0 = 3$. The slope is

$$\frac{\text{vertical change}}{\text{horizontal change}} = \frac{2}{3}.$$

Note that the slope is the same as the coefficient of x in the equation $y = \frac{2}{3}x + 3$.

To get the **formula** for **the slope of a line**, suppose the line contains the points (x_1, y_1) and (x_2, y_2). The vertical change is $y_2 - y_1$ and the horizontal change is $x_2 - x_1$. The slope is

$$\frac{\text{vertical change}}{\text{horizontal change}}$$

$$m = \frac{y_2 - y_1}{x_2 - x_1}.$$

Note that the letter m is used to denote the slope of a line.

Example 4 *Find the slope of the line given by* $\{(x, y) \mid 3x + 4y = 12\}$.

SOLUTION: From Example 2, the points $(0, 3)$ and $(4, 0)$ are on the line.

Let $(0, 3) = (x_1, y_1)$ and $(4, 0) = (x_2, y_2)$. Then

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{0 - 3}{4 - 0} = -\frac{3}{4}.$$

Notice that the line that is rising from left to right has **positive slope** indicating that y increases as x increases. The line that is falling from left to right has **negative slope** indicating that y decreases as x increases.

Example 5 *Find the slope of the line passing through the points* $(4, -3)$ *and* $(2, 5)$. *Is y increasing as x increases?*

SOLUTION: Using the slope formula gives

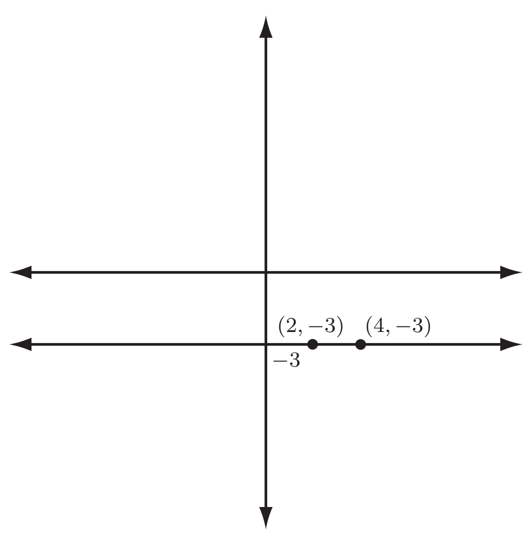
$$\begin{aligned} m &= \frac{5-(-3)}{2-4} \\ &= \frac{8}{-2} = -4. \end{aligned}$$

The slope is negative indicating that y is decreasing as x increases.

Example 6 *Find the slope of the line passing through the points* $(4, -3)$ *and* $(2, -3)$. *Plot and describe the line through these points.*

SOLUTION: The slope is

$$m = \frac{-3-(-3)}{2-4} = \frac{0}{-2} = 0.$$



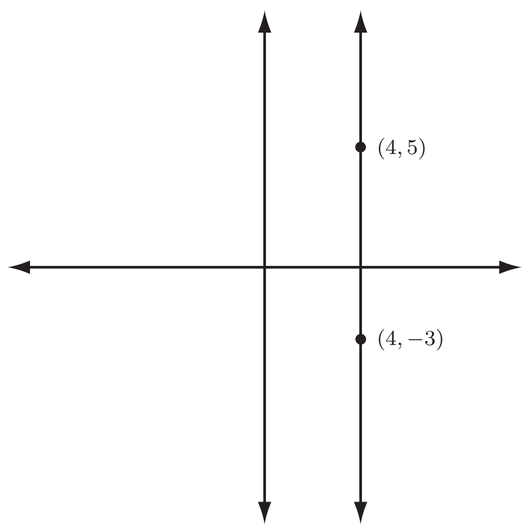
The line is a horizontal line through $y = y - 3$. Note that the value of y does not change from $(2, -3)$ to $(4, -3)$. It remains constant. Therefore, since all horizontal lines have a constant value for y, their slope will always equal zero. **The conclusion is that all horizontal lines have slope equal to zero**.

Example 7 *Find the slope of the line passing through* $(4, -3)$ *and* $(4, 5)$. *Plot & describe the graph of the line.*

SOLUTION: The slope is

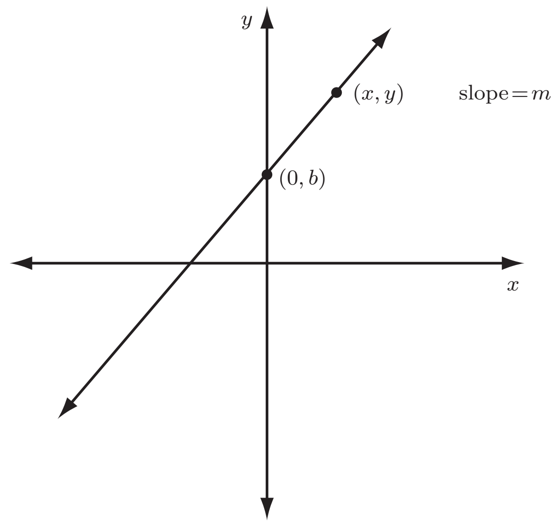
$$m = \frac{5 - (-3)}{4 - 4} = \frac{8}{0}.$$

This is undefined.



The graph is a vertical line. In this example the value of x stays constant from $(4, -3)$ to $(4, 5)$. All vertical lines have a constant value for x. Their slope is undefined. **The conclusion is that all vertical lines have undefined slope**.

Next, we look at describing the graph of a line algebraically by finding the equation of a given line. First, given the slope of the line and the y-intercept we can write down the **slope-intercept form** of the equation of the line. Consider the graph below:



The line passes through the y-intercept $(0, b)$ and has slope m. The point (x, y) is a general point on the line. Using the slope formula and points $(0, b)$, (x, y) we get

$$\begin{aligned} \frac{y-b}{x-0} &= m \\ \frac{y-b}{x} &= m, \\ y - b &= mx \\ y &= mx + b. \end{aligned}$$

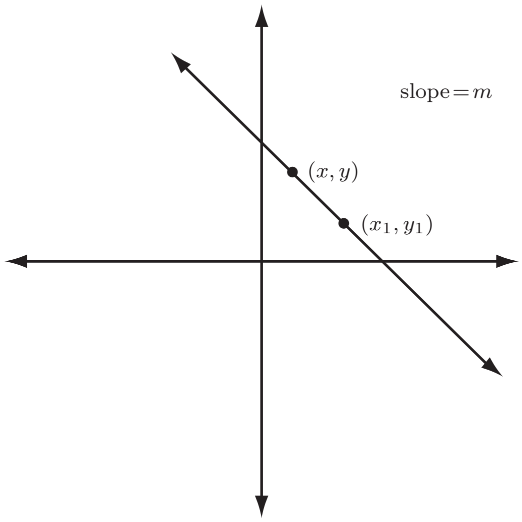
Thus the slope-intercept form of the equation of the line is $y = mx + b$. Note that the slope is the coefficient of x and the y-value of the y-intercept is the constant term.

Example 8 *Find the slope-intercept form of the equation of a line with slope* $\frac{2}{3}$ *and y-intercept* $(0, -4)$.

SOLUTION: The slope-intercept form of the equation of the line is $y = mx + b$. Here, $m = \frac{2}{3}$ and $b = -4$. So the equation is

$$y = \frac{2}{3}x + (-4) \qquad \text{or} \qquad y = \frac{2}{3}x - 4.$$

If we are given the slope of the line and one point other than the y-intercept, we can use the **point-slope form** of the equation of the line. Consider the graph below:



The line passes through the point (x_1, y_1) and has slope $= m$. Let (x, y) be a general point on the line such that $x \neq x_1$. Then

$$\frac{(y - y_1)}{(x - x_1)} = m \qquad \text{or} \qquad y - y_1 = m(x - x_1).$$

This is the **point-slope form** of the equation of the line.

Example 9 *Find the equation of the line with slope* $= -\frac{2}{3}$ *and passing through* $(5, -3)$.

SOLUTION: The point-slope form of the equation of the line is

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - (-3) &= -\frac{2}{5}(x - 5) \\ y + 3 &= -\frac{2}{5}(x - 5) \end{aligned}$$

We can leave the equation like this. to change it to the slope-intercept form, solve the equation for y to get

$$\begin{aligned} y+3 &= -\frac{2}{5}x+2 \\ y &= -\frac{2}{5}x-1. \end{aligned}$$

Example 10 *Find the equation of the line passing through the points* $(4,-3)$ *and* $(-6,2)$.

SOLUTION: The slope has not been given. First find the slope.

$$\begin{aligned} m &= \frac{y_2-y_1}{x_2-x_1} \\ &= \frac{2-(-3)}{-6-4} \\ &= \frac{5}{-10} \\ &= -\frac{1}{2}. \end{aligned}$$

Now use one of the points $(4,-3)$ and the point-slope form equation to get

$$\begin{aligned} y-(-3) &= -\frac{1}{2}(x-4) \\ y+3 &= -\frac{1}{2}(x-4). \end{aligned}$$

Solve for y to put the equation in the slope intercept form.

$$\begin{aligned} y+3 &= -\frac{1}{2}x+2 \\ y &= -\frac{1}{2}x-1. \end{aligned}$$

2.2 More on Straight Line Graphs

In this section you learn how to transform a linear equation from the **standard form** to the slope-intercept form and graph the equation. **Parallel lines** and perpendicular lines are discussed.

Linear Equations: A linear equation in two variables x and y in standard form is

$$ax + by = c$$

where a, b, and c are integers. It is preferable to have $a \geq 0$.

A linear equation is slope-intercept form or point-slope form can be transformed into the standard form.

Example 1 *Transform the following linear equations to* ***standard form***

a. $y = \frac{2}{5}x - 3$

b. $y - 4 = -3/4(x + 2)$

SOLUTION:

a.

$$\begin{aligned} y &= \frac{2}{5}x - 3 \\ 5y &= 2x - 15 \qquad \textit{Multiply each term by 5} \\ -2x + 5y &= -15 \qquad \textit{Subtract } 2x \textit{ from each side} \\ 2x - 5y &= 15 \qquad \textit{Multiply by } -1 \textit{ to have leading coeff. positive} \end{aligned}$$

Note that the equation $-2x + 5y = -15$ is also acceptable. It can be written as

$$5y - 2x = -15.$$

b.

$$\begin{aligned} y - 4 &= -\frac{3}{4}(x + 2) \\ 4y - 16 &= -3(x + 2) \qquad \textit{Multiply through by 4} \\ 3x + 4y &= 10 \qquad \textit{Bring variables to one side and const. to another} \end{aligned}$$

Transforming a linear equation from standard form to slope-intercept form is equivalent to solving the equation for y.

Example 2 *Transform the linear equation $3x + 4y = 12$ to slope-intercept form. Identify the slope and the y-intercept.*

SOLUTION:

$$\begin{aligned} 3x + 4y &= 12 \\ 4y &= -3x + 12 && \textit{Subtract } 3x \textit{ from each side} \\ y &= -\frac{3x}{4} + \frac{12}{4} && \textit{Divide each term by 4} \\ y &= -\frac{3}{4}x + 3. \end{aligned}$$

So $m = -\frac{3}{4}$ (coefficient of x) and $(0, 3)$ is the y-intercept [y-intercept has $x = 0$, and $y = b$].

Section 2.1 had example showing how to graph straight line equations by plotting a sample of points and using the x-intercept and y-intercept. Alternatively, we can use the slope and one point to plot the graph of a linear equation.

Example 3 *Plot the graph of each of the following equations.*

a. $y = 3x - 4$

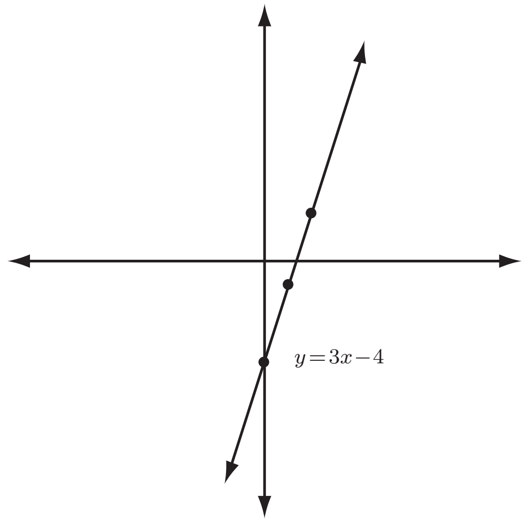
b. $y - 2 = -\frac{3}{4}(x + 2)$

c. $3x + 4y = -12$

SOLUTION:

a. $y = 3x - 4$

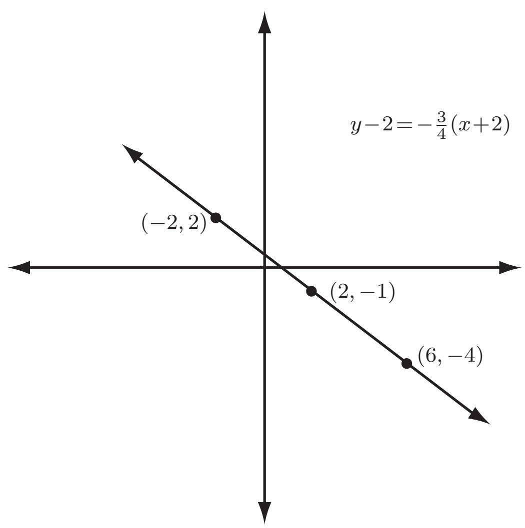
This equation is in the slope-intercept form. The slope $m = 3$ and the y-intercept is the point $(0, -4)$. The slope $m = 3 = \frac{3}{1}$. This means that as x increases by 1 unit, y increases by 3 units. Start at $(0, -4)$. Go up 3 units and over 1 unit to get the next point $(1, -1)$.



b. $y-2=-\frac{3}{4}(x+2)$

This equation is in point-slope form. The point is $(-2,2)$ and the slope is $-\frac{3}{4}$.

One way to interpret the slope is to say: Go down 3 units for every 4 units you go to the right. Use the point $(-2,2)$ as the starting point.

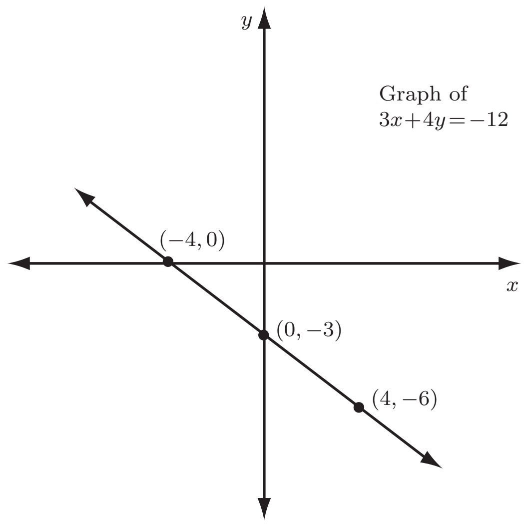


c. $3x + 4y = -12$

Solve the equation for y to get

$$4y = -3x - 12$$
$$y = -\frac{3}{4}x - 3$$

$m = -\frac{3}{4}$, y-intercept is the point $(0, -3)$. the graph is given below.



Note that we could have determined the x-intercept and y-intercept as we did in Section 2.1.

In fact a linear equation can be put in the form

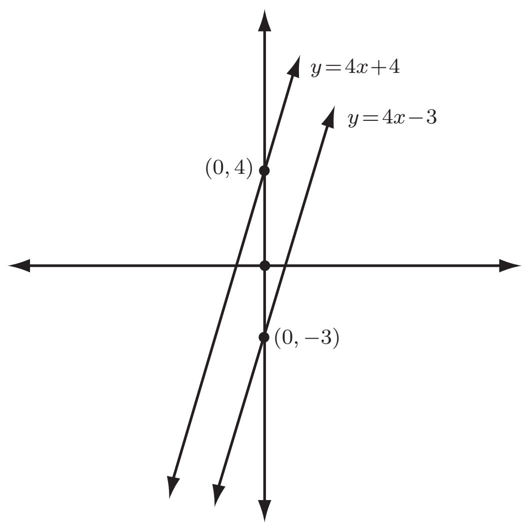
$$\frac{x}{a} + \frac{y}{b} = 1$$

where $(a, 0)$ is the x-intercept and $(0, b)$ is the y-intercept. In Example 3c, the equation $3x+4y = -12$ is equivalent to $\frac{x}{-4} + \frac{y}{-3} = 1$, obtained by dividing through by -12 and simplifying. this gives $(-4, 0)$ as the x-intercept and $(0, -3)$ as the y-intercept.

Parallel Lines: Parallel lines have the same slope but different y-intercepts.

Example 4 *Plot the graphs of the linear equation* $y_1 = 4x-3$ *and* $y_2 = 4x+4$ *on the same coordinate plane.*

SOLUTION: Both equations have slope $m = 4$, and the y-intercepts are $(0,-3)$ and $(0,4)$ respectively.



The two straight lines are parallel.

Example 5 *Find the equation of the straight line graph that is parallel to the graph of* $2x - 3y = 7$ *and passes through the point* $(-2, 5)$. *Give the answer in standard form.*

SOLUTION: First find the slope of the given equation:

$$2x - 3y = 7$$
$$-3y = -2x + 7$$
$$y = \frac{2}{3}x - \frac{7}{3}$$

$m = \frac{2}{3}$.

The line passing through $(-2, 5)$ has $m = 2/3$. The point-slope form gives

$$y - 5 = \frac{2}{3}(x + 2)$$

or

$$\begin{aligned} 3y - 15 &= 2x + 4 \\ 3y - 2x &= 19 \\ 2x - 3y &= 19. \end{aligned}$$

> **Perpendicular Lines:** Perpendicular lines intersect at a right angle. Furthermore, two lines are perpendicular if the product of their slopes is -1.

Example 6 *Determine whether the graphs of the linear equations $3x - y = 10$ and $x + 3y = 3$ are perpendicular. Plot the lines on the same coordinate axes.*

SOLUTION: We need to show whether the product of the slopes is -1. Solve each equation for y

$$\begin{aligned} 23x - y &= 10 & x + 3y &= 3 \\ -y &= -3x + 10 & 3y &= -x + 3 \\ y &= 3x - 10 & y &= -\frac{1}{3}x + 1 \end{aligned}$$

The slopes are 3 and $-\frac{1}{3}$ respectively. The product of the slopes is $3(-\frac{1}{3}) = -1$. Thus the two lines are perpendicular. Their graphs also show they are perpendicular.

> **Negative reciprocal:** Two lines are perpendicular if the **slope** of one is the **negative reciprocal** of the other.

Example 7 *find the equation of the line passing through $(3, -4)$ and perpendicular to the line $2x - 5y = 10$. Leave answer in standard form.*

SOLUTION: First find the slope of the given line by solving for y.

$$\begin{aligned} 2x - 5y &= 10 \\ -5y &= -2x + 10 \\ y &= \frac{2}{5}x - 2. \end{aligned}$$

Let m be the slope of the line passing through $(3, -4)$. Then $\frac{2}{5}m = -1$, $m = -\frac{5}{2}$. Using **point-slope** equation, gives

$$\begin{aligned} y + 4 &= -\frac{5}{2}(x - 3) \\ 2y + 8 &= -5(x - 3) \\ 2y + 8 &= -5x + 15 \\ 5x + 2y &= 7 \end{aligned}$$

Horizontal Lines and Vertical Lines: All points on a horizontal line have the same y-value. the equation of a horizontal line is $y = b$ where b is a constant. The slope is zero and the y-intercept is $(0, b)$.

All points on a vertical line have the same x-value. The equation of a vertical line is $x = a$ where a is a constant. the slope is undefined and the x-intercept is $(a, 0)$.

Example 8 *Plot the graphs of following equations. Determine the slope if possible.*

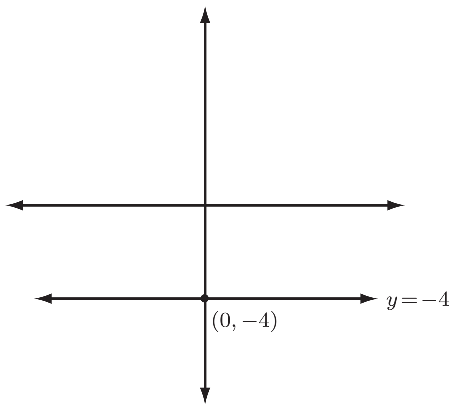
a. $y = -4$

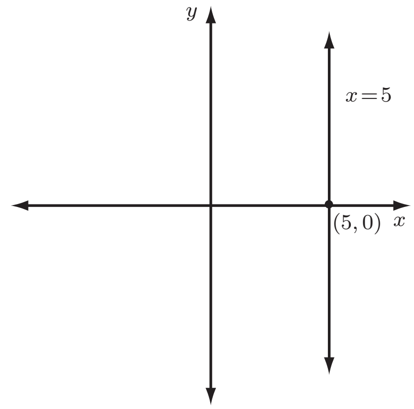
b. $x = 5$

SOLUTION:

a. $y = -4$ is a horizontal line through $(0, -4)$. the slope is 0.

b. $x = 5$ is a vertical line through $(5, 0)$. The slope is undefined.





2.3 Linear Inequality in Two Variable

Linear Inequality: A linear inequality in two variables is similar to a linear equation except that the equality sign is replaced by an inequality sign such as $<$, $>$, $\leq$, or *geq*.

The solution of a linear inequality in two variables is a half plane consisting of coordinate points that satisfy the inequality.

Example 1 *Is the point* $(-2, 3)$ *in the solution set of the inequality* $2x-3y \geq 2$*?*

SOLUTION: $2x - 3y \geq 2$

Is $2(-2) - 3(3) \geq 2$? Substitute $x = -2$ and $y = 3$.

Is $-4 - 6 = -10 \geq 2$? No.

Since $-10 \ngeq 2$, $(-2, 3)$ is not in the solution set of the inequality.

To solve a linear inequality, replace the inequality sign by an equals sign. Plot the resulting linear equation. Use inequality ($<$ or $>$), otherwise use a solid line. The straight line divides the plane into two half planes. Pick a test point from one half plane to decide which half plane is the solution set. Shade that half plane.

Example 2 *Solve the following linear inequalities.*

a. $2x - 3y \leq 6$

b. $y > -\frac{3}{4}x + 6$

c. $x < -4$

d. $y \geq 7$

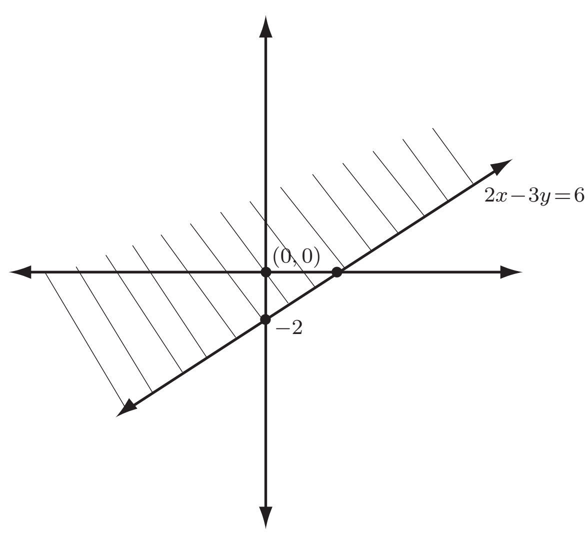
SOLUTION:

a. $2x - 3y \leq 6$

$$2x - 3y = 6 \qquad \text{Replace } \leq \text{ by } =$$
$$-3y = -2x + 6$$
$$y = \frac{2}{3}x - 2$$

y-intercept is $(0, -2)$ and slope is $\frac{2}{3}$.

The line is solid because the inequality sign is $\leq$. the graph is plotted below.



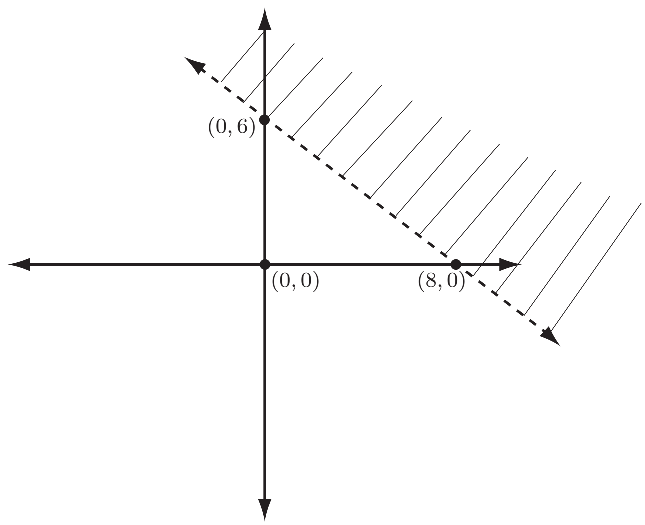
The two half planes are above and below the graph of the equation $2x - 3y = 6$.

The point $(0,0)$ is a convenient test point. Substituting $(0,0)$ in the inequality gives Is $2(0)-3(0)\leq 6$? Since $0\leq 6$. So the line half plane above the line.

b. $y>-\frac{3}{4}x+6$

$$y=-\frac{3}{4}x+6 \qquad \text{Replace } > \text{ by } =.$$

Plot the graph of $y=-\frac{3}{4}x+6$. Use dashed line since the inequality is strict.



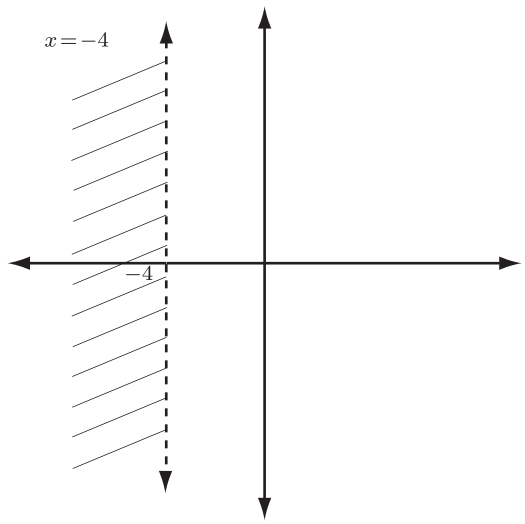
Use $(0,0)$ as test point.

Is $0>-\frac{3}{4}(0)+6$? Since 0 is not greater than 6, $(0,0)$ is not in the solution set. shade the other side.

c. $x<-4$

$$x=-4$$

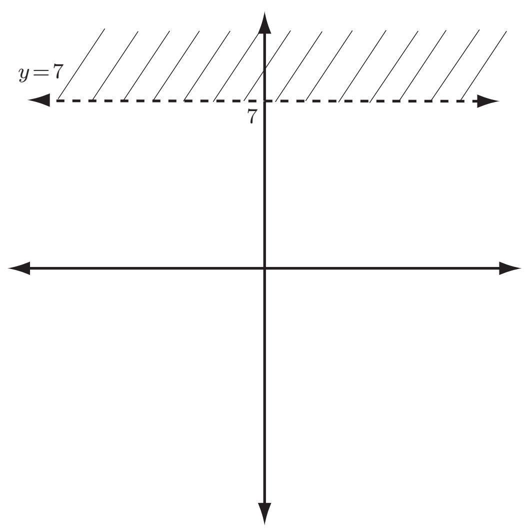
This is a vertical line through $x=-4$



Solution set consists of the plane to the left of $x = -4$.

d. $y \geq 7$

$$y = 7$$



Solution set consists of points on $y = 7$ and above $y = 7$.