IV Wavelets and Multiresolution Approximations

A brief exposition

C S Salimath
Research Scholar
Department of Mathematics,
Karnatak University, Dharwad – 580003
E-mail: salimathcs@hotmail.com
URL: http://www.geocities.com/salimaths

Abstract:

Approximation theory developed as an independent discipline dealing mainly with representation/approximation of functions and their properties. The development of enormous amount of computing ability in recent decades, has given a new-thrust, heralding a rejuvenation of the theory and reinforcement of its techniques. The birth of Wavelet theory has helped interpret many of the concepts and formulations, in view of the changing scenario in the field of applications. The purpose of this article is to review the traditional methods of Approximation theory and introduce and highlight recent advances, under the framework of Wavelet theory, and show how it can be treated in a unified way. An attempt is made to focus on related topics e.g.; Interpolation, Least square approximation, Multi-resolution, B-splines and Coiflets etc.

Key words:

The best approximation, Estimation of error, least square - approximation, Orthogonal projection, B-splines, Cubic splines and Coiflets.

4.1 Introduction:

Every measurement whether manual or mechanical (whether the result obtained from an impression of the human eye or ear or from a highly sophisticated instrument), it is merely an approximation. We know and our computing machines know, only a finite number of decimal places, in a numerical representation of a distance, weight, force, temperature, and a host of other physical quantities and their derivatives. The eye has a limited resolving power and the ear has a limited frequency response and these limitations prevent us in achieving mathematical accuracy in measurements of all types whether human or instrumental. These limitations are not only due to "manufacturing defects"(biological or mechanical) as Heisenberg's Uncertainty Principle tells that they are inherent in the essence of things and some compromise(trade-off) between precision and mathematical convenience; perception and reality; reason and observation; is unavoidable. The certainty of error in every measurement, in every physical phenomenon or a theoretical model of that, emphasizes the importance of knowing how accurate a particular approximation can or should be. In some applications, the best approximation is not always the most desirable one. It may be computationally expensive or might not share certain essential features of the problem at hand. So the problem is how well one can construct an approximation which is good (at least good enough") but not necessarily the best. Other things being equal, a better approximation is preferred to a worse one. For a fixed expenditure of computational resources, measurements and analyses expressed in terms of "compactly supported" wavelets give a better approximation to speech signals, turbulence and other transient and localized phenomena than conventional methods. Recent studies on wavelets have established, beyond anybody's doubt that wavelet series approximate abrupt transitions much more accurately than Fourier series, which otherwise, reproduce perfectly steady and stable signals. If one method produces a much better approximation than another, then fewer data points will be required to provide the desired solution accurately leading to "data compression" which has reduced dramatically the cost of storing and sending information from anywhere to anywhere on the planet. Moreover, wavelet approximation does not cost more to calculate than an ordinary Fourier approximation. Today, life and Universe depend more on approximations than ever before and so do Science and Technology and it goes without saying that each one of us is dependent on technology for sustaining our life-style and possibly life itself.

4.2 Genesis of the theories:

It is common knowledge that functions are basic mathematical tools for describing and analyzing physical phenomena. However, it is very rarely that functions are known explicitly. One of the basic ideas and philosophies of Approximation and Wavelet theories is to represent an arbitrary function in terms of other functions which are nicer, simpler, more familiar and more easily computable. There are at least two approaches to Wavelet theory, the first is the interpolation of Wavelet transform as a time-frequency analysis and the second approach uses the Wavelet transform as a mathematical microscope. The second approach is closely linked to Approximation theory. Let us agree to call, functions used for representing arbitrary functions as analyzing functions. For instance, when we try to expand a function in a power series, we are trying to represent the function in terms of polynomials, namely the partial sums of the power series. Such a way of representing gives a simple way of obtaining information about the function. The value of a polynomial can easily be evaluated on a computer taking advantage of the fact that it involves only three arithmetic operations. There are standard packages on polynomial evaluation, by means of which we can sometimes obtain, a more accurate answer, instead of writing a difficult program and getting only approximate value. The problem of representing a given function normally splits into various sub-problems. They can broadly be categorized into:

- Best choice of representing functions
- Approximating the value of representing functions
- Approximating the coefficients of the representation
- Approximating the representation itself e.g. truncating(chopping) the power series to a suitable size
- Fixing criteria for a good approximation
- Choosing approximable functions

It may be noted that, there are several methods of representing functions and each is suitable for certain tasks. There is no single representation which is suitable for all applications. Wavelets; because of their versatility and adaptability are the most suitable representing functions.

4.3 A little history:

Classical Approximation theory dealt with the problem of whether a given function could be accurately approximated using some norm, by an element from a prescribed set of functions, which generally possessed some noteworthy properties. Quantitative approximation theory, in particular, attempted to determine as precisely as possible, the size of the error in the approximation, given specific information about the function to be approximated and the set of functions from which the approximation is taken. Thus the Weierstrass's famous theorem of 1883 asserted that each continuous function on a closed interval in the real line could be approximated to within any specified tolerance by a polynomial. Historically, the first examples of representations (approximations) of functions appeared in the form of tables e.g. Trigonometric and Logarithmic tables in 17th century. The power function x^n was first described by Descartes. After the invention of the exponential notation, there were many examples of power series in the 17th and 18th centuries, notably Newton's binomial expansion: $(1+x)^n$ and extended form of it for non-integral powers. After the advent of Calculus we find Taylor's and Fourier's series to be the most important and useful representations (approximations) of functions.

4.4 Some rudiments of Approximation theory:

The main theme of this article is linear approximation of functions i.e. approximating by linear combination of functions (Superpositions). The problem of linear approximation can be formulated in the following way:
Let ? be a sub- set of functions of a fixed function space A. If a function f € A, can one find a linear combination P = $\sum a_i f_i$ which is close to f? Two problems arise: We must select the set ? and second, decide how the deviation of P from f should be measured. To fix the ideas, let A be a compact Housdorff space and let C = C [A] be the set of all continuous functions on A. Then C is a normed linear space over R where $\|f\| = Sup|f(x)|$, x € A The convergence of $f_n \to f$ in the norm of C, i.e. $\|f_n - f\| \to o$ as $n \to \infty$ is equivalent to the uniform convergence of f_n (x) to f(x), $\forall$ x$\in$ A.It fallows from this interpretation that C is complete. Complete normed spaces are called Banach spaces which are very important in the theory of approximation.For example, $L^p[a,b]$, $p \geq 1$.However, for all practical purposes, approximation in the spaces over R or C remains both interesting and useful special case.
The following definitions apply to any Banach space with elements f and a distinguished subset ? .We call f approximable by P = $\sum a_i f_{i,} f_i \in$? , $a_i \in R$ if $\forall \in > o, \exists P$ such that

$\|f-P\|<\in$, and in this case $E_n(f) \,\square\, \inf\|f-P\|$ is called the nth order (degree) approximation of f by $a_i \in Rand$? (*Estimation of error*). If the infimum is attained for some P then P is called the "*the best approximation*" to f. For the space C[a,b], a natural choice of ? is given by the power functions 1,x, $x^2, --x^n --$.Another important compact set K is the additive group of reals R modulo $2p$ .We shall follow the traditional practice of identifying $f \in C$ [K], with the continuous $2p$ -periodic functions on R.A tool of approximation for -functions $f \in C[K]$ is the following set of trigonometric polynomials,

$$T_n(x) = a_o/2 + \sum a_k \cos kx + b_k \sin kx$$, k = 1, 2, ----n which form the

partial sums of the famous *Fourier series*.

4.5 Interpolation and Approximation:

The intimate connection between interpolation and approximation has been recognized and exploited by mathematicians in recent years to develop new theories and techniques. In retrospect, consider a set of real valued functions, $f_{1,}f_2 ---f_n$ on A and c_k k=1,2,--n be given real numbers(data points).The polynomial P(x) = $\sum_{i=1}^{n} a_i f_i$ (x) is said to interpolate the *values* c_k '*s* if $P(x_k) = c_k$ .Usually the c_k '*s* are the values at the points a_k of some given function f(x), then P is said to interpolate f. Assume that, we can find polynomials $l_k(x)$ such that $l_j(x_k) = d_{jk}$ .Then P(x) = $\sum c_i l_i(x)$ interpolates c_k .Interpolation with algebraic polynomials is probably the most common form, because they are the easiest to evaluate on a computer.

. In fact, Taylor's and Chebysheve's theorems bear ample testimony to this fact. Taylor's theorem (a beautiful and remarkably useful result) exhibits the value of a function at a point in terms of its values of all its derivatives at some nearby point, in the form of an infinite series. It also provides solution to the problem of constructing the best approximation by polynomials. This allows one, to represent classical functions quite readily, for instance, $e^x = \sum \frac{x^n}{n!}$.

Now suppose, we are given a function f, know its value $f(x_o)$ along with its first n derivatives $f^{'}(x_0), --- f^n(x_0)$ at some base point x_o.In order to obtain a polynomial $P_n(x)$ which approximates f, it is reasonable to ask whether it can interpolate $f^k(x_o), k = o,1, --n$ .Indeed, in accordance with Taylor's theorem

$$P_n(x) = \sum_{k=o}^{n} \frac{f^k(x)}{k!}(x-x_o)^k + R_n(x) \text{ where } R_n(x) = \frac{f^{n+1}(c)}{(n+1)!}(x-x_o)^{n+1}, x_o \le c \le x;$$ Thus

Taylor's theorem not only gives a nice formula of interpolation, it gives a method of approximating functions by means of polynomials (truncated Taylor's series).

Here, $E_n(f) = \max\left|f(x) - P_n(x)\right|$, the degree of approximation(estimation of error).Now, if $E_n(f) \to o as n \to \infty$,we can say that the approximation is worthwhile. Again, it is not enough to have $E_n(f) \to o as n \to \infty$,we must know how fast it tends to o. That will tell us where to stop for the desired accuracy. Thus, we see how Taylor's series helps us in approximating a function by a polynomial, once we have settled the question of the rate of convergence. In general, the "smoother" the function the faster $E_n(f) \to o as n \to \infty$. Theorems that guarantee this are called "direct" theorems [A] of approximation theory. Conversely, "inverse" theorems assert that, a function f has certain smoothness properties if $E_n(f) \to o$ rapidly enough. In this way, we are able to characterize the functions by the order of magnitude of their degree of approximation. Cases when there is an explicit formula for the degree of approximation, or for that matter, for the polynomial of best approximation, are exceptional and are of special interest.

Now, to establish the close ties Legendre polynomials have with approximation of functions, one should consider the notion of *Least square approximation* .If f(x) be defined on [-1,1], it is but natural to ask whether f(x) could be approximated as closely as possible by a polynomial p(x) of degree < or = n, in the sense of least squares. The answer is affirmative if one can interpret $\int_{-1}^{1} (f(x) - p(x))^2 dx$ as representing the sum of the squares of the deviations of p(x) from f(x). Then the strategy is to minimize the value of the integral by a suitable choice of p(x). It turns out that, the minimizing polynomial is precisely the partial sum of the first n+1 terms of the Legendre series.[G] A significant parallel to interpolation by polynomials consists of interpolation by splines developed by Schoenberg and others. Theory of splines dramatically generalizes the power series representation of a function. One of the main drawbacks of a power series is that, representation is usually accurate only in the neighborhood of a given point. Another disturbing feature of polynomial approximation is that, if the function to be approximated(approximant) is " badly behaved" anywhere in the interval of approximation, then the approximation is "poor" everywhere. This *global* dependence on *local* properties can be avoided using splines. Splines are piece-wise polynomial functions of a given degree and are patched together at critical points called nodes(knots).One of the basic methods for constructing wavelets involves use of "cardinal B-splines".Thease are arguably, the simplest functions with small supports that are most efficient for both hardware and software implementation. As noted earlier, a function ? € $L^2(R)$ is said to generate a MRA, if it generates a nested sequence of subspaces V_j that satisfies certain conditions laid down by Mallat and Meyer. ? as we know is called the "scaling function" and typical examples of scaling functions are precisely the mth order cardinal *B-splines* $B_m where B_m(x) = \int_{-\infty}^{\infty} B_{m-1}(x-t) B_1(t) dt$ -- (1)

and $B_o(x) = c[-1/2, 1/2](x)$, $B_1(x) = (1-|x|)c[-1,1]$(x), the well-known "hat"or "tent"function. It is possible to find a sequence of functions with increasing smoothness, iterating (1).In addition to their computational simplicity; the small supports of B-splines make them most desirable for local interpolation schemes of approximating functions.

4.6 Approximation from a Geometrical stand-point:

The question is this: If $f : R \to C$ is any arbitrary function, with a period a, can we find a decomposition of f, of the form $f(t) = \sum c_n e^{2p\,\mathrm{int}/a}$? The immediate answer is "no" if one considers only finite sum. The sum on the right is infinitely differentiable, while there is no reason for f to be. If equality cannot hold for a finite sum exactly, we can always try to have it hold "as well as possible". More precisely, we will try to answer the following question:

Given any integer N, is it possible to find coefficients x_n such that $\left\| \sum_{-N}^{N} x_n e^n - f \right\|$ attains minimum in L_2-norm. Geometrically this amounts to finding an element f_N in the subspace T_N of $L_2[o,a]$ that has minimum "distance" from f. When such an element f_N exists we say that it is the "the best approximation" of f in T_N. Geometrically, it is the *orthogonal projection* of f onto the subspace T_N, with respect to the *Fourier basis*.

The best approximation not only exists, but to our good fortune, it is unique. Indeed, it is given by $f_N(t) = \sum_{-N}^{N} c_n e_n(t) where c_n = \frac{1}{a}\int_o^a f(t) e^{-2p\,\mathrm{int}/a} dt$ -----------(2)

An immediate consequence of (2) is the inequality $\sum_{-N}^{N} |c_n|^2 \le \frac{1}{a}\int_0^a |f(t)|^2 dt$, traditionally known as *Bessel's inequality.*

One can ask what happens to f_N as N tends to 8 .We have, in fact, the most general result: If $f \in L_p^2[0,1], p \ge 1$, then the best approximation of f in the subspace T_N is given by $f_N(t) = \sum_{-N}^{N} c_n e_n(t) where c_n = \frac{1}{a}\int_o^a f(t) e^{-2p\,\mathrm{int}/a} dt$ and tends to f as N tends to 8 .

Expressed otherwise, $\int_o^a |f(t) - f_N(t)|^2 dt \to o as N \to \infty$ .A more sophisticated (scholarly) way of expressing this is to say that, the family of functions $(e_n)_{n \in Z}$ is a topological basis for the space $L_p^2[o,a]$.Moreover, the series $\sum_{-\infty}^{\infty} c_n e_n$ is summable to f in this space.

Incidentally, c_n are called *Fourier Coefficients* of the periodic function f.

4.7 Generality and Centrality:

Let us see, how these ideas fit into a broader pattern in an arbitrary Hilbert space, which generalizes several aspects of R^n .Hilbert spaces are, basically endowed with a "Euclidian"geometry in the sense that, there is a distance function and the notion of an angle between two vectors. Their completeness allows one to develop the notion of an infinite dimensional basis, giving flexibility and generality. Most importantly, we need to

consider Hilbert spaces instead of Banach spaces, to study wavelets from Approximation-theory point of view, about which, the article centres around.
Given a Hilbert space H, a subspace V of H and $f \in H$,we can ask the following:

i) Does there exist an $f^* \in V$ *such that* $\| f - f^* \| = \min \| f - v \|, v \in V$?

ii) If the answer to i) is "yes" can we characterize f?

If f^* exists, it is called "the best approximation" of in V. For example, take
H= $L_p^2[o, 2p]$, and V the subspace of trigonometric polynomials generated by
1, *sinx, cosx, -------sinmx, cosmx;*
The next theorem [B] answers our first question.
Suppose H is a Hilbert space and V a complete subspace of H.
Then given an arbitrary function: $f \in H, \exists f^* \in V, s.t. \| f - f^* \| = \min \| f - v \|, \forall v \in V$.

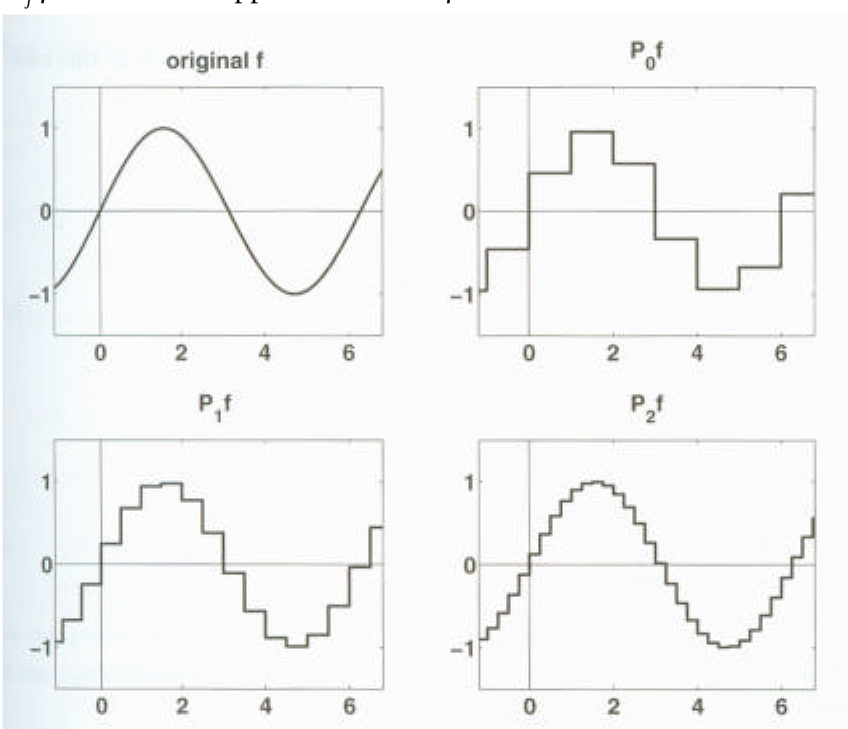
4.8 Wavelet-approximation:

The fundamental idea behind wavelet is to approximate and analyze a function according to scale. Some researchers feel that, by using wavelets one is adopting a new mindset or perspective in approximating functions. Wavelets are basically functions that satisfy certain mathematical requirements, and are used in representing data or functions. Though the idea is not new, as approximation of functions using superposition of simpler functions, has existed since 1800's when Joseph Fourier discovered that, he could superpose sines and cosines to represent other functions. However, in Wavelet theory the scale, that one uses plays a special role. The main goal of wavelet algorithms is to process data at different scales or resolutions, which is at the heart of "multiresolution"technique. Loosely speaking, if we look at a function with a large "window" we would see gross features. If, on the other hand, we look at a function with a small "window" we would notice details. These features peculiar to wavelets, make them interesting and useful. For many decades, scientists have wanted more appropriate functions than the sines and cosines to approximate transient signals. By their very definition, these functions are non-local (and stretch to infinity) do a poor job in approximating sharp spikes. But, with Wavelet Analysis, we can use approximating functions, which are contained in smaller domains resulting in "compression" of the data and at the same time, reducing the cost of computation.

A Multiresolution analysis (MRA), more justifiably called a " multiresolution-approximation" by its "intellectual father" Stephane Mallat in his paper[C], is a sequence of embedded (nested) subspaces V_j for approximating functions of $L^2(R)$.He claims that, from any multiresolution approximation, we can derive a function ?,the translates and dilates of which: $2^{j/2} y\,(2^j x - k)_{j,k \in Z}$ form an an o.n.b of $L^2(R)$.The approximation of a function $f \in L^2(R)$ at the scale (resolution level) 2^j is defined as the *orthogonal projection* of f on V_j.To compute this orthogonal projection, we show that there exists a unique function ? , the integral translates of which form an o.n.b.of V_o.The additional

information called *detail* available in the approximation at a resolution 2^{j+1} as compared to the resolution 2^{j} is given by an orthogonal projection on the orthogonal complement of V_j in V_{j+1} i.e. W_j. An important problem of approximation theory as stated earlier is to measure the decay of the approximation, when the resolution is increased given *a priori* knowledge about the function's smoothness. We estimate the decay for functions in Sobolev spaces, which answers our second question. [D]
Because practical measurements of real phenomena require time and resources, they provide not all values but only a finite sequence of values, called a *sample*, of the function representing the phenomenon under consideration. Therefore, the fist in the first step in the wavelet approximation is to approximate a function by its sample alone. One of the simplest methods of approximation uses Haar wavelets. The resulting function is called a *step function* which approximates the sampled function.
Although approximations more accurate than simple steps exist, they demand more sophisticated mathematics. In contrast to Haar wavelets, which exhibits jump discontinuities, Daubechies wavelets are continuous. As a consequence of their continuity, Daubechies wavelets approximate continuous signals more accurately with fewer wavelets than do Haar wavelets, but at the cost of intricate algorithms based upon a complicated theory. Daubechies wavelets provide a smoother overall approximation of a function known only from its sample.

We, describe briefly examples of multiresolution approximations using Haar, Daubechie's wavelets. The orthogonal projection of an arbitrary function $f \in L^2$ is $P_j f = \sum_k \langle f, f_{jk} \rangle f_{jk}$. The basis functions f_{jk} are shifted in steps of 2^{-j} as k varies. $P_j f$ is called an approximation to f at resolution 2^{-j}.



The function f(x) = sinx and its approximations $P_0 f, P_1 f, P_2 f$ at resolutions 1,1/2,1/4, respectively based on the *one dimensional Haar wavelets.*
The following are the successive multiresolution approximations of an image using *two dimensional Daubechie's wavelets* at different resolutions each finer than the preceding.

In cubic-splines found independently by G.Battle and P.Lamerie who have done pioneering work in wavelets, the subspace V_o is the set of functions which are C^2 and equal to cubic polynomials defined on intervals [k,k+1], k€ Z.It is well-known that a unique cubic spline $g(x) \in V_0$ exists such that $g(k) = d_{ko}$.The *Fourier transform* of g(x) is given by $\hat{g}(w) = \left(\frac{\sin w/2}{w/2}\right)^4 \left(1 - \frac{2}{3}\sin^2 \frac{w}{2}\right)^{-1}$. Any functin $f€V_0$ can be decomposed in a unique way: $f(x) = \sum_{k=-\infty}^{\infty} f(k) g(x-k)$.The approximation of a function at a resolution

2^j is equal to its orthogonal projection on V_j.The additional precision of the approximation when the resolution increases from $2^j to 2^{j+1}$ is given by orthogonal projection on W_j. We next, introduce the notion of *Coifman wavelet systems*(in short Coiflets) and state the basic approximation theorem for the wavelet series.[E] The notion of Coiflets is similar to Daubechies wavelets in that, they have a maximum number of vanishing moments, but the vanishing moments are equally distributed between the scaling function and the wavelet function. For such wavelets, one has a very good approximation theory, one of the reasons this virtue of wavelets was harnessed in the first place. The fundamental result is that, if the sample values of a smooth discretized function are used as scaling function coefficients, at a finer scale, then the resulting wavelet series approximates the underlying function, with exponentially increasing accuracy as the "genus" of the Coiflets gets larger. We call this series "The Wavelet Sampling Approximation" of a given function. It differs from the usual orthogonal projection approximation, in that the coefficients are samples of a function rather than L^2-inner products of the function with the given basis elements. These are the type of wavelet-approximations used systematically in almost all applications of wavelets.
Let us discuss in detail how this works-out. Consider A = $(a, b)^/$, a wavelet matrix of rank 2 where $a = (h_k), b = (g_k)$,are the scaling and wavelet – vectors respectively. As we know, $g_k = (-1)^k h_{1-k}$ which is symmetric about k = o, the theorems connected with MRA guarantee the existence of $\{f_{jk}\} and \{y_{jk}\}$ [E].Assuming that a has a finite length,

$\exists N_1, N_2 s.t. \sum_{N_1}^{N_2} g_k = 2 \; \& \sum_{N_1}^{N_2} g_k g_{k+2l} = 2d_{ol}$.The scaling and wavelet vectors as usual satisfy

$$G(x) = \frac{1}{2}\sum h_k e^{2p ikx}, H(x) = \frac{1}{2}\sum g_k e^{2p ikx} - (3).$$

We recall further that, the m^th moment of an integrable function is defined by Mom(f) = $\int x^m f(x)dx$, whenever the integral makes sense. Evidently, for an o.n.b. system, Mom (f) = 1, Mom (y) = o where m=o. Further, $\sum f(x-k) = 1$.With this necessary background, we are in a position to define:
An orthonormal system with compact support is called Coifman wavelet system of degree N, if $f and y$ satisfy:

$$Mom(f) = 1, Mom(f) = \int x^m f(x)dx = 1, m = 1,2,--N, Mom(y) = \int x^m y(x)dx = o, m = 0,1,--N$$

To understand and appreciate the significance of Coiflets, we are led to ask: Given samples of a continuous signal equally spaced in time, is it possible to recover the signal or to put it in another way, how close the original signal is approximated from the knowledge of samples? For Coiflets, we can obtain "exponential approximation" as we see in the next theorem which serves as a fitting *finale* to our discussion.

Coifman Basic Approximation Theorem:

For an orthogonal Coifman system of degree N with the scaling function ? and the scaling vector a, if $f \in C^N(R), define for j \in \square, S^j(f)(x) = 2^{-j/2} \sum f(\frac{k}{2^j}) f_{jk}(x)$ .Then $\| f(x) - S^j(f)(x) \| \leq C 2^{-jN}$, where C depends only on f and a.

The wavelet sampling approximation is what is used in most applications of wavelets because it is the easiest approximation to compute. The significance of the above result is that, the degree of approximation is comparable to that obtained by using Daubechies wavelet system and orthogonal projection.

Conclusion:

The theories of Approximation and Wavelets have attracted and engaged the attention of mathematicians and scientists alike, for the last 40 years or so, to develop new and exciting ideas and techniques. After the advent of computers, the research in these areas has gained momentum spectacularly. Moreover, they are playing increasingly important roles in applications to many branches of applied sciences and engineering. Approximation theory has widely influenced areas of mathematics namely Special functions, Partial Differential Equations, Harmonic Analysis and Wavelet theory. Some modern applications include, Computer graphics, Signal processing, Pattern recognition and Economic forecasting. During the last decade, the theory has stretched its arms to embrace the theoretical and computational aspects of several exciting areas such as Neural-networks and Computer Aided Graphic Designs. Wavelet theory in turn touches, Image compression, Channel coding, under water communications, Fractals etc. In any country or company Research and Development by trial and error was suited to the time when technology was in its infancy. This approach is no longer adequate. The need to predict performance and consequences and to optimize design for safety, speed, accuracy, quality and cost have become key factors of science and technology in the 21st century. These factors call for new theories, ideas, conceptualizations; formulations etc. Wavelet theory has ushered in a new era holding "the master-key" (so to speak), to almost all problems of Information and Communication technologies.

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Bibliography:

[A]G.G.Lorentz : Approximation of functions –. Holt, Rinehart and Winston (1966)

[B]C.Gasquet & P.Witomski. Fourier analysis and Applications -Springer (1998)

[C] S .Mallat. Multiresolution Approximations and Wavelet Orthonormal bases of $L^2(R)$ - Trans. Amer. Math. Sc.(1989)

[D]Mhusker& Pai. Fundamentals of Approximation Theory - Narosa (2000)

[E]Resinkoff & Wells. Wavelet Analysis - Springer (1998)

[F]Charles chui. Wavelet Theory - Academic Press Cambridge (1991)