Home Assignment 3

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Problem Given a gaussian distribution for $\psi(x)$ calculate the values of $\langle \Delta x^2 \rangle$ and $\langle \Delta p^2 \rangle$ in the momentum representation.

$$\psi(x) = \frac{1}{\pi^{\frac{1}{4}}\sqrt{\sigma}} exp\left(ikx - \frac{x^2}{2\sigma^2}\right) \tag{1}$$

Solution For evaluating the required quantities in the momentum representation we need to find out $\phi(p)$ which is the wavefunction in the momentum space representation and is related to $\psi(x)$ through a Fourier transform. Taking the Fourier transform of (1) gives:

$$\phi(p) = \frac{1}{\pi^{\frac{1}{4}}}\sqrt{\frac{\sigma}{\hbar}} exp\left(-\frac{1}{2}\frac{(p-\hbar k)^2\sigma^2}{\hbar^2}\right) \tag{2}$$

Now we calculate the required quantities. We have

$$\langle x \rangle = \langle \phi | \hat{x} | \phi \rangle \tag{3}$$

And also

$$\hat{x} = i\hbar \frac{\partial}{\partial p} \tag{4}$$

Using the above two equations we get

$$\langle x \rangle = \int_{-\infty}^{\infty} dp \phi^* i\hbar \frac{\partial \phi}{\partial p} \tag{5}$$

As we get an odd function in $p - \hbar k$ on differentiating $\phi(p)$ and shifting origin by $\hbar k$ we have

$$\langle x \rangle = 0 \tag{6}$$

Now we calculate $\langle x^2 \rangle$.

$$\langle x^2 \rangle = -\int_{-\infty}^{\infty} dp\phi^* \hbar^2 \frac{\partial^2 \phi}{\partial (p - \hbar k)^2} \tag{7}$$

which gives

$$\langle x^2 \rangle = -\frac{\hbar^2}{\pi^{\frac{1}{2}}} \int_{-\infty}^{\infty} dp \frac{\sigma}{\hbar} exp\left(-\frac{(p-\hbar k)^2 \sigma^2}{\hbar^2}\right)\left(\frac{(p-\hbar k)^2 \sigma^4}{\hbar^4} - \frac{\sigma^2}{\hbar^2}\right) \tag{8}$$

which gives on integrating using standard formulae

$$\langle x^2 \rangle = -\frac{\sigma^2}{2} + \sigma^2 = \frac{\sigma^2}{2} \tag{9}$$

$$\langle \Delta x^2 \rangle = \langle x^2 \rangle - \langle x \rangle^2 = \frac{\sigma^2}{2} \tag{10}$$

Now we evaluate $\langle p \rangle$

$$\langle p \rangle = -\int_{-\infty}^{\infty} dp\phi^* p\phi \tag{11}$$

$$\langle p \rangle = -\int_{-\infty}^{\infty} dp \frac{\sigma}{\hbar\sqrt{\pi}} p exp\left(-\frac{(p-\hbar k)^2 \sigma^2}{\hbar^2}\right) \tag{12}$$

Again shifting origin to $\hbar k$ we get the p term making the integral odd while the $\hbar k$ term contributing to the integral thus giving

$$\langle p \rangle = \hbar k \tag{13}$$

$$\langle p^2 \rangle = -\int_{-\infty}^{\infty} dp \frac{\sigma}{\hbar\sqrt{\pi}} p^2 exp\left(-\frac{(p-\hbar k)^2 \sigma^2}{\hbar^2}\right) \tag{14}$$

Again shifting the origin by $\hbar k$ we will have both the square terms in shifted p^2 contributing and the middle one being an odd function will yield zero thus giving (first term from p^2 next from $\hbar^2 k^2$ in $(p + \hbar k)^2$).

$$\langle p^2 \rangle = \frac{\hbar^2}{2\sigma^2} + \hbar^2 k^2 \tag{15}$$

Thus giving

$$\langle \Delta p^2 \rangle = \langle p^2 \rangle - \langle p \rangle^2 = \frac{\hbar^2}{2\sigma^2} \tag{16}$$

And this tells us that we have

$$\langle \Delta x^2 \rangle \langle \Delta p^2 \rangle = \frac{\hbar^2}{4} \tag{17}$$

the uncertainty principle...

Standard Formulae used in the above derivations are:

$$\phi(p) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} exp\left(-i\frac{px}{\hbar}\right) \langle x|\psi\rangle dx \tag{18}$$

And the standard integrals.

$$\int_{-\infty}^{\infty} exp\left(-\alpha x^2\right) dx = \sqrt{\frac{\pi}{\alpha}} \tag{19}$$

$$\int_{-\infty}^{\infty} x^2 exp\left(-\alpha x^2\right) dx = \frac{1}{2}\frac{\sqrt{\pi}}{\alpha^{\frac{3}{2}}} \tag{20}$$