Problem Statement Consider a system whose Hamiltonian $\hat{H}$ and an operator in its ket space are given by the following matrices:

$$\hat{H} = \varepsilon_0 \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{1}$$

and

$$\hat{A} = a \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} \tag{2}$$

where ε_0 has dimensions of energy. The system is initially prepard in a state given by $|\psi\rangle$,

$$|\psi\rangle = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \tag{3}$$

a) What is the result of a particular measurement and with what probability? What is the expectation value of the Hamiltonian in this state?
b) If as a consequence of measurement mentioned in a) the system is left in a state with energy -ε_0, what are the possible results of measurement of the observable A immediately thereafter on the same system? Calculate the probabilities of occurence of each possible result and calculate the uncertainty Δ A.

Solution: The Hamiltonian leads to eigenvalues 0, -1 and 2. The eigen vector corresponding to the eigenvalues are

$$|\psi\rangle_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \tag{4}$$

$$|\psi\rangle_{-1} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{5}$$

$$|\psi\rangle_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \tag{6}$$

We have

$$|\psi\rangle = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{6}} \left[-\frac{1}{\sqrt{2}} |\psi\rangle_2 + \frac{3}{\sqrt{2}} |\psi\rangle_0 + |\psi\rangle_{-1} \right] \tag{7}$$

The probabilities hence being given by

$$P_2 = \frac{1}{12} \quad P_0 = \frac{9}{12} \quad P_{-1} = \frac{1}{6} \tag{8}$$

The expectaion value of the Hamiltonian is

$$\langle \hat{H} \rangle = P_0 * 0 + P_{-1} * -\varepsilon_0 + P_2 * 2\varepsilon_0 = 0 \tag{9}$$

If the measurement done here collapses the system into state $|\psi\rangle_0$, the possible results of measurement of A are the eigenvalues of A which are 0 or $a\sqrt{2}$ or $-a\sqrt{2}$.The corresponding eigenvectors are

$$|\psi_A\rangle_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \tag{10}$$

$$|\psi_A\rangle_{\sqrt{2}} = \frac{1}{2} \begin{pmatrix} 1 \\ \sqrt{2} \\ -1 \end{pmatrix} \tag{11}$$

$$|\psi_A\rangle_{-\sqrt{2}} = \frac{1}{2} \begin{pmatrix} 1 \\ -\sqrt{2} \\ -1 \end{pmatrix} \tag{12}$$

Now we note that we have

$$|\psi\rangle_0 = \frac{1}{2}|\psi_A\rangle_0 + (\frac{1}{2} + \frac{1}{2\sqrt{2}})|\psi_A\rangle_{\sqrt{2}} + (\frac{1}{2} - \frac{1}{2\sqrt{2}})|\psi_A\rangle_{-\sqrt{2}} \tag{13}$$

The corresponding probabilities given by

$$P_{A0} = \frac{1}{4} \quad P_{A\sqrt{2}} = \frac{3}{8} + \frac{1}{2\sqrt{2}} \quad P_{A-\sqrt{2}} = \frac{3}{8} - \frac{1}{2\sqrt{2}} \tag{14}$$

The expectation value of A is hence given by

$$\langle \hat{A} \rangle = P_A 0 * 0 + P_{A\sqrt{2}} * a\sqrt{2} + P_{A-\sqrt{2}} * -a\sqrt{2} = a \tag{15}$$

So the uncertainty matrix $\Delta\hat{A}$ is given by

$$\Delta\hat{A} = \hat{A} - \langle \hat{A} \rangle \tag{16}$$

$$\Delta\hat{A} = a \begin{pmatrix} -1 & 1 & 0 \\ 1 & -1 & -1 \\ 0 & -1 & -1 \end{pmatrix} \tag{17}$$