Home Assignment 4

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Problem An operator $\hat{A}$ has two degenerate eigenstates with an eigenvalue a. If $\hat{B}$ is an operator such that $[\hat{A},\hat{B}]=0$. Are the two eigenstates of $\hat{A}$ simultaneous eigenstates of $\hat{B}$? If yes prove it else how would you construct the eigenstates of $\hat{B}$?

Solution First let us try to understand what do we mean by having degenerate states. Given $\hat{A}$, it has degenerate eigenstates if their exist linearly independent orthogonal states which satisfy the eigenequation with the same eigenvalue. If this is the case then it is true that the same eigen equation is satisfied by any state in the space spanned by the linearly independent states. Now comes the question whether these states represent the same states or different states.

The crux is if we have $\hat{B}$ such that $[\hat{A},\hat{B}]=0$ and $\hat{B}$ has nondegenerate eigenstates in the space spanned by degenerate eigenkets of $\hat{A}$ then because we can have the nondegenerate eigen kets of $\hat{B}$ span the space, and these also being the eigenkets for $\hat{A}$ we have that $\hat{A}$ and $\hat{B}$ have simultaneous eigen kets given by the nondegenerate eigen kets of $\hat{B}$ and each of which represents a different physical state. If we have that $\hat{B}$ also has degeneracy in its eigenstates, then we first find the nondegenerate eigenkets in $\hat{B}$, resolve the conflict in a subset of space spanned by the $\hat{A}$ eigenkets. Then try and hunt for another $\hat{C}$ which will help out in resolving the conflict.

In what follows next I demonstrate the thing for the problem.

Let the eigenplane corresponding to $\hat{A}$ be spanned by two eigenkets $|a_1\rangle$ and $|a_2\rangle$ with eigenvalue a. Hence we have

$$\hat{A}|a_1\rangle = a|a_1\rangle \tag{1}$$

$$\hat{A}|a_2\rangle = a|a_2\rangle \tag{2}$$

And note that these two eigenket that we chose are not unique that is we can have many other eigenkets spanning the eigenplane. Now as we have$[\hat{A},\hat{B}]$=0

$$\hat{A}\hat{B}|a_1\rangle = \hat{B}\hat{A}|a_1\rangle = a\hat{B}|a_1\rangle \tag{3}$$

and

$$\hat{A}\hat{B}|a_2\rangle = \hat{B}\hat{A}|a_2\rangle = a\hat{B}|a_2\rangle \tag{4}$$

which implies that $\hat{B}$ acting on any of the eigen kets $|a_1\rangle$ and $|a_2\rangle$ yields a vector in the eigenplane. This in turns means that $\hat{B}$ has the form

$$\hat{B} = \begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix} \tag{5}$$

Note here that due to the above form in the eigenplane of $\hat{A}$, $\hat{B}$ has to have eigenvectors in the plane if the operator $\hat{B}$ has a form which doesnot lead to collapse of a dimension(i.e. the algebraic multiplicity is the same as geometric multiplicity.)

Diagonalize this matrix to get the eigenvalues of $\hat{B}$. Now two cases arise either $\hat{B}$ has different eigenvalues or the same eigenvalues. If $\hat{B}$ has different eigenvalues then the corresponding eigenkets fall in the space spanned by $|a_1\rangle$ and $|a_2\rangle$. But we can call $|b_1\rangle$ and $|b_2\rangle$ as simultaneous eigenkets of both $\hat{A}$ and $\hat{B}$. And they represent two different physical states. If however we have again a degeneracy that is the eigenplane spanned by $|a_1\rangle$ and $|a_2\rangle$ is also an eigenplane for $\hat{B}$ then we again have simultaneous eigenkets for both $\hat{A}$ and $\hat{B}$ given by $|a_1\rangle$ and $|a_2\rangle$. Now whether they define two different physical states is a matter of whether another $\hat{C}$ satisfying commutator relationship with $\hat{A}$ and $\hat{B}$ exists having nondegenerate eigenstates in the eigenplane of $\hat{A}$ and $\hat{B}$.