New General Analytic and Numerical Methods in Constrained Optimization with Applications to Optimal Consumption

Dean Banerjee* and John Gregory
Department of Mathematics
Southern Illinois University
Carbondale, IL
USA 62901- 4408

ABSTRACT New analytic methods are given to solve general economic models which occur as constrained optimization problems in the calculus of variations or optimal control theory. These methods allow us to solve problems including equality and inequality differential and integral constraints with a complete variety of endpoint conditions. They also allow us to determine the multipliers or shadow variables. For economic problems which do not yield closed formed solutions, general, accurate, and efficient numerical methods, which go with these analytic methods, given by the first author are available. These methods have pointwise, maximum, numerical errors proportional to h^2, where h is the discretization step size.

These new methods are applied to a general optimal consumption model and then used to solve this model. A specific example of this model is considered.

1 Introduction

Constrained problems in the calculus of variations and optimal control have had a long and distinguished history in the theory of economic modeling (for example, see [5]). Recently, the second author has given new analytical and numerical techniques to solve these problems. The main purpose of this paper is to consider a general class of optimal consumption problems and solve them with these techniques. An important secondary purpose is to present these techniques which can be used to solve a wide variety of economic problems.

In Section 2, we briefly review the major ideas for the techniques. In Section 3, we define a general optimal consumption problem and give a formal solution. In Section 4 we consider a specific example problem and give an explicit solution.

2 Reformulation Techniques

The purpose of this section is to give a method which will allow one to solve general constrained optimization problems by reformulating them as equivalent unconstrained calculus of variations problems. The unconstrained problems are them solved by the first necessary condition of the calculus of variations (the first variation is zero), which implies the Euler - Lagrange equations, the corner conditions, and the transversality conditions[2].

The basic general constrained problem is: Among all arcs

$$y_i(x) \quad (i = 1,...n; \quad x_1 \le x \le x_2) \tag{1}$$

satisfying differential equations and end-conditions of the form

$$\phi_\beta(x,y,y') = 0 \quad (\beta=1,...,m < n) \tag{2}$$

$$\psi_\beta(x_1 ,y(x_1),x_2, y(x_2)] = 0 \quad (\mu = 1,...,p < 2b+2) \tag{3}$$

find the arc which minimizes

$$J = g[x_1 ,y(x_1),x_2 , y(x_2)] + \int_{x}^{x_2} f(x, y, y')dx \tag{4}$$

The solution to this problem is found in [4] and is called the *multiplier rule* for the problem of Bolza.

Our unconstrained reformulation of this problem is as follows:

$$\text{Let } y(x) = (y, ..., y_n) \tag{5}$$

$$I[Y] = g[x_1, y(x_1), x_2, y(x_2)] + \int_{x_1}^{x_2} [\lambda_0 f(x,y,y') + z_\beta'(x)\phi_\beta(x,y,y')]dx$$

$$= \boldsymbol{g} + \int_{x_1}^{x_2} \boldsymbol{F(x,Y,Y')dx},$$

$$\psi_\mu[x_1, y(x_1), x_2, y(x_2)] = 0 \quad (\mu = 1,...,p < 2n+2) \qquad (6a)$$

$$\psi_{\beta+p}[x_1, z(x_1), x_2, z(x_2)] \cong z_\beta(x_1) = 0 \quad (\beta = 1,2,...,m) \qquad (6b)$$

$$Y^T = (y_1, \dots, y_n, z_1, \dots, z_n) \qquad (6c)$$

Note that we have used F in two different settings. It will be obvious by the context which one is being used. We now define Y to be a critical point of (5) if I'(Y,W) = 0 for all variations W compatible with (6). The next result is immediate.

Theorem A critical point solution to problem (6) - (7) satisfies the Euler - Langrange equations

$$\frac{d}{dt} F_{Y'} = F_Y$$

(7)

between corners of the solution, the transversality conditions

$$\begin{aligned} &[(F - Y_j' F_{Y_j'})dx + F_{Y_j'} dY_j]_1^2 + \lambda_0 dg + e_\mu d\psi_\mu \equiv 0 \\ &\psi_\mu = 0 \qquad \mu = 1,\dots,p,p+1,\dots,p+m \end{aligned} \qquad (8)$$

at the endpoints of Y for every choice of the differentials dx_1, dY_{j1}, dx_2, and dY_{j2} and the corner condition. Note that $F_{Y'}$ is continuous at the corners of Y(x).

Theorem A necessary condition for y(x) to solve (1) - (4) is that there exist Y(x) which is a solution to (7) - (8) where the first components of Y(x) is equal to y(x).

Theorems as well as their proofs are contained in [4]. The Euler - Lagrange equations (7), the corner conditions, and the transversality conditions (8) are enough to find a solution if one exists.

Reference [4] also contains the results when inequality constraints of the form $\phi_\alpha \leq 0$ α are included. General optimal control problems are solved by similar methods and a complete discussion is in [4]. Finally, [4] contains complete numerical methods for both the constrained calculus of variations or optimal control.

3 A General Optimal Consumption Problem

In this section, a general class of optimal consumption problems is considered. This problem involves expending some resource over a fixed period of time in an optimal way. It is assumed that there is diminishing utility for each unit of the resource spent. If this was the only factor involved, then this would dictate that the resource should spent in a constant fashion. However, there is also constraints in the problem which can alter the optimal solution. A formal statement of the problem is as follows:

$$\text{maximize} \quad \int_0^L f(s,\dot{s},t)dt$$

$$\text{subject to} \quad \int_0^L g(s,\dot{s},t)dt = 0$$

Here the first equation represents the total utility from spending the resource over time and the second equation represents a general budget constraint. The method presented in section two will now be applied to the problem. First, the integral constraint is replaced by a differential constraint and the problem is reformulated.

$$\text{maximize} \quad \int_0^L f(s,\dot{s},t)dt$$

$$\text{subject to} \quad \dot{w}(t) = g(s,\dot{s},t)$$

$$w(L) = 0$$

$$w(0)=0$$

$$H = f(s, \dot{s}, t) + \dot{\lambda}\,[\dot{w}(t) - g(s, \dot{s}, t)]$$

$$\frac{d}{dt}\begin{bmatrix} f_{\dot{s}}(s, \dot{s}, t) - \dot{\lambda}\, g_{\dot{s}}(s, \dot{s}, t) \\ \dot{\lambda} \\ \dot{w}(t) - g_{\dot{s}}(s, \dot{s}, t) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The transversality conditions can be stated as follows

$$\left.\begin{bmatrix} f_{\dot{s}}(s, \dot{s}, t) + \dot{\lambda}\,[\dot{w}(t) - g_{\dot{s}}(s, \dot{s}, t)]\dot{\lambda} \\ \dot{w}(t) - g(s, \dot{s}, t) \end{bmatrix}\right|_{t=L} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The only fixed condition which is necessary is

$$w(L) = 0$$

Note that the transversality equation does not apply to $\dot{\lambda}(L)$ since w(L) is fixed. This is a set of differential equations with sufficient boundary conditions to determine a unique solution.

4 A Specific Example

For this example, the resource to be considered will be an individual person's money. This money is to be spent over a lifetime of known length, L, in an optimal way.

It will be assumed that if the spending rate is $\dot{s}(t)$, then the instantaneous utility gained is of the form $k\dot{s}^{\alpha}(t)$. For any realistic model, $0 < \alpha < 1$. The value of k will be assumed to be positive and therefore does not have any bearing on the solution. The constant α can be determined from a power regression which is done on the data of the specific individual in question.

It will be assumed that the interest rate is *i*. In this case, if one has one dollar at time t, then the future value of this dollar at time L is $e^{\beta(L-t)}$. Here, the constant β is Ln(1+*i*).

The budget constraint in this problem is simply the statement that the future value of all money spent over the lifetime of length L years is equal to the future value at time L of all money earned. $e^{-\beta L} E$ will be used to represent this number. It therefore follows that

$$Ee^{-\beta L} = \int_0^L [e(t) - h(t)]e^{\beta(L-t)}dt$$

Here e(t) represents the individual's earned income and h(t) represents the money given away for such reasons such as inheritance, donations, and taxes.

The statement of the problem is as follows:

$$\text{maximize} \int_0^L \dot{s}^{\alpha}(t)dt$$

$$\text{subject to} \int_0^L [\dot{s}(t)e^{-\beta t} - \frac{E}{L}]dt = 0$$

Applying the variational methods discussed in section 3, the following equations are obtained:

$$\frac{d}{dt}\begin{bmatrix} \alpha\dot{s}^{\alpha-1}(t) - \dot{\lambda}e^{-\beta t} \\ \dot{\lambda} \\ \dot{w}(t) - \dot{s}(t)e^{-\beta t} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The transversality conditions can be stated as follows

$$\left.\begin{matrix} \alpha\dot{s}^{\alpha-1}(t) - \dot{\lambda}e^{-\beta t} \\ \dot{w}(t) - \dot{s}(t)e^{-\beta t} \end{matrix}\right|_{t=L} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The only fixed condition which is necessary is

$$w(L) = 0$$

This can be expressed as an algebraic expression in $\dot{s}(t)$ and is easily solved with all of the necessary conditions. The resulting solution is

$$\dot{s}(t) = \frac{E\beta}{(1-\alpha)[\ \exp(\frac{\beta L}{1-\alpha}) - 1\ ]} \exp(\frac{\beta}{1-\alpha} t)$$

This closed form solution is an exponential curve of steepness determined by the parameters α and β. It has the property that if the interest rate is 0, then it is a constant and in the limit as $\alpha \rightarrow 1$, it becomes a delta function concentrated at time L. These properties are what would be expected.

References

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