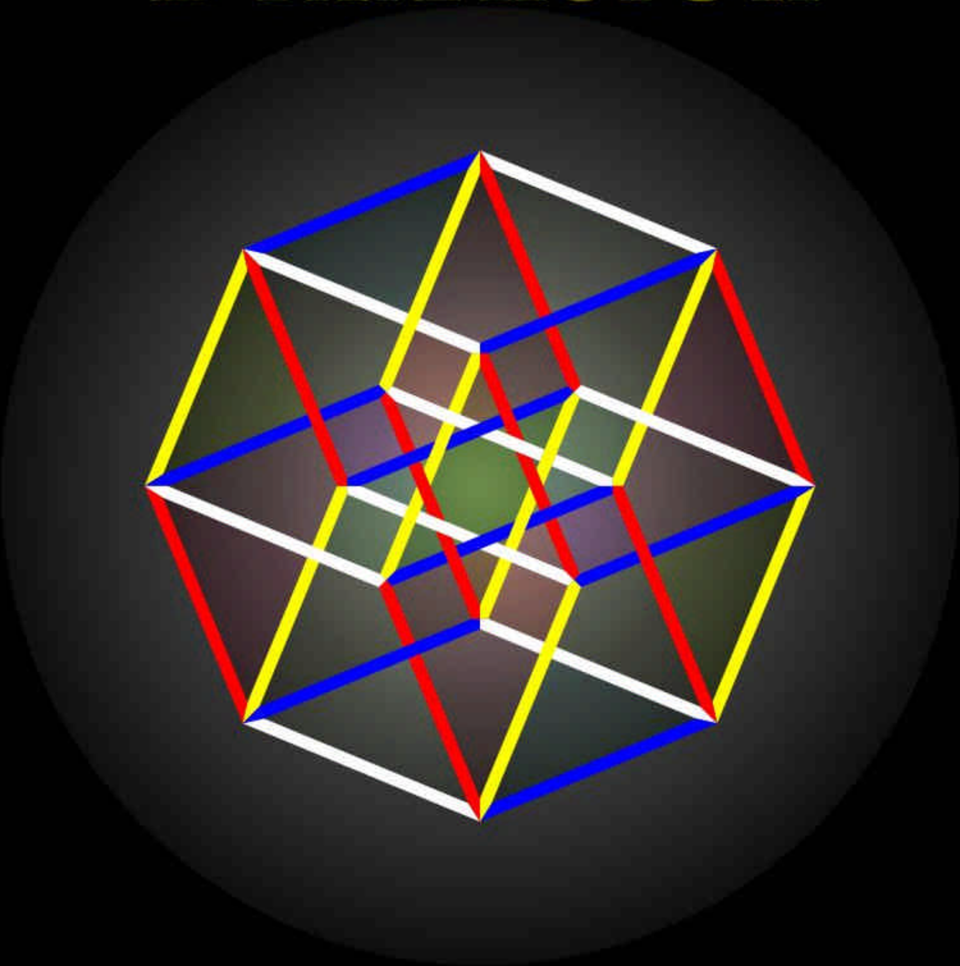
The Fourth Dimension



C.H. Hinton

THE FOURTH DIMENSION

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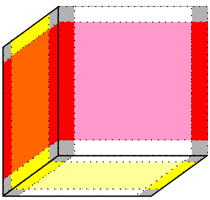

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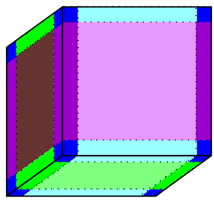

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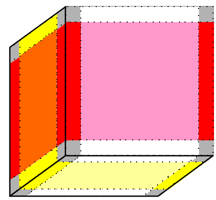

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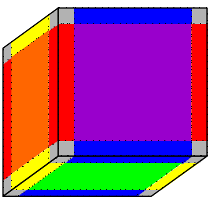

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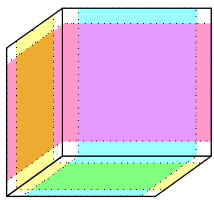

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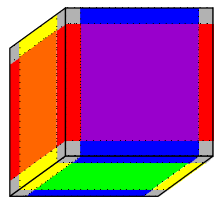

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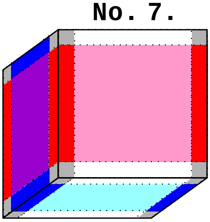

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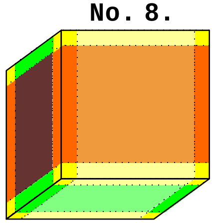

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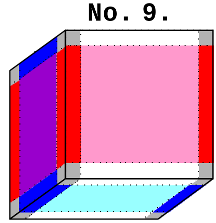

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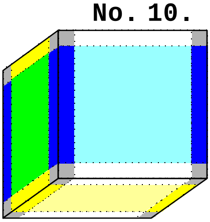

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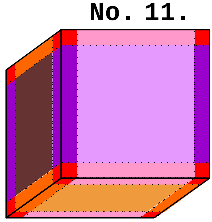

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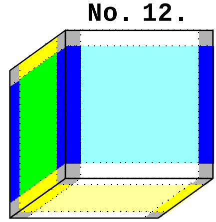

THE

FOURTH DIMENSION

BY

C. HOWARD HINTON

TO WHICH IS ADDED

A LANGUAGE OF SPACE

Celephaïs Press
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PREFACE

I HAVE endeavoured to present the subject of the higher dimensionality of space in a clear manner, devoid of mathematical subtleties and technicalities. In order to engage the interest of the reader, I have in the earlier chapters dwelt on the perspective the hypothesis of a fourth dimension opens, and have treated of the many connections there are between this hypothesis and the ordinary topics of our thoughts.

A lack of mathematical knowledge will prove of no disadvantage to the reader, for I have used no mathematical processes of reasoning. I have taken the view that the space which we ordinarily think of, the space of real things (which I would call permeable matter), is different from the space treated of by mathematics. Mathematics will tell us a great deal about space, just as the atomic theory will tell us a great deal about the chemical combinations of bodies. But after all, a theory is not precisely equivalent to the subject with regard to which it is held. There is an opening, therefore, from the side of our ordinary space perceptions for a simple, altogether rational, mechanical, and observational way of treating this subject of higher space, and of this opportunity I have availed myself.

The details introduced in the earlier chapters, especially in Chapters VIII., IX., X., may perhaps be found wearisome. They are of no essential importance in the main line of argument, and if left till Chapters XI. and XII. have been read, will be found to afford interesting and obvious illustrations of the properties discussed in the later chapters.

My thanks are due to the friends who have assisted me in designing and preparing the modifications of my previous models, and in no small degree to the publisher of this volume, Mr. Sonnenschein, to whose unique appreciation of the line of thought of this, as of my former essays, their publication is owing. By the provision of a coloured plate, in addition to the other illustrations, he has added greatly to the convenience of the reader.

C. HOWARD HINTON.

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THE FOURTH DIMENSION

CHAPTER I

FOUR-DIMENSIONAL SPACE

THERE is nothing more indefinite, and at the same time more real, than that which we indicate when we speak of the "higher." In our social life we see it evidenced in a greater complexity of relations. But this complexity is not all. There is, at the same time, a contact with, an apprehension of, something more fundamental, more real.

With the greater development of man there comes a consciousness of something more than all the forms in which it shows itself. There is a readiness to give up all the visible and tangible for the sake of those principles and values of which the visible and tangible are the representations. The physical life of civilised man and of a mere savage are practically the same, but the civilised man has discovered a depth in his existence, which makes him feel that that which appears all to the savage is a mere externality and appurtenage to his true being.

Now, this higher—how shall we apprehend it? It is generally embraced by our religious faculties, by our idealising tendency. But the higher existence has two sides. It has a being as well as qualities. And in trying to realise it through our emotions we are always taking the subjective view. Our attention is always fixed on what we feel, what we think. Is there any way of apprehending the higher after the purely objective method of a natural science? I think that there is.

Plato, in a wonderful allegory, speaks of some men living in such a condition that they were practically reduced to be the denizens of a shadow world. They were chained, and perceived but the shadows of themselves and all real objects projected on a wall, towards which their faces were turned. All movements to them were but movements on the surface, all shapes by the shapes of outlines with no substantiality.

Plato uses this illustration to portray the relation between true being and the illusions of the sense world. He says that just as a man liberated from his chains could learn and discover that this world was solid and real, and could go back and tell his bound companions of this greater higher reality, so the philosopher who has been liberated, who has gone into the thought of the ideal world, into the world of ideas greater and more real than the things of sense, can come and tell his fellow men of that which is more true than the visible sun—more noble than Athens, the visible state.

Now, I take Plato's suggestion; but literally, not metaphorically. He imagines a world which is lower than this world, in that shadow figures and shadow motions are its constituents; and to it he contrasts the real world. As the real world is to this shadow world, so is the higher world to our world. I accept his analogy. As our world in three dimensions is to a shadow or plane world, so is the higher world to our three-dimensional world. That is, the higher world is four-dimensional; the higher being is, so far as its existence is concerned apart from its qualities, to be sought through the conception of an actual existence spatially higher than that which we realise with our senses.

Here you will observe I necessarily leave out all that gives its charm and interest to Plato's writings. All those conceptions of the beautiful and good which live immortally in his pages.

All that I keep from his great storehouse of wealth is this one thing simply—a world spatially higher than this world, a world which can only be approached through the stocks and stones of it, a world which must be apprehended laboriously, patiently, through the material things of it, the shapes, the movements, the figures of it.

We must learn to realise the shapes of objects in this world of the higher man; we must become familiar with the movements that objects make in his world, so that we can learn something about his daily experience, his thoughts of material objects, his machinery.

The means for the prosecution of this enquiry are given in the conception of space itself.

It often happens that that which we consider to be unique and unrelated gives us, within itself, those relations by means of which we are able to see it as related to others, determining and determined by them.

Thus, on the earth is given that phenomenon of weight by means of which Newton brought the earth into its true relation to the sun and other planets. Our terrestrial globe was determined in regard to other bodies of the solar system by means of a relation which subsisted on the earth itself.

And so space itself bears within it relations of which we can determine it as related to other space. For within space are given the conceptions of point and line, line and plane, which really involve the relation of space to a higher space.

Where one segment of a straight line leaves off and another begins is a point, and the straight line itself can be generated by the motion of the point.

One portion of a plane is bounded from another by a straight line, and the plane itself can be generated by the straight line moving in a direction not contained in itself.

Again, two portions of solid space are limited with regard to each other by a plane; and the plane, moving in a direction not contained in itself, can generate solid space.

Thus, going on, we might say that space is that which limits two portions of higher space from each other, and that our space will generate the higher space by moving in a direction not contained in itself.

Another indication of the nature of four-dimensional space can be gained by considering the problem of the arrangement of objects.

If I have a number of swords of varying degrees of brightness, I can represent them in respect of this quality by points arranged along a straight line.

If I place a sword at A, fig. 1., and regard it as having a certain brightness, then the other swords can be arranged in a series along the line, as at A, B, C, etc., according to their degree of brightness.

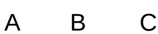


Fig. 1.

If I now take account of another quality, say length, they can be arranged in a plane. Starting from A, B, C, I can find points to represent different degrees of length along such lines as AF, BD, CE, drawn from A and B and C. Points on these lines represent different degrees of length with the same degree of brightness. Thus the whole plane is occupied by points representing all conceivable varieties of brightness and length.

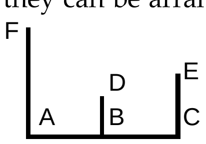


Fig. 2.

Bringing in a third quantity, say sharpness, I can draw, as in fig. 3, any number of upright lines. Let distances along these upright lines represent degrees of sharpness, thus the points F and G will represent swords of certain definite degrees of the three qualities mentioned, and the whole of space will serve to represent all conceivable degrees of these three qualities.

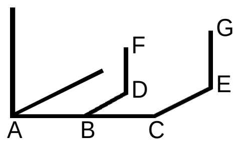


Fig. 3.

If I now bring in a fourth quality, such as weight, and try to find a means of representing it as I did the other three qualities, I find a difficulty. Every point in space is taken up by some conceivable combination of the three qualities already taken.

To represent four qualities in the same way as that in which I have represented three, I should need another dimension of space.

Thus we may indicate the nature of four-dimensional space by saying that it is a kind of space which would give positions representative of four qualities, as three-dimensional space gives positions representative of three qualities.

CHAPTER II

THE ANALOGY OF A PLANE WORLD

AT the risk of some prolixity I will go fully into the experience of a hypothetical creature confined to motion on a plane surface. By so doing I shall obtain an analogy which will serve in our subsequent enquiries, because the change in our conception, which we make in passing from the shapes and motions in two dimensions to those in three, affords a pattern by which we can pass on still further to the conception of an existence in four-dimensional space.

A piece of paper on a smooth table affords a ready image of a two-dimensional existence. If we suppose the being represented by the piece of paper to have no knowledge of the thickness by which he projects above the surface of the table, it is obvious that he can have no knowledge of objects of a similar description, except by the contact with their edges. His body and the objects in his world have a thickness of which however he has no consciousness. Since the direction stretching up from the table is unknown to him he will think of the objects of his world as extending in two dimensions only. Figures are to him completely bounded by their lines, just as solid objects are to us by their surfaces. He cannot conceive of approaching the centre of a circle, except by breaking through the circumference, for the circumference encloses the centre in the direction in which motion is possible to him. The plane surface over which he slips and with which he is always in contact will be unknown to him; there are no differences by which he can recognise its existence.

But for the purpose of our analogy this representation is deficient.

A being has thus described has nothing about him to push off from, the surface over which he slips affords no means by which he can move in one direction rather than another. Placed on a surface over which he slips freely, he is in a condition analogous to that in which we should be if we were suspended in space. There is nothing which he can push off from in any direction known to him.

Let us therefore modify our representation. Let us suppose a vertical plane against which particles of this matter slip, never leaving the surface. Let these particles possess an attractive force and cohere together into a disk; this disk will represent the globe of a plane being. He must be conceived as existing on the rim.

Let 1 represent this vertical disk of flat matter and 2 the plane being on it, standing upon its rim as we stand on the surface of our earth. The direction of the attractive force of his matter will give the creature a knowledge of up and down, determining for him one direction in plane space. Also, since he can move along the surface of his earth, he will have the sense of a direction parallel to its surface, which we may call forwards and backwards.

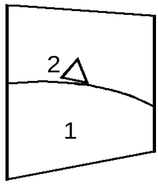


Fig. 4.

He will have no sense of right and left—that is, of the direction which we recognise and extending out from the plane to our right and left.

The distinction of right and left is the one that we must suppose to be absent, in order to project ourselves into the condition of a plane being.

Let the reader imagine himself, as he looks along the plane, fig. 4, to become more and more identified with the thin body on it, till he finally looks along parallel to the surface of the plane earth, and up and down, losing the sense of the direction which stretches right and left. This direction will be an unknown dimension to him.

Our space conceptions are so intimately connected with those which we derive from the existence of gravitation that it is difficult to realise the condition of a plane being, without picturing him as in material surroundings with a definite direction of up and down. Hence the necessity of our somewhat elaborate scheme of representation, which, when its import has been grasped, can be dispensed with for the simpler one of a thin object slipping over a smooth surface, which lies in front of us.

It is obvious that we must suppose some means by which the plane being is kept in contact with the surface on which he slips. The simplest supposition to make is that there is a transverse gravity, which keeps him to the plane. This gravity must be thought of as different to the attraction exercised by his matter, and as unperceived by him.

At this stage of our enquiry I do not wish to enter into the question of how a plane being could arrive at a knowledge of the third dimension, but simply to investigate his plane consciousness.

It is obvious that the existence of a plane being must be very limited. A straight line standing up from the surface of his earth affords a bar to his progress. An object like a wheel which rotates round an axis would be unknown to him, for there is no conceivable way to which he can get to the centre without going through the circumference. He would have spinning disks, but could not get to the centre of them. The plane being can represent the motion from any one point of his space to any other, by means of two straight lines drawn at right angles to each other.

Let AX and AY be two such axes. He can accomplish the translation from A to B by going along AX to C, and then from C along CB parallel to AY.

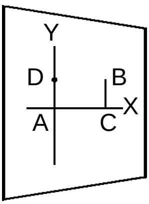


Fig. 5.

The same result can of course be obtained by moving to D along AY and then parallel to AX from D to B, or of course by and diagonal movement compounded by these axial movements.

By means of movements parallel to these two axes he can proceed (except for material obstacles) from any one point of his space to any other.

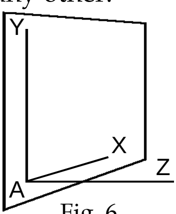


Fig. 6

If now we suppose a third line drawn out from A at right angles to the plane it is evident that no motion in either of the two dimensions he knows will carry him in the least degree in the direction represented by A Z.

The lines AZ and AX determine a plane. If he could be taken off his plane, and transferred to the plane AXZ, he would be in a world exactly like his own. From every line in his world there goes off a space world exactly like his own.

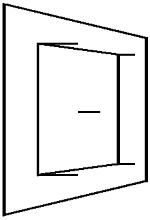


Fig. 7

From every point in his world a line can be drawn parallel to AZ in the direction unknown to him. If we suppose the square in fig. 7 to be a geometrical square from every point of it, inside as well as on the contour, a straight line can be drawn parallel to AZ. The assemblage of these lines constitute a solid figure, of which the square in the plane is the base. If we consider the square to represent an object in the plane being's world then we must attribute to it a very small thickness, for every real thing must possess all three dimensions. This thickness he does not perceive, but thinks of this real object as a geometrical square. He thinks of it as possessing area only, and no degree of solidity. The edges which project from the plane to a very small extent he thinks of as having merely length and no breadth—as being, in fact, geometrical lines.

With the first step in the apprehension of a third dimension there would come to a plane being the conviction that he had previously formed a wrong conception of the nature of his material objects. He had conceived them as geometrical figures of two dimensions only. If a third dimension exists, such figures are incapable of real existence. Thus he would admit that all his real objects had a certain, though very small thickness in the unknown dimension, and that the conditions of his existence demanded the supposition of an extended sheet of matter, from contact with which in their motion his objects never diverge.

Analogous conceptions must be formed by us on the supposition of a four-dimensional existence. We must suppose a direction in which we can never point extending from every point of our space. We must draw a distinction between a geometrical cube and a cube of real matter. The cube of real matter we must suppose to have an extension in an unknown direction, real, but so small as to be imperceptible by us. From every point of a cube, interior as well as exterior, we must imagine that it is possible to draw a line in the unknown direction. The assemblage of these lines would constitute a higher solid. The lines going off in the unknown direction from the face of a cube would constitute a cube starting from that face. Of this cube all that we should see in our space would be the face.

Again, just as the plane being can represent any motion in his space by two axes, so we can represent any motion in our three-dimensional space by means of three axes. There is no point in our space to which we cannot move by some combination of movements on the directions marked out by these axes.

On the assumption of a fourth dimension we have to suppose a fourth axis, which we will call AW. It must be supposed to be at right angles to each and every one of the three axes AX, AY, AW. Just as the two axes, AX, AZ, determine a plane which is similar to the original plane on which we supposed the plane being to exist, but which runs off from it, and only meets it in a line; so in our space if we take any three axes such as AX, AY, and AW, they determine a space like our space world. This space runs off from our space, and if we were transferred to it we should find ourselves in a space exactly similar to our own.

We must give up any attempt to picture this space in its relation to ours, just as a plane being would have to give up any attempt to picture a plane at right angles to his plane.

Such a space and ours run in different directions from the plane of AX and AY. They meet in this plane but have nothing else in common, just as the plane space of AX and AY and that of AX and AZ run in different directions and have but the line AX in common.

Omitting all discussion of the manner on which a plane being might be conceived to form a theory of a three-dimensional existence, let us imagine how, with the means at his disposal, he could represent the properties of three-dimensional space.

There are two ways in which the plane being can think of one of our solid bodies. He can think of the cube, fig. 8, as composed of a number of sections parallel to his plane, each lying in the third dimension a little further off from his plane than the preceding one. These sections he can represent as a series of plane figures lying in his plane, but in so representing them he destroys the coherence of them in the higher figure. The set of squares, A, B, C, D, represents the sections parallel to the plane of the cube shown in figure, but they are not in their proper relative positions.

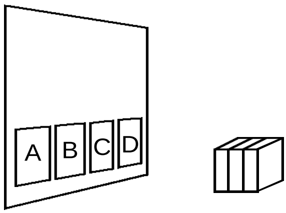


Fig. 8

The plane being can trace out a movement in the third dimension by assuming discontinuous leaps from one section to another. Thus, a motion along the edge of the cube from left to right would be represented in the set of sections in the plane as the succession of the corners of the sections A, B, C, D. A point moving from A through BCD in our space must be represented in the plane as appearing in A, then in B, and so on, without passing through the intervening plane space.

In these sections the plane being leaves out, of course, the extension in the third dimension; the distance between any two sections is not represented. In order to realise this distance the conception of motion can be employed.

Let fig. 9 represent a cube passing transverse to the plane. It will appear to the plane being as a square object, but the matter of which this object is composed will be continually altering. One material particle takes the place of another, but it does not come from anywhere or go anywhere in the space which the plane being knows.

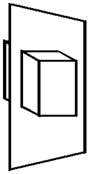


Fig 9.

The analogous manner of representing a higher solid in our case, is to conceive it as composed of a number of sections, each lying a little further off in the unknown direction than the preceding.

We can represent these sections as a number of solids.

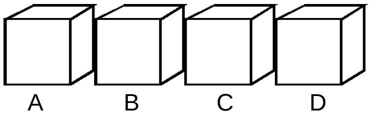


Fig. 10.

Thus the cubes A, B, C, D, may be considered as the sections at different intervals in the unknown dimension of a higher cube. Arranged thus their coherence in the higher figure is destroyed, they are mere representations.

A motion in the fourth dimension from A through B, C, etc., would be continuous, but we can only represent it as the occupation of the positions A, B, C, etc., in succession. We can exhibit the results of the motion at different stages, but no more.

In this representation we have left out the distance between one section and another; we have considered the higher body merely as a series of sections, and so left out its contents. The only way to exhibit its contents is to call in the aid of the conception of motion.

If a higher cube passes transverse to our space, it will

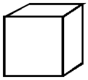


Fig 11.

appear as a cube isolated in space, the part that has not come into our space and the part that has passed through it will not be visible. The gradual passing through our space would appear as the change of the matter of the cube before us. One material particle in it is succeeded by another, neither coming nor going in any direction we can point to. In this manner, by the duration of the figure, we can exhibit the higher dimensionality of it; a cube of our matter, under the circumstances supposed, namely, that it has a motion transverse to our space, would instantly disappear. A higher cube would last till it had passed transverse to our space by its whole distance of extension in the fourth dimension.

As the plane being can think of the cube as consisting of sections, each like a figure he knows, extending away from his plane, so we can think of a higher solid as composed of sections, each like a solid which we know, but extending away from our space.

Thus, taking a higher cube, we can look on it as starting from a cube in our space and extending in the unknown dimension.

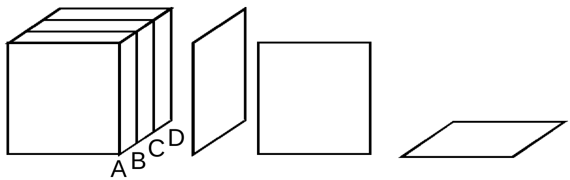


Fig. 12.

Take the face A and conceive it to exist as simply a face, a square with no thickness. From this face the cube in our space extended by the occupation of space which we can see.

But from this face there extends equally a cube in the unknown dimension. We can think of the higher cube, then, by taking the set of sections A, B, C, D, etc., and considering that from each of them there runs a cube. These cubes have nothing in common with each other, and of each of them in its actual position all that we can have in our space is an isolated square. It is obvious that we can take our series of sections in any manner we please. We can take them parallel, for instance, to any one of the three isolated faces shown in the figure. Corresponding to the three series of sections at right angles to each other, which we can make of the cube in space, we must conceive of the higher cube, as composed of cubes starting from squares parallel to the faces of the cube, and of these cubes all that exist in our space are the isolated squares from which they start.

CHAPTER III

THE SIGNIFICANCE OF A FOUR-DIMENSIONAL EXISTENCE

HAVING now obtained the conception of a four-dimensional space, and having formed the analogy which, without any further geometric difficulties, enables us to enquire into its properties, I will refer the reader, whose interest is principally in the mechanical aspect, to Chapters VI. and VII. In the present chapter I will deal with the general significance of the enquiry, and in the next with the historical origin of the idea.

First, with regard to the question of whether there is any evidence that we are really in four-dimensional space, I will go back to the analogy of the plane world.

A being in a plane world could not have any experience of three-dimensional shapes, but he could have an experience of three-dimensional movements.

We have seen that his matter must be supposed to have an extension, though a very small one, in the third dimension. And thus, in the small particles of his matter, three-dimensional movements may well be conceived to take place. Of these movements he would only perceive the resultants. Since all movements of an observable size in the plane world are two-dimensional, he would only perceive the resultants in two dimensions of the small three-dimensional movements. Thus, there would be phenomena which he could not explain by his theory of mechanics—motions would take place which he could not explain by his theory of motion. Hence, to determine if we are in a four-dimensional world, we must examine the phenomena of motion in our space. If movements occur which are not explicable on the suppositions of our three-dimensional mechanics, we should have an indication of a possible four-dimensional motion, and if, moreover, it could be shown that such movements would be a consequence of a four-dimensional motion in the minute particles of bodies or of the ether, we should have a strong presumption in favour of the reality of the fourth dimension.

By proceeding in the direction of finer and finer subdivision, we come to forms of matter possessing properties different to those of the larger masses. It is probably that at some stage in this process we should come to a form of matter of such minute subdivision that its particles possess a freedom of movement in four dimensions. This form of matter I speak of as four-dimensional ether, and attribute to it properties approximating to those of a perfect liquid.

Deferring the detailed discussion of this form of matter to Chapter VI., we will now examine the means by which a plane being would come to the conclusion that three-dimensional movements existed in his world, and point out the analogy by which we can conclude the existence of four-dimensional movements in our world. Since the dimensions of the matter in his world are small in the third direction, the phenomena in which he would detect the motion would be those of the small particles of matter.

Suppose that there is a ring in his plane. We can imagine currents flowing round the ring in either of two opposite directions. These would produce unlike effects, and give rise to two different fields of influence. If the ring with a current in it in one direction be taken up and turned over, and put down again on the plane, it would be identical with the ring having a current in the opposite direction. An operation of this kind would be impossible to the plane being. Hence he would have in his space two irreconcilable objects, namely, the two fields of influence due to the two rings with currents in them in opposite directions. By irreconcilable objects in the plane I mean objects which cannot be thought of as transformed one into the other by any movement in the plane.

Instead of currents flowing in the rings we can imagine a different kind of current. Imagine a number of small rings strung on the original ring. A current round these secondary rings would give two varieties of effect, or two different fields of influence, according to its direction. These two varieties of current could be turned one into the other by taking one of the rings up, turning it over, and putting it down again in the plane. This operation is impossible to the plane being, hence in this case also there would be two irreconcilable fields in the plane. Now, if the plane being found two such irreconcilable fields and could prove that they could not be accounted for by currents in the rings, he would have to admit the existence of currents round the rings—that is, in rings strung on the primary ring. Thus he would come to admit the existence of a three-dimensional motion, for such a disposition of currents is in three dimensions.

Now in our space there are two fields of different properties, which can be produced by an electric current flowing in a closed circuit or ring. These two fields can be changed one into the other by reversing the currents, but they cannot be changed one into the other by any turning about of the rings in our space; for the disposition of the field with regard to the ring itself is different when we turn the ring over, and when we reverse the direction of the current in the ring.

As hypotheses to explain the differences of these two fields and their effects we can suppose the following kinds of space motions:—First, a current along the conductor; second, a current round the conductor—that is, of rings of current strung on the conductor as an axis. Neither of these suppositions accounts for facts of observation.

Hence we have to make the supposition of a four-dimensional motion. We find that a four-dimensional rotation of the nature explained in a subsequent chapter has the following characteristics:—First, it would give us two fields of influence, the one of which could be turned into the other by taking the circuit up into the fourth dimension, turning it over, and putting it down in our space again, precisely as the two kinds of fields in the plane could be turned one into the other by a reversal of the current in our space. Second, it involves a phenomenon precisely identical with that most remarkable and mysterious feature of an electric current, namely that it is a field of action, the rim of which necessarily abuts on a continuous boundary formed by a conductor. Hence, on the assumption of a four-dimensional movement in the region of the minute particles of matter, we should expect to find a motion analogous to electricity.

Now, a phenomenon of such universal occurrence as electricity cannot be due to matter and motion in any very complex relation, but ought to be seen as a simple and natural consequence of their properties. I infer that the difficulty in its theory is due to the attempt to explain a four-dimensional phenomenon by a three-dimensional geometry.

In view of this piece of evidence we cannot disregard that afforded by the existence of symmetry. In this connection I will allude to the simple way of producing the images of insects, sometimes practised by children. They put a few blots of ink in a straight line on a piece of paper, fold the paper along the blots, and on opening it the lifelike presentiment of an insect is obtained. If we were to find a multitude of these figures, we should conclude that they had originated from a process of folding over; the chances against this kind of reduplication of parts is too great to admit of the assumption that they had been formed in any other way.

The production of the symmetrical forms of organised beings, though not of course due to a turning over of bodies of any appreciable size in four-dimensional space, can well be imagined as due to a disposition in that manner of the smallest living particles from which they are built up. Thus, not only electricity, but life, and the processes by which we think and feel, must be attributed to that region of magnitude in which four-dimensional movements take place.

I do not mean, however, that life can be explained as a four-dimensional movement. It seems to me that the whole bias of thought, which tends to explain the phenomena of life and volition, as due to matter and motion in some peculiar relation, is adopted rather in the interests of the explicability of things than with any regard to probability.

Of course, if we could show that life were a phenomenon of motion, we should be able to explain a great deal that is at present obscure. But there are two great difficulties in the way. It would be necessary to show that in a germ capable of developing into a living being, there were modifications of structure capable of determining in the developed germ all the characteristics of its form, and not only this, but of determining those of all the descendants of such a form in an infinite series. Such a complexity of mechanical relations, undeniable though it be, cannot surely be the best way of grouping the phenomena and giving a practical account of them. And another difficulty is this, that no amount of mechanical adaptation would give that element of consciousness which we possess, and which is shared in a modified degree by the animal world.

In those complex structures which men build up and direct, such as a ship or railway train (and which, if seen by an observer of such a size that the men guiding them were invisible, would seem to present some of the phenomena of life) the appearance of animation is not due to any diffusion of life in the material parts of the structure, but to the presence of a living being.

The old hypothesis of a soul, a living organism within the visible man, appears to me much more rational than the attempt to explain life as a form of motion. And when we consider the region of extreme minuteness characterised by four-dimensional motion the difficulty of conceiving such an organism alongside the bodily one disappears. Lord Kelvin suggests that matter is formed from the ether. We may very well supposed that the living organisms directing the material ones are co-ordinate with them, not composed of matter, but consisting of etherial bodies, and as such capable of motion through the ether, and able to originate material living bodies throughout the mineral.

Hypotheses such as these find no immediate ground for proof or disproof in the physical world. Let us, therefore, turn to a different field, and, assuming that the human soul is a four-dimensional being, capable in itself of four dimensional movements, but in its experience through the senses limited to three dimensions, ask if the history of thought, of those productivities which characterise man, correspond to our assumption. Let us pause to review those steps by which man, presumably a four-dimensional being, despite his bodily environment, has come to recognise the fact of four-dimensional existence.

Deferring this enquiry to another chapter, I will here recapitulate the argument in order to show that our purpose is entirely practical and independent of any philosophical or metaphysical considerations.

If two shots are fired at a target, and the second bullet hits it in a different place to the first, we suppose that there was some difference in the conditions under which the second shot was fired from those affecting the first shot. The force of the powder, the direction of aim, the strength of the wind, or some condition must have been different in the second case, if the course of the bullet was not exactly the same as in the first case. Corresponding to every difference in a result there must be some difference in the antecedent material conditions. By tracing out this chain of relations we explain nature.

But there is also another mode of explanation which we apply. If we ask what was the cause that a certain ship was built, or that a certain structure was erected, we might proceed to investigate the changes in the brain cells of the men who designed the works. Every variation in one ship or building from another ship or building is accompanied by a variation in the processes that go on in the brain matter of the designers. But practically this would be a very long task.

A more effective mode of explaining the production of the ship or building would be to enquire into the motives, plans, and aims of the men who constructed them. We obtain a cumulative and consistent body of knowledge much more easily and effectively in the latter way.

Sometimes we apply the one, sometimes the other mode of explanation.

But it must be observed that the method of explanation founded on aim, purpose, volition, always presupposes a mechanical system in which the volition and aim works. The conception of men as willing and acting from motives involves that of a number of uniform processes of nature which he can modify, and of which he can make application. In the mechanical conditions of the three-dimensional world, the only volitional agency which we can demonstrate is the human agency. But when we consider the four-dimensional world the conclusion remains perfectly open.

The method of explanation founded on purpose and aim does not, surely, suddenly begin with man and end with him. There is as much behind the exhibition of will and motive which we see in man as there is behind the phenomena of movement: they are co-ordinate, neither to be resolved into the other. And the commencement of the investigation of that will and motive which lies behind the will and motive manifested in the three-dimensional mechanical field is in the conception of a soul—a four-dimensional organism, which expresses its higher physical being in the symmetry of the body, and gives the aims and motives of human existence.

Our primary task is to form a systematic knowledge of the phenomena of a four-dimensional world and find those points in which this knowledge must be called in to complete our mechanical understanding of the universe. But a subsidiary contribution towards the verification of the hypothesis may be made by passing in review the history of human thought, and enquiring if it presents such features as would be naturally expected on this assumption.

CHAPTER IV

THE FIRST CHAPTER IN THE HISTORY OF FOUR SPACE

PARMENIDES, and the Asiatic thinkers with whom he is in close affinity, propound a theory of existence which is in close accord with the conception of a possible relation between a higher and a lower dimensional space. This theory, prior and in marked contrast to the mains stream of thought, which we shall afterwards describe, forms a closed circle by itself. It is one which in all ages has had a strong attraction for pure intellect, and is the natural mode of thought for those who refrain from projecting their own volition into nature under the guise of causality.

According to Parmenides of the school of Ela the all is one, unmoving and unchanging. The permanent amid the transient—that foothold for thought, that solid ground for feeling on the discovery of which depends all our life—is no phantom; it is the image amidst deception of true being, the eternal, the unmoved, the one. Thus says Parmenides.

But how explain the shifting scene, these mutations of things?

"Illusion," answers Parmenides. Distinguishing between truth and error, he tells of the true doctrine of the one—the false opinion of a changing world. He is no less memorable for the manner of his advocacy than for the cause he advocates. It is as if from his firm foothold of being he could play with the thoughts under the burden of which others laboured, for from him springs that fluency of opposition and hypothesis which forms the texture of Plato's dialectic.

Can the mind conceive a more delightful intellectual picture than that of Parmenides, pointing to the one, the true, the unchanging, and yet on the other hand ready to discuss all manner of false opinion, forming a cosmogony too, false "but mine own" after the fashion of the time.

In support of the true opinion he proceeded by the negative way of showing the self-contradiction in the ideas of change and motion. It is doubtful if his criticism, save in minor points, has ever been successfully refuted. To express his doctrine in the ponderous modern way we must make the statement that motion is phenomenal, not real.

Let us represent his doctrine.

Imagine a sheet of still water into which a slanting stick is being lowered with a motion vertically downwards. Let 1, 2, 3 (Fig. 13), be three consecutive positions of the stick, A, B, C, will be three consecutive positions of the meeting of the stick, with the surface of the water. As the stick passes down, the meeting will move from A on to B and C.

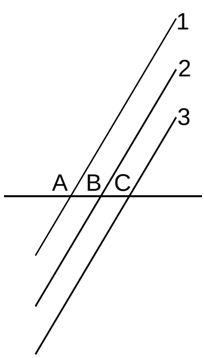


Fig. 13.

Suppose now all the water to be removed except a film. At the meeting of the film and the stick there will be an interruption of the film. If we suppose the film to have a property, like that of a soap bubble, of closing up round any penetrating object, then as the stick goes vertically downwards the interruption in the film will move on.

If we pass a spiral through the film the intersection will give us a point moving in a circle shown by the dotted lines in the square. Suppose now the spiral to be still and the film to move vertically upwards, the whole spiral will be represented in the film of the consecutive positions of the point of intersection. In the film the permanent evidence of the spiral is experienced as a time series—the record of traversing the spiral is a point moving in a circle. If now we suppose a consciousness connected with the film in such a way that the intersection of the spiral with the film gives rise to a conscious experience, we shall see that we have in the film a point moving in a circle, conscious of its motion, knowing nothing of that real spiral the record of the successive interactions of which by the film is the motion of the point.

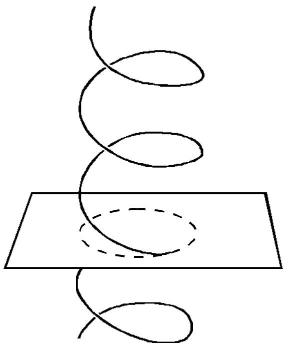


Fig. 14.

It is easy to imagine complicated structures of the nature of the spiral, structures consisting of filaments, and to suppose also that these structures are distinguishable from each other at every section. If we consider the intersections of these filaments with the film as it passes to be the atoms constituting a filmar universe, we shall have bodies corresponding to the filamentary structure, and the positions of these structures with regard to one another will give rise to bodies in the film moving amongst one another. This mutual motion is apparent merely. The reality is of permanent structures stationary, and all the relative motions accounted for by one steady movement of the film as a whole.

Thus we can imagine a plane world, in which all the variety of motion is the phenomenon of structures consisting of filamentary atoms traversed by a plane of consciousness. Passing to four dimensions and our space, we can conceive that all things and movements in our world are the reading off of a permanent reality by a space of consciousness. Each atom at every moment is not what it was, but a new part of that endless line which is itself. And all this system successively revealed in the time which is but the succession of consciousness, separate as it is in parts, in its entirety is one vast unity. Representing Parmenides' doctrine thus, we gain a firmer hold on it than if we merely let his words rest, grand and massive, in our minds. And we have gained the means also of representing phases of that Eastern thought to which Parmenides was no stranger. Modifying his uncompromising doctrine, let us suppose, to go back to the plane of consciousness and the structure of filamentary atoms, that these structures are themselves moving—are acting, living. Then, in the transverse motion of the film, there would be two phenomena of motion, one due to the reading off in the film of the permanent existences as they are in themselves, and another phenomenon of motion due to the modification of the record of the things themselves, by their proper motion during the process of traversing them.

Thus a conscious being in the plane would have, as it were, a two-fold experience. In the complete traversing of the structure, the intersection of which with the film gives his conscious all, the main and principal movements and actions which he went through would be the record of his higher self as it existed unmoved and unacting. Slight modifications and derivations from these movements and actions would represent the activity and self-determination of the complete being, of his higher self.

It is admissible to suppose that the consciousness in the plane has a share in that volition by which the complete existence determines itself. Thus the motive and will, the initiative and life, of the higher being, would be represented in the case of the being in the film by an initiative and a will capable, not of determining any great things or independent movements in his existence, but only of small and relatively insignificant activities. In all the main features of his life his experience would be representative of one state of the higher being whose existence determines his as the film passes on. But in his minute and apparently unimportant actions he would share in that will and determination by which the whole of the being he really is acts and lives.

An alteration of the higher being would correspond to a different life history for him. Let us now make the supposition that film after film traverses these higher structures, that the life of the real being is read off again and again in successive waves of consciousness. There would be a succession of lives in the different advancing planes of consciousness, each differing from the preceding and differing in virtue of that will and activity which in the preceding had not been devoted to the greater and apparently most significant things in life, but the minute and apparently unimportant. In all great things the being of the film shares in the existence of his higher self as it is at any one time. In the small things he shares in that volition by which the higher being alters and changes, acts and lives.

Thus we gain the conception of a life changing and developing as a whole, a life in which our separation and cessation and fugitiveness are merely apparent, but which in its events and course alters, changes, develops; and the power of altering and changing this whole lies in the will and power the limited being has of directing, guiding, altering himself in the minute things of his existence.

Transferring our conceptions to those of an existence in a higher dimensionality traversed by a space of conciousness, we have an illustration of a thought which has found frequent and varied expression. When, however, we ask ourselves what degree of truth there lies in it, we must admit that, at far as we can see, it is merely symbolical. The true path in the investigation of a higher dimensionality lies in another direction.

The significance of the Parmenidean doctrine lies in this: that here, as again and again, we find that those conceptions which man introduces of himself, which he does not derive from the mere record of his outward experience, have a striking and significant correspondence to the conception of a physical existence in a world of a higher space. How close we come to Parmenides' thought by this manner of representation it is impossible to say. What I want to point out is the adequateness of the illustration, not only to give a static model of his doctrine, but one capable as it were, of a plastic modification into a correspondence into kindred forms of thought. Either one of two things must be true—that four-dimensional conceptions give a wonderful power of representing the thought of the East, or that the thinkers of the East must have been looking at and regarding four-dimensional existence.

Coming now to the main stream of thought we must dwell in some detail on Pythagoras, not because of his direct relation to the subject, but because of his relation to the investigators who come later.

Pythagoras invented the two-way counting. Let us represent the single-way counting by the points *aa*, *ab*, *ac*, *ad*, using these pairs of letters instead of the numbers 1, 2, 3, 4. I put an *a* in each case first for a reason which will immediately appear.

We have a sequence and order. There is no conception of distance necessarily involved. The difference between the posits is one of order not of distance--only when identified with a number of equal material things in juxtaposition does the notion of distance arise.

Now, besides the simple series I can have, starting from *aa, ba, ca, da,* from *ab, bb, cb, db,* and so on, and forming a scheme:

da	*db*	*dc*	*dd*
ca	*cb*	*cc*	*cd*
ba	*bb*	*bc*	*bd*
aa	*ab*	*ac*	*ad*

This complex or manifold gives a two-way order. I can represent it by a set of points, if I am on my guard against assuming any relation of distance.

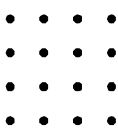


Fig. 15.

Pythagoras studied this two-fold way of counting on reference to material bodies, and discovered that most remarkable property of the combination of number and matter that bears his name.

The Pythagorean property of an extended material system can be exhibited in a manner which will be of use to us afterwards, and which therefore I will employ now instead of using the kind of figure which he himself employed.

Consider a two-fold field of points arranged in regular rows. Such a field will be presupposed in the following argument.

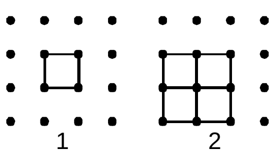


Fig. 16.

It is evident that in fig. 16 four of the points determine a square, which square we may take as the unit of measurement for areas. But we can also measure areas in another way.

Fig. 16 (1) shows four points determining a square.

But four squares also meet in a point, fig. 16 (2).

Hence a point at the corner of a square belongs equally to four squares.

Thus we may say that the point value of the square shown is one point, for if we take the square in fig. 16 (1) it has four points, but each of these belong equally to four other squares. Hence one fourth of each of them belongs to the square (1) in fig. 16. Thus the point value of the square is one point.

The result of counting the points is the same as that arrived at by reckoning the square units enclosed.

Hence, if we wish to measure the area of any square we can take the number of points it encloses, count these as one each, and take one-fourth the number of points at its corners.

Now draw a diagonal square as shown in fig. 17. It contains one point and the four corners count for one point more; hence its point value is 2. The value is the measure of its area—the size of this square is two of the unit squares.

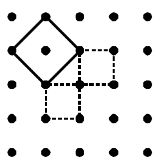


Fig. 17.

Looking now at the sides of this figure we see that there is a unit square on each of them—the two squares contain no points, but have four corner points each, which gives the point value of each as one point.

Hence we see that the square on the diagonal is equal to the squares on the two sides; or as it is generally expressed, the square on the hypotenuse is equal to the sum of the squares on the sides.

Noticing this fact we can proceed to ask if it is always true. Drawing the square shown in fig. 18, we can count the number of its points. There are five altogether. There are four points inside the square on the diagonal, and hence, with the four points at its corners the point value is 5—that is, the area is 5. Now the squares on the sides are respectively of the area 4 and 1. Hence in this case also the square

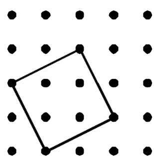


Fig. 18.

on the diagonal is equal to the sum of the squares on the sides. This property of matter is one of the first great discoveries of applied mathematics. We shall prove afterwards that it is not a property of space. For this present it is enough to remark that the positions in which the points are arranged is entirely experimental. It is by means of equal pieces of some material, or the same piece of material moved from one place to another, that the points are arranged.

Pythagoras next enquired what the relation must be so that a square drawn slanting-wise should be equal to one straight-wise. He found that a square whose side is five can be placed either rectangularly along the lines of points, or in a slanting position. And this square is equivalent to two squares of sides 4 and 3.

Here he came upon a numerical relation embodied in a property of matter. Numbers immanent in the objects produced the equality so satisfactory for intellectual apprehension. And he found that numbers when immanent in sound—when the strings of a musical instrument were given certain definite proportions of length—were no less captivating to the ear than the equality of squares was to the reason. What wonder then that he ascribed an active power to number.

We must remember that, sharing like ourselves the search for the permanent in changing phenomena, the Greeks had not that conception of the permanent in matter that we have. To them material things were not permanent. In fire solid things would vanish; absolutely disappear. Rock and earth had a more stable existence, but they too grew and decayed. The permanence of matter, the conservation of energy, were unknown to them. And that distinction which we draw so readily between the fleeting and permanent causes of sensation, between a sound and a material object, for instance, had not the same meaning to them which it has for us. Let us but imagine for a moment that material things are fleeting, disappearing, and we shall enter with a far better appreciation into that search for the permanent which, with the Greeks, as with us, is the primary intellectual demand.

What is that which amid a thousand forms is ever the same, which we can recognise under all its vicissitudes, of which the diverse phenomena are the appearances?

To think that this is number is not so very wide of the mark. With an intellectual apprehension which far outran the evidences for its application, the atomists asserted that there were everlasting material particles, which, by their union, produced all the varying forms and states of bodies. But in view of the observed facts of nature as then known, Aristotle, with perfect reason, refused to accept this hypothesis.

He expressly states that there is a change of quality, and that the change due to motion is only one of the possible modes of change.

With no permanent material world about us, with the fleeting, the unpermanent, all around we should, I think, be ready to follow Pythagoras in his identification of number with that principle which subsists amidst all changes, which in multitudinous forms we apprehend immanent in the changing and disappearing substance of things.

And from the numerical idealism of Pythagoras there is but a step to the more rich and full idealism of Plato. That which is apprehended by the sense of touch we put as primary and real, and the other senses we say are merely concerned with appearances. But Plato took them all as valid, as giving qualities of existence. That the qualities were not permanent in the world as given to the senses forced him to attribute to them a different kind of permanence. He formed the conception of a world of ideas, in which all that really is, all that affects us and gives the rich and wonderful wealth of our experience, is not fleeting and transitory, but eternal. And of this real and eternal we see in the things about us the fleeting and transitory images.

And this world of ideas was no exclusive one, wherein was no place for the innermost convictions of the soul and its most authoritative assertions. Therein existed justice, beauty—the one, the good, all that the soul demanded to be. The world of ideas, Plato's wonderful creation preserved for man, for his deliberate investigation and their sure development, all that the rude incomprehensible changes of a harsh experience scatters and destroys.

Plato believed in the reality of ideas. He meets us fairly and squarely. Divide a line into two parts, he says: one to represent the real objects in the world, the other to represent the transitory appearances, such as the image in still water, the glitter of the sun on a bright surface, the shadows on the clouds.

A	B
Real things: *e.g.* the sun	Appearances: *e.g.* the reflection of the sun

Take another line and divide it into two parts, one representing our ideas, the ordinary occupants of our minds, such as whiteness, equality, and the other representing our true knowledge, which is of eternal principles, such as beauty, goodness.

A′	B′
Eternal principles, as beauty	Appearances in the mind, as whiteness, equality

Then as A is to B, so is A′ to B′.

That is, the soul can proceed, going away from real things to a region of perfect certainty, where it beholds what is, not the scattered reflections; beholds the sun, not the glitter on the sands; true being, not chance opinion.

Now, this is to us, as it was to Aristotle, absolutely inconceivable from a scientific point of view. We can understand that a being is known in the fulness of his relations; it is in his relations to his circumstances that a man's character is known; it is in his acts under his conditions that his character exists. We cannot grasp or conceive any principle of individuation apart from the fulness of the relations to the surroundings.

But suppose now that Plato is talking about the higher man—the four-dimensional being that is limited in our external experience to a three-dimensional world. Do not his words being to have a meaning? Such a being would have a consciousness of motion which is not as the motion he can see with the eyes of the body. He, in his own being, knows a reality to which the outward matter of this too solid earth is flimsy superficiality. He too knows a mode of being, the fulness of relations, in which can only be represented in the limited world of sense, as the painter unsubstantially portrays the depths of woodland, plains, and air. Thinking of such a being in man, was not Plato's line well divided?

It is noteworthy that, if Plato omitted his doctrine of the independent origin of ideas, he would present exactly the four-dimensional argument; a real thing as we think it is an idea. A plane being's idea of a square object is the idea of an abstraction, namely a geometrical square. Similarly our idea of a solid thing is an abstraction, for in our idea there is not the four-dimensional thickness which is necessary, however slight, to give reality. The argument would then run, as a shadow is to a solid object, so is the solid object to the reality. Thus A and B would be identified.

In the allegory which I have already alluded to, Plato in almost as many words shows forth the relation between existence in a superficies and in solid space. And he uses this relation to point to the conditions of a higher being.

He imagines a number of men prisoners, chained so that they look at the wall of a cavern in which they are confined, with their backs to the road and the light. Over the road pass men and women, figures and processions, but of all this pageant all that the prisoners behold is the shadow of it on the wall whereon they gaze. Their own shadows and the shadows of the things in the world are all that they see, and identifying themselves with their shadows related as shadows to a world of shadows, they live in a kind of dream.

Plato imagines one of their number to pass out from amongst them into the real space world, and then returning to tell them of their condition.

Here he presents most plainly the relation between existence in a plane world and existence in a three-dimensional world. And he uses this illustration as a type of the manner in which we are to proceed to a higher state from the three-dimensional life we know.

It must have hung upon the weight of a shadow which path he took!—whether the one we shall follow toward the higher solid and the four-dimensional existence, or the one which makes ideas the higher realities, and the direct perception of them the contact with the truer world.

Passing on to Aristotle, we will touch on the points which most immediately concern our enquiry.

Just as a scientific man of the present day in reviewing the speculations of the ancient world would treat them with a curiosity half amused but wholly respectful, asking of each and all wherein lay their relation to fact, so Aristotle, in discussing the philosophy of Greece as he found it, asks, above all other things: "Does this represent the world? In this system is there an adequate presentation of what is?"

He finds them all defective, some for the very reasons which we esteem them most highly, as when he criticises the Atomic theory for its reduction of all change to motion. But in the lofty march of his reason he never loses sight of the whole; and that wherein our views differ from his lies not so much in a superiority of our point of view, as in the fact which he himself enunciates—that it is impossible for one principle to be valid in all branches of enquiry. The conceptions of one method of investigation are not those of another; and our divergence lies in our exclusive attention to the conceptions useful in one way of apprehending nature rather than in any possibility we find in our theories of giving a view of the whole transcending that of Aristotle.

He takes account of everything; he does not separate matter and the manifestation of matter; he fires all together in a conception of a vast world process in which everything takes part—the motion of a grain of dust, the unfolding of a leaf, the ordered motion of the spheres in heaven—all are parts of one whole which he will not separate into dead matter and adventitious modifications.

And just as our theories, as representations of actuality, fall before his unequalled grasp of fact, so the doctrine of ideas fell. It is not an adequate account of existence, as Plato himself shows in his "Parmenides"; it only explains things by putting their doubles beside them.

For his own part Aristotle invented a great marching definition which, with a kind of power of its own, cleaves its way through phenomena to limiting conceptions on either hand, towards whose existence all experience points.

In Aristotle's definition of matter and form as the constituent of reality, as in Plato's mystical vision of the kingdom of ideas, the existence of the higher dimensionality is implicitly involved.

Substance according to Aristotle is relative, not absolute. In everything that is there is the matter of which it is composed, the form which it exhibits; but these are indissolubly connected, and neither can be thought without the other.

The blocks of stone out of which a house is built are the material for builder; but, as regards the quarryman, they are the matter of the rocks with the form he has imposed on them. Words are the final product of the grammarian, but the mere matter of the orator or poet. The atom is, with us, that out of which chemical substances are built up, but looked at from another point of view is the result of complex processes.

Nowhere do we find finality. The matter in one sphere is the matter, plus form, of another sphere of thought. Making an obvious application to geometry, plane figures exist as the limitation of different portions of the plane by one another. In the bounding lines the separated matter of the plane shows its determination into form. And as the plane is the matter relatively to determinations in the plane, so the plane itself exists in virtue of the determination of space. A plane is that wherein formless space has form superimposed on it, and gives an actuality of real relations. We cannot refuse to carry this process of reasoning a step farther back, and say that space itself is that which gives form to higher space. As a line is the determination of a plane, and a plane of a solid, so solid space itself is the determination of a higher space.

As a line by itself is inconceivable without that plane which it separates, so the plane is inconceivable without the solids which it limits on either hand. And so space itself cannot be positively defined. It is the negation of the possibility of movement in more than three dimensions. The conception of space demands that of a higher space. As a surface is thin and mathematical without the substance of which it is the surface, so matter itself is thin without the higher matter.

Just as Aristotle invented that algebraical method of representing unknown quantities by mere symbols, not by lines necessarily determinate in length as what the habit of the Greek geometers, and so struck out the path towards those objectifications of thought which, like independent machines for reasoning, supply the mathematician with his analytical weapons, so in the formulation of the doctrine of matter and form, of potentiality and actuality, of the relativity of substance, he produced another kind of objectification of mind—a definition which had a vital force and an activity of its own.

In none of his writings, as far as we know, did he carry it to its legitimate conclusion on the side of matter, but in the direction of the formal qualities he was led to his limiting conception of that existence of pure form which lies beyond all known determination of matter. The unmoved mover of all things is Aristotle's highest principle. Towards it, to partake of its perfection all things move. The universe, according to Aristotle, is an active process—he does not adopt the illogical conception that it was once set in motion and has kept on ever since. There is room for activity, will, self-determination, in Aristotle's system, and for the contingent and accidental as well. We do not follow him, because we are accustomed to find in nature infinite series, and do not feel obliged to pass on to a belief in the ultimate limits to which they seem to point.

But apart from the pushing to the limit, as a relative principle this doctrine of Aristotle's as to the relativity of substance is irrefragible in its logic. He was the first to show the necessity of that path of thought which when followed leads to a belief in a four-dimensional space.

Antagonistic as he was to Plato in his conception of the practical relation of reason to the world of phenomena, yet in one point he coincided with him. And in this he showed the candour of his intellect. He was more anxious to lose nothing than to explain everything. And that wherein so many have detected an inconsistency, an inability to free himself from the school of Plato, appears to us in connection with our enquiry as an instance of the acuteness of his observation. For beyond all knowledge given by the senses Aristotle held that there is an active intelligence, a mind not the passive recipient of impressions from without, but an active and originative being, capable of grasping knowledge at first hand. In the active soul Aristotle recognised something in man not produced by his physical surroundings, something which creates, whose activity is a knowledge underived from sense. This, he says, is the immortal and undying being in man.

Thus we see that Aristotle was not far from the recognition of the four-dimensional existence, both without and within man, and the process of adequately realising the higher dimensional figures to which we shall come subsequently is a simple reduction to practice of his hypothesis of a soul.

The next step in the unfolding of the drama of the recognition of the soul as connected with out scientific conception of the world, and, at the same time, the recognition of that higher of which a three-dimensional world presents the superficial appearance, took place many centuries later. If we pass over the intervening time without a word it is because the soul was occupied with the assertion of itself in other ways than that of knowledge. When it took up the task in earnest of knowing this material world in which it found itself, and of diverting the course of inanimate nature, from that most objective aim came, reflected back as from a mirror, its knowledge of itself.

CHAPTER V

THE SECOND CHAPTER IN THE HISTORY OF FOUR SPACE

LOBATCHEWSKY, BOLYAI, AND GAUSS

BEFORE entering on a description of the word of Lobatchewsky and Bolyai it will not be out of place to give a brief account of them, the materials for which are to be found in an article by Franz Schmidt in the forty-second volume of the *Mathematische Annalen*, and in Engel's edition of Lobatchewsky.

Lobatchewsky was a man of the most complete and wonderful talents. As a youth he was full of vivacity, carrying his exuberance so far as to fall into serious trouble for hazing a professor, and other freaks. Saved by the good offices of the mathematician Bartels, who appreciated his ability, he managed to restrain himself within the bounds of prudence. Appointed professor at his own University, Kasan, he entered on his duties under the regime of a pietistic reactionary, who surrounded himself with sycophants and hypocrites. Esteeming probably the interests of his pupils as higher than any attempt at a vain resistance, he made himself the tyrant's right-hand man, doing an incredible amount of teaching and performing the most varied official duties. Amidst all his activates he found time to make important contributions to science. His theory of parallels is most closely connected with his name, but a study of his writings shows that he was a man capable of carrying on mathematics in its main lines of advance, and of a judgement equal to discerning what those lines were. Appointed rector of his University, he died at an advanced age, surrounded by friends, honoured, with the results of his beneficent activity all around him. To him no subject came amiss, from the foundations of geometry, to the improvement of the stoves by which the peasants warmed their homes.

He was born in 1793. His scientific work was unnoticed till, in 1867, Houel, the French mathematician, drew attention to its importance.

Johann Bolyai de Bolyai was born in Klausenburg, a town in Transylvania, December 18th, 1802.

His father, Wolfgang Bolyai, a professor in the Reformed College of Maros Vasarhely, retained the ardour in mathematical studies which had made him a chosen companion of Gauss in their early student days at Göttingen.

He found an eager pupil in Johann. He relates that the boy sprang before him like a devil. As soon as he had enunciated a problem the child would give the solution and command him to go on further. As a thirteen-year-old boy his father sometimes sent him to fill his place when incapacitated from taking his classes. The pupils listened to him with more attention than to his father for they found him clearer to understand.

In a letter to Gauss Wolfgang Bolyai writes:—

"My boy is strongly built. He has learned to recognise many constellations, and the ordinary figures of geometry. He makes apt applications of his notions, drawing for instance the positions of the stars with their constellations. Last winter in the country, seeing Jupiter he asked: 'How is it that we can see him from here as well as from the town? He must be far off.' And as to three different places to which he had been he asked me to tell him about them in one word. I did not know what he meant, and then he asked me if one was in a line with the other and all in a row, or if they were in a triangle.

"He enjoys cutting paper figures with a pair of scissors, and without my ever having told him about triangles remarked that a right-angled triangle which he had cut out was half of an oblong. I exercise his body with care, he can dig well in the earth with his little hands. The blossom can fall and no fruit left. When he is fifteen I want to send him to you to be your pupil."

In Johann's autobiography he says:—

"My father called my attention to the imperfections and gaps in the theory of parallels. He told me he had gained more satisfactory results than his predecessors, but had obtained no perfect and satisfying conclusion. None of his assumptions had the necessary degree of geometrical certainty, although they sufficed to prove the eleventh axiom and appeared acceptable on first sight.

"He begged of me, anxious not without a reason, to hold myself aloof and to shun all investigation on this subject, if I did not wish to live all my life in vain."

Johann, in the failure of his father to obtain any response from Gauss, in answer to a letter in which he asked the great mathematician to make of his son "an apostle of truth in a far land," entered the Engineering School at Vienna. He writes from Temesvar, where he was appointed sub-lieutenant, 1823:—

"Temesvar, November 3rd, 1923.

"DEAR GOOD FATHER,

"I have so overwhelmingly much to write about my discovery that I know no other way of checking myself than taking a quarter of a sheet only to write on. I want an answer to my four-sheet letter.

"I am unbroken in my determination to publish a work on Parallels, as soon as I have put my materials in order and have the means.

"At present I have not made any discovery, but the way I have followed almost certainly promises me the attainment of my object if any possibility of it exists.

"I have not got my object yet, but I have pondered such stupendous things that I was overwhelmed myself, and it would be an eternal shame if they were lost. When you see them you will find that it is so. Now I can only say that I have made a new world out of nothing. Everything that I have sent you before is a house of cards in comparison with a tower. I am convinced that it will be no less to my honour than if I had already discovered it."

The discovery of which Johann here speaks was published as an appendix to Wolfgang Bolyai's *Tentamen*.

Sending the book to Gauss, Wolfgang writes, after an interruption of eighteen years in his correspondence:—

"My son is first lieutenant of Engineers and will soon be captain. He is a fine youth, a good violin player, a skilful fencer, and brave, but has had many duels, and is wild even for a soldier. Yet he is distinguished—light in darkness and darkness in light. He is an impassioned mathematician with extraordinary capacities. . . . He will think more of your judgement on his work than that of all Europe."

Wolfgang received no answer from Gauss to this letter, but sending a second copy of the book received the following reply:—

"You have rejoiced me, my unforgotten friend, by your letters. I delayed answering the first because I wanted to wait for the arrival of the promised little book.

"Now something about your son's work.

"If I begin with saying that 'I ought not to praise it,' you will be staggered for a moment. But I cannot say anything else. To praise it is to praise myself, for the path your son has broken in upon and the results to which he has been led are almost exactly the same as my own reflections, some of which date from thirty to thirty-fire years ago.

"In fact I am astonished to the uttermost. My intention was to let nothing be known in my lifetime about my own work, of which, for the rest, but little is committed to writing. Most people have but little perception of the problem, and I have found very few who took any interest in the views I expressed to them. To be able to do that one must first of all have had a real live feeling for what is wanting, and as to that most men are completely in the dark.

"Still it was my intention to commit everything to writing in the course of time, so that at least it should not perish with me.

"I am deeply surprised that this task can be spared me, and I am most of all pleased in this that it is the son of my old friend who has in so remarkable a manner preceded me."

The impression which we receive from Gauss's inexplicable silence towards his old friend is swept away by this letter. Hence we breathe the clear air of the mountain tops. Gauss would not have failed to perceive the vast significance of his thoughts, sure to be all the greater in their effect on future ages from the want of comprehension of the present. Yet there is not a word or a sign in his writing to claim the thought for himself. He published no single line on the subject. By the measure of what he thus silently relinquishes, by such a measure of a world-transforming thought, we can appreciate his greatness.

It is a long step from Gauss's serenity to the disturbed and passionate life of Johann Bolyai—he and Gauss, the two most interesting figures in the history of mathematics. For Bolyai, the wild solider, the duellist, fell at odds with the world. It is related of him that he was challenged by thirteen officers of his garrison, a thing not unlikely to happen considering how differently he thought from everyone else. He fought them all in succession—making it his only condition that he should be allowed to play on his violin for an interval between meeting each opponent. He disarmed or wounded all his antagonists. It can be easily imagined that a temperament such as his was one not congenial to his military superiors. He was retired in 1833.

His epoch-making discovery awoke no attention. He seems to have conceived the idea that his father had betrayed him in some inexplicable way by his communications with Gauss, and he challenged the excellent Wolfgang to a duel. He passed his life in poverty, many a time, says his biographer, seeking to snatch himself from dissipation and apply himself again to mathematics. But his efforts had no result. He died January 27th, 1860, fallen out with the world and with himself.

METAGEOMETRY

The theories which are generally connected with the names of Lobatchewsky and Bolyai bear a singular and curious relation to the subject of higher space.

In order to show what this relation is, I must ask the reader to be at the pains to count carefully the sets of points by which I shall estimate the volumes of certain figures.

No mathematical processes beyond this simple one of counting will be necessary.

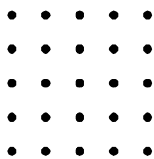


Fig. 19.

Let us suppose we have before us in fig. 19 a plane covered with points at regular intervals, so placed that every four determine a square.

Now it is evident that as four points determine a square, so four squares meet in a point.

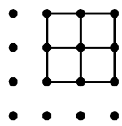


Fig. 20.

Thus, considering a point inside a square as belonging to it, we may say that a point on the corner of a square belongs to it and to four others equally: belongs a quarter of it to each square.

Thus the square ACDE (fig. 21) contains one point, and has four points at the four corners. Since one-fourth of each of these four belongs to the square, the four together count as one point, and the point value of the square is two points—the one inside and the four at the corner make two points belonging to it exclusively.

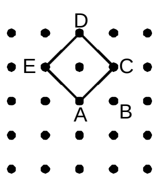


Fig. 21.

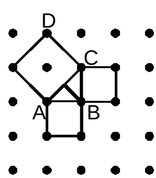


Fig. 22.

Now the area of this square is two unit squares, as one can see by drawing two diagonals in fig. 22.

We also notice that the square in question is equal to the sum of the squares on the sides AB, BC, of the right-angled triangle ABC. Thus we recognise the proposition that the square on the hypothenuse is equal to the sum of the squares on the two sides of a right-angled triangle.

Now suppose we set ourselves the question of determining the whereabouts in the ordered system of points, the end of a line would come when it turned about a point keeping one extremity fixed at the point.

We can solve this problem in a particular case. If we can find a square lying slantwise amongst the dots which is equal to one which goes regularly, we shall know that the two sides are equal, and that the slanting side is equal to the straight-way side. Thus the volume and shape of a figure remaining unchanged will be the test of its having rotated about the point, so that we can say that its side in its first position would turn into its side in the second position.

Now, such a square can be found in the one whose side is five units in length.

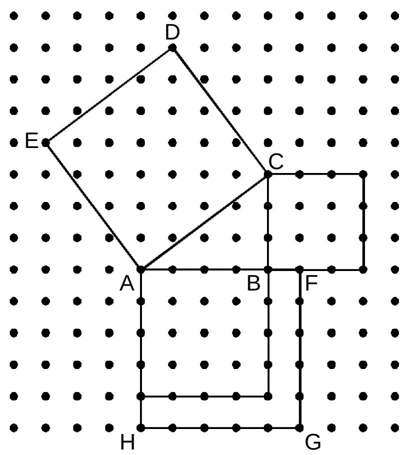


Fig. 23.

In fig. 23, in the square on AB, there are—

9 points interior 9
4 at the corners 1
4 sides with 3 on each side, considered as 1½ on each side, because belonging equally to two squares. . . . 6

The total is 16. There are 9 points in the square on BC.

In the square on AC there are—

24 points inside	24
4 at the corners	1

or 25 altogether.

Hence we see again that the square on the hypothenuse is equal to the squares on the sides.

Now take the square AFHG, which is larger than the square on AB. It contains 25 points.

16 inside.	24
16 on the sides, counting as . .	8
4 on the corners	1

making 25 altogether.

If two squares are equal we conclude the sides are equal. Hence, the line AF turning round A would move so that it would after a certain turning coincide with AC.

This is preliminary, but it involves all the mathematical difficulties that will present themselves.

There are two alternations of a body by which its volume is not changed.

One is the one we have considered, rotation, the other is what is called shear.

Consider a book, or heap of loose pages. They can be slid so that each one slips over the preceding one, and the whole assumes the shape *b* in fig. 24.

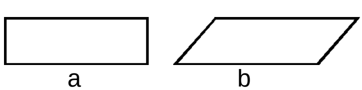


Fig. 24.

This deformation is not shear alone, but shear accompanied by rotation.

Shear can be considered as produced in another way.

Take the square ABCD (fig. 25). and suppose that it is pulled out from along one of its diagonals both ways, and proportionately compressed along the other diagonal. It will assume the shape in fig. 26.

This compression and expansion along two lines at right angles is what is called shear; it is equivalent to the sliding illustrated above, combined with a turning round.

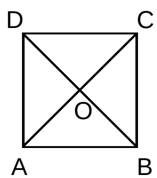


Fig. 25.

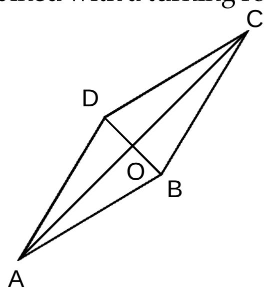


Fig. 26.

In pure shear a body is compressed and extended in two directions at right angles to each other, so that its volume remains unchanged.

Now we know that material bodies resist shear—shear does violence to the internal arrangement of their particles, but they turn as whole without such internal resistance.

But there is an exception. In a liquid shear and rotation take place equally easily, there is no more resistance against a shear than there is against a rotation.

Now, suppose all bodies were to be reduced to the liquid state, in which they yield to shear and to rotation equally easily, and then were to be reconstructed as solids, but in such a way that shear and rotation had interchanged places.

That is to say, let us suppose that when they had become solids again they would shear without offering any internal resistance, but a rotation would do violence to their internal arrangement.

That is, we should have a world in which shear would have taken the place of rotation.

A shear does not alter the volume of a body: thus an inhabitant living in such a world would look on a body sheared as we look on a body rotated. He would say that it was the same shape, but had turned a bit round.

Let us imagine a Pythagoras in this world going to work to investigate, as is his wont.

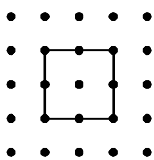


Fig. 27.

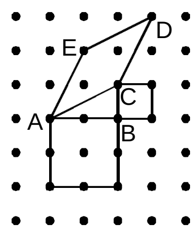


Fig. 28.

Fig. 27 represents a square unsheared. Fig. 28 represents a square sheared. It is not the figure into which the square in fig. 27 would turn, but the result of shear on some square not drawn. It is a simple slanting placed figure, taken now as we took a simple slanting placed square before. Now, since bodies in this world of shear offer no internal resistance to shearing, and keep their volume when sheared, an inhabitant accustomed to them would not consider that they altered their shape under shear. He would call ACDE as much as square as the square in fig. 27. We will call such figures shear squares. Counting the dots in ACDE, we find—

2 inside = 2
4 at corners = 1

or a total of 3.

Now, the square on the side AB has 4 points, that on BC has 1 point. Here the shear square on the hypothenuse has not 5 points by 3; it is not the sum of the squares on the side, but the difference.

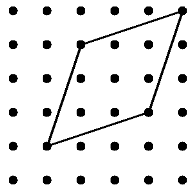


Fig. 29.

This relation always holds. Look at fig. 29.

Shear square on hypothenuse—

7 internal . . .	7
4 at corners . . .	1
	8

Square on one side—which the reader can draw for himself—

4 internal . .	4
8 on sides. .	4
4 at corners .	1
	9

and the square on the other side is 1. Hence in this case again the difference is equal to the shear square on the hypothenuse, 9 – 1 = 8.

Thus in a world of shear the square on the hypothenuse would be equal to the difference of the square on the sides of a right-angled triangle.

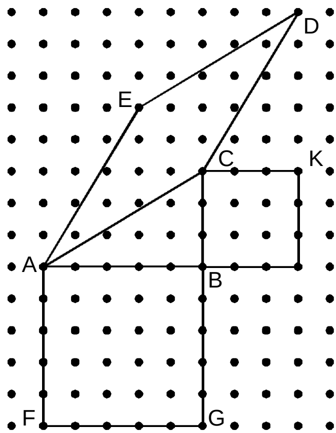


Fig. 29 *bis*.

In fig. 29 *bis* another shear square is drawn on which the above relation can be tested.

What now would be the position a line on turning by shear would take up?

We must settle this in the same way as previously with our turning.

Since a body sheared remains the same, we must find two equal bodies, one in the straight way, one in the slanting way, which have the same volume. Then the side of one will be turning become the side of the other, for the two figures are each what the other becomes by a shear turning.

We can solve the problem in a particular case—

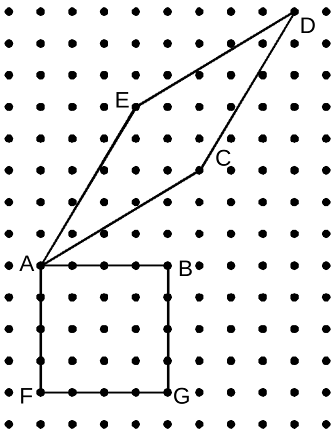


Fig. 30.

In the figure ACDE (fig. 30) there are—

15 inside	.	15
4 at corners	.	1

a total of 16.

Now in the square ABGF there are 16—

9 inside	.	9
12 on sides	.	6
4 at corners	.	1
		—
		16

Hence the square on AB would, by the shear turning, become the shear square ABDE.

And hence the inhabitant of this world would say that the line AB turned into the line AC. These two lines would be to him two lines of equal length, one turned a little way round from the other.

That is, putting shear in place of rotation, we get a different kind of figure, as the result of the shear rotation, from what we got with our ordinary rotation. And as a consequence we get a position for the end of a line of invariable length when it turns by the shear rotation, different from the position which it would assume on turning by our rotation.

A real material rod in the shear world would, on turning about A, pass from the position AB to the position AC. We say that its length alters when it becomes AC, but this transformation of AB would seem to an inhabitant of the shear world like a turning of AB without altering its length.

If we now suppose a communication of ideas that takes place between one of ourselves and an inhabitant of the shear world, there would evidently be a difference between his views of distance and ours.

We should say that his line AB increased in length in turning to AC. He would say that our line AF (fig. 23) decreased in length in turning to AC. He would think that what we called an equal line was in reality a shorter one.

We should say that a real turning round would have its extremities in the positions we call at equal distances. So would he—but the positions would be different. He could, like us, appeal to the properties of matter. His rod to him alters as little as ours does to us.

Now, is there any standard to which we could appeal, to say which of the two is right in this argument? There is no standard.

We should say that, with a change of position, the configuration and shape of his objects altered. He would say that the configuration and shape of our objects altered in what we called merely a change of position. Hence distance independent of position is inconceivable, or practically distance is solely a property of matter.

There is no principle to which either party in this controversy could appeal. There is nothing to connect the definition of distance with our ideas rather than with his, except the behaviour of an actual piece of matter.

For the study of the processes which go on in our world the definition of distance given by taking the sum of the squares is of paramount importance to us. But as a question of pure space without making any unnecessary assumptions the shear world is just as possible and just as interesting as our world.

It was the geometry of such conceivable worlds that Lobatchewsky and Bolyai studied.

This kind of geometry has evidently nothing to do directly with four-dimensional space.

But a connection arises in this way. It is evident that, instead of taking a simple shear as I have done, and defining it as that change of the arrangement of the particles of a solid which they will undergo without offering any resistance due to their mutual action, I might take a complex motion, composed of a shear and a rotation together, or some other kind of deformation.

Let us suppose such an alteration picked out and defined as the one which means simple rotation, then the type, according to which all bodies will alter by this rotation, is fixed.

Looking at the movements of this kind, we should say that the objects were altering their shape as well as rotating. But to the inhabitants of that world they would seem to be unaltered, and our figures in their motions would seem to them to alter.

In such a world the features of geometry are different. We have seen one such difference in the case of our illustration of the world of shear, where the square on the hypothenuse was equal to the difference, not the sum, of the squares on the sides.

In our illustration we have the same laws of parallel lines as in our ordinary rotation world, but in general the laws of parallel lines are different.

In one of these worlds of a different constitution of matter through one point there can be two parallels to a given line. In another of them there can be none, that is, although a line be drawn parallel to another it will meet it after a time.

Now it was precisely in this respect of parallels that Lobatchewsky and Bolyai discovered these different worlds. They did not think of them as worlds of matter, but they discovered that space did not necessarily mean that our law of parallels is true. They made the distinction between laws of space and laws of matter, although that is not the form in which they stated their results.

The way in which they were led to these results was the following. Euclid had stated the existence of parallel lines as a postulate—putting frankly this unproved proposition—that one line and only one parallel to a given straight line can be drawn, as a demand, or something that must be assumed. The words of his ninth postulate are these: "If a straight line meeting two other straight lines makes the interior angles on the same side of it equal to two right angles, the two straight lines will never meet."

The mathematicians of later ages did not like this bald assumption, and not being able to prove the proposition they called it an axiom—the eleventh axiom.

Many attempts were made to prove this axiom; no one doubted of its truth, but no means could be found to demonstrate it. At last an Italian, Sacchieri, unable to find a proof, said: "Let us suppose it not true." He deduced the results of there being possible two parallels to one given line through a given point, but feeling the waters too deep for the human reason, he devoted the latter half of his book to disproving what he had assumed in the first part.

Then Bolyai and Lobatchewsky with firm step entered on the forbidden path. There can be no greater evidence of the indomitable nature of the human spirit, or of its manifest destiny to conquer all those limitations which bind it down within the sphere of sense than this grand assertion of Bolyai and Lobatchewsky.

Fig. 31.

Take a line AB and a point C. We say and see and know that through C can only be drawn one line parallel to AB.

But Bolyai said: "I will draw two." Let CD be parallel to AB, that is, not meet AB however far produced, and let lines beyond CD also not meet AB; let there be a certain region between CD and CE, in which no line drawn meets AB. CE and CD produced backwards through C will give a similar region on the other side of C.

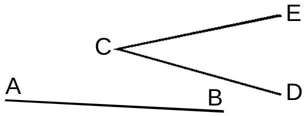


Fig. 32.

Nothing so triumphantly, one may almost say so insolently, ignoring of sense had ever been written before. Men had struggled against the limitations of the body, fought them, despised them, conquered them. But no one had ever thought simply as if the body, the bodily eyes, the organs of vision, all this vast experience of space, had never existed. The age-long contest of the soul with the body, the struggle for mastery, had come to a culmination. Bolyai and Lobatchewsky simply thought as if the body was not. The struggle for dominion, the strife and combat of the soul were over; they had mastered, and the Hungarian drew his line.

Can we point out any connection, as in the case of Parmenides, between these speculations and higher space? Can we suppose it was any inner perception by the soul of a motion not known to the senses, which resulted in this theory so free from the bonds of sense? No such supposition appears to be possible.

Practically, however, metageometry had a great influence in bringing the higher space to the front as a working hypothesis. This can be traced to the tendency of the mind to move in the direction of least resistance. The results of the new geometry could not be neglected, the problem of parallels had occupied a place too prominent in the development of mathematical thought for its final solution to be neglected. But this utter independence of all mechanical considerations, this perfect cutting loose from the familiar intuitions, was so difficult that almost any other hypothesis was more easy of acceptance, and when Beltrami showed that the geometry of Lobatchewsky and Bolyai was the geometry of shortest lines drawn on certain curved surfaces, the ordinary definitions of measurement being retained, attention was drawn to the theory of a higher space. An illustration of Beltrami's theory is furnished by the simple consideration of hypothetical beings living on a spherical surface.

Let ABCD be the equator of a globe, and AP, BP, meridian lines drawn to the pole, P. The lines AB, AP, BP would seem to be perfectly straight to a person moving on the surface of the sphere, and unconscious of its curvature. Now AP and BP both make right angles with AB. Hence they satisfy the definition of parallels. Yet they meet in P. Hence a being living on a spherical surface, and unconscious of its curvature, would find that parallel lines would meet. He would also find that the angles in a triangle were greater than two right angles. In the triangle PAB, for instance, the angles at A and B are right angles, so the three angles of the triangle PAB are greater than two right angles.

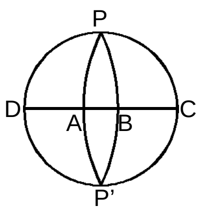


Fig. 33.

Now in one of the systems of metageometry (for after Lobatchewsky had shown the way it was found that other systems were possible besides his) the angles of a triangle are greater than two right angles.

Thus a being on a sphere would form conclusions about his space which are the same as he would form if he lived on a plane, the matter in which had such properties as are presupposed by one of these systems of geometry. Beltrami also discovered a certain surface on which there could be drawn more than one "straight" line through a point which would not meet another given line. I use the word straight as equivalent to the line having the property of giving the shortest path between any two points on it. Hence, without giving up the ordinary methods of measurement, it was possible to find conditions in which a plane being would necessarily have an experience corresponding to Lobatchewsky's geometry. And by the consideration of a higher space, and a solid curved in such a higher space, it was possible to account for a similar experience in a space of three dimensions.

Now, it is far more easy to conceive of a higher dimensionality to space than to imagine that a rod in rotating does not move so that its end described a circle. Hence, a logical conception having been found harder than that of a four-dimensional space, thought turned to the latter as a simple explanation of the possibilities to which Lobatchewsky had awakened it. Thinkers became accustomed to deal with the geometry of higher space—it was Kant, says Veronese, who first used the expression of "different spaces"—and with familiarity the inevitableness of the conception made itself felt.

From this point it is but a small step to adapt the ordinary mechanical conceptions to a higher spatial existence, and then the recognition of its objective existence could be delayed no longer. Here, too, as in so many cases, it turns out that the order and connection of our ideas is the order and connection of things.

What is the significance of Lobatchewsky's and Bolyai's work?

It must be recognised as something totally different from the conception of a higher space; it is applicable to spaces of any number of dimensions. By immersing the conception of distance in matter to which it properly belongs, it promises to be of the greatest aid in analysis for the effective distance of any two particles is the product of complex material conditions and cannot be measured by hard and fast rules. Its ultimate significance is altogether unknown. It is a cutting loose from the bonds of sense, not coincident with the recognition of a higher dimensionality, but indirectly contributory thereto.

Thus, finally, we have come to accept what Plato held in the hollow of his hand; what Aristotle's doctrine of the relativity of substance implies. The vast universe, too, has its higher, and in recognising it we find that the directing being within us no longer stands inevitably outside our systematic knowledge.

CHAPTER VI

THE HIGHER WORLD

It is indeed strange, the manner in which we must being to think about the higher world.

Those simplest objects analogous to those which are about us on every side in our daily experience such as a door, a table, a wheel are remote and unrecognisable in the world of four dimensions, while the abstract ideas of rotation, stress and strain, elasticity into which analysis resolves the familiar elements of our daily experience are transferable and applicable with no difficulty whatever. Thus we are in the unwonted position of being obliged to contrast the daily and habitual experience of a four-dimensional being, from a knowledge of the abstract theories of the space, the matter, the motion of it; instead of, as in our case, passing to the abstract theories form the richness of sensible things.

What would be a wheel in four dimensions? What the shafting for the transmission of power which a four-dimensional being would use.

The four-dimensional wheel, and the four-dimensional shafting are what will occupy us for these few pages. And it is no futile or insignificant enquiry. For in the attempt to penetrate into the nature of the higher, to grasp within our ken that which transcends all analogies, because what we know are merely partial views of it, the purely material and physical path affords a means of approach pursuing which we are in less likelihood of error than if we use the more frequently trodden path of framing conceptions which in their elevation and beauty seem to us ideally perfect.

For where we are concerned with our own thoughts, the development of our own ideals, we are as it were on a curve, moving at any moment in a direction of tangency. Whither we go, what we set up and exalt as perfect, represents not the true tread of the curve, but our own direction at the present—a tendency conditioned by the past, and by a vital energy of motion essential but only true when perpetually modified. That eternal corrector of our aspirations and ideals, the material universe draws sublimely away from the simplest things we can touch or handle to the infinite depths of starry space, in one and all uninfluenced by what we think or feel, presenting unmoved fact to which, think it good or think it evil, we can but conform, yet out of all that impassivity with a reference to something beyond our individual hopes and fears supporting us and giving us our being.

And to this great being we come with the question: "You, too, what is your higher?"

Or to put it in a form which will leave our conclusions in the shape of no barren formula, and attacking the problem on its most assailable side: "What is the wheel and the shafting of the four-dimensional mechanic?"

In entering on this enquiry we must make a plan of procedure. The method which I shall adopt is to trace out the steps of reasoning by which a being confined to movement in a two-dimensional world could arrive at a conception of our turning and rotation, and then to apply an analogous process to the consideration of the higher movements. The plane being must be imagined as no abstract figure, but as a real body possessing all three dimensions. His limitation to a plane must be the result of physical conditions.

We will therefore think of him as of a figure cut out of paper placed on a smooth plane. Sliding over this plane, and coming into contact with other figures equally thin as he in the third dimension, he will apprehend them only by their edges. To him they will be completely bounded by lines. A "solid" body will be to him a two-dimensional extent, the interior of which can only be reached by penetrating through the bounding lines.

Now such a plane being can think of our three-dimensional existence in two ways.

First, he can think of it as a series of sections, each like the solid he knows of extending in a direction unknown to him, which stretches transverse to his tangible universe, which lies in a direction at right angles to every motion which he made.

Secondly, relinquishing the attempt to think of the three-dimensional solid body in its entirety he can regard it as consisting of a number of plane sections, each of them in itself exactly like the two-dimensional bodies he knows, but extending away from his two-dimensional space.

A square lying in his space he regards as a solid bounded by four lines, each of which lies in his space.

A square standing at right angles to his plane appears to him as simply a line in his plane, for all of it except the line stretches in the third dimension.

He can think of a three-dimensional body as consisting of a number of such sections, each of which starts from a line in his space.

Now, since in his world he can make any drawing or model which involves only two dimensions, he can represent each such upright section as it actually is, and can represent a turning from a known into the unknown dimension as a turning from one to another of his known dimensions.

To see the whole he must relinquish part of that which he has, and take the whole portion by portion.

Consider now a plane being in front of a square, fig. 34. The square can turn about any point in the plane—say the point A. But it cannot turn about a line, as AB. For, in order to turn about the line AB, the square must leave the plane and move in the third dimension. This motion is out of his range of observation, and is therefore, except for a process of reasoning, inconceivable to him.

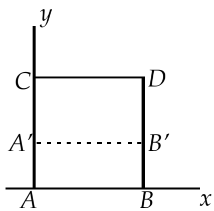


Fig. 34.

Rotation will therefore be to him rotation about a point. Rotation about a line will be inconceivable to him.

The result of rotation about a line he can apprehend. He can see the first and last positions occupied in a half revolution about the line AC. The result of such a half revolution is to place the square ABCD on the left hand instead of on the right hand of the line AC. It would correspond to a pulling of the whole bode ABCD through the line AC, or to the production of a solid body which was the exact reflection of it in the line AC. It would be as if the square ABCD turned into its image, the line acting as a mirror. Such a reversal of the positions of the parts of the square would be impossible in his space. The occurrence of it would be a proof of the existence of a higher dimensionality.

Let him now, adopting the conception of a three-dimensional body as a series of sections lying, each removed a little farther than the preceding one, in direction at right angles to his plane, regard a cube, fig. 36, as a series of sections, each like the square which forms its base, all rigidly connected together.

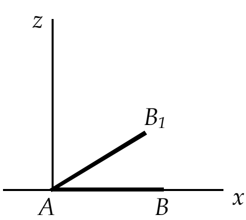


Fig. 35.

If now he turns the square about the point A in the plane of zy, each parallel section turns with the square he moves. In each of the sections there is a point of rest, that vertically over A. Hence he would conclude that in the turning of a three-dimensional body there is one line which is at rest. That is, a three-dimensional turning is a turning about a line.

In a similar way let us regard ourselves as limited to a three-dimensional world by a physical condition. Let us imagine that there is a direction at right angles to every direction in which we can move, and that we are prevented from passing in this direction by a vast solid, that against which in every movement we make we slip as the plane being slips against his plane sheet.

We can then consider a four-dimensional body as consisting of a series of sections, each parallel to our space, and each a little further off than the preceding on the unknown dimension.

Take the simplest four-dimensional body—one which begins as a cube, fig. 36, in our space, and consists of sections, each a cube like fig. 36, lying away from our space. If we turn the cube which is its base in our space about a line, if, *e.g.*, in fig. 36 we turn the cube about the line AB, not only it but each of the parallel cubes moves about a line. The cube we see moves about the line AB, the cube beyond it about a line parallel to AB and so on. Hence the whole four-dimensional body moves about a plane, for the assemblage of these lines is our way of thinking about the plane which, starting from the line as in our space, runs off in the unknown direction.

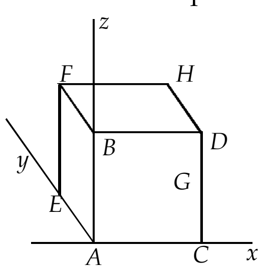


Fig. 36.

In this case all that we see of the plane about which the turning takes place is the line AB.

But it is obvious that the axis plane may lie in our space. A point near the plane determines with it a three-dimensional space. When it begins to rotate around the plane it does not move anywhere in this three-dimensional space, but moves out of it. A point can no more rotate round a plane in three-dimensional space than a point can move round a line in two-dimensional space.

We will now apply the second of the modes of representation to this case of turning about a plane, building up our analogy step by step from the turning in a plane about a point and that in space about a line, and so on.

In order to reduce our considerations to those of the greatest simplicity possible, let us realise how the plane being would think of the motion by which a square is turned round a line.

Let fig. 34, ABCD be a square on his plane, and represent the two dimensions of his space by the axes Ax, Ay.

Now the motion in which the square is turned over about the line AC involves the third dimension.

He cannot represent the motion of the whole square in its turning, but he can represent the motions of parts of it. Let the third axis perpendicular to the plane of the paper be called the axis of z. Of the three axes, x, y, z, the plane being can represent any two in his space. Let him then drawn, in fig. 35, two axes, x and z. Here he has in his plane a representation of what exists in the plane which goes off perpendicularly to his space.

In this representation the square would not be shown, for in the plane of xy simple the line AB of the square is contained.

The plane being then would have before him, in fig. 35, the representation of one line AB of his square and two axes, x and z, at right angles. Now it would be obvious to him that, by a turning such as he knows, by a rotation about a point, the line AB can turn round A, and occupying all the intermediate positions, such as AB_1, come, after half a revolution to lie as Ax produced through A.

Again, just as he can represent the vertical plane through AB, so he can represent the vertical plane through A′B′, fig. 34, and in a like manner can see that the line A′B′ can turn about the point A′ till it lies in the opposite direction from that which it runs at first.

Now these two turnings are not inconsistent. In his plane, if AB is turned about A, and A′B′ about A′, the consiststency of the square would be destroyed, it would be an impossible motion for a rigid body to perform. But in the turning which he studies portion by portion there is nothing inconsistent. Each line in the square can turn in this way, hence he would realise the turning of the whole square as the sum of a number of turnings of isolated parts. Such turnings, if they took place in his plane, would be inconsistent, but by virtue of a third dimension they are consistent, and the result of them all is that the square turns about the line AC and lies in a position in which it is the mirror image of what it was in its first position. Thus he can realise a turning about a line by relinquishing one of his axes, and representing his body part by part.

Let us apply this method to the turning of a cube so as to become the mirror image of itself. In our space we can construct three independent axis, x, y, z, shown in fig. 36. Suppose that there is a fourth axis, w, at right angles to each and every one of them. We cannot, keeping all three axes, x, y, z, represent w in our space; but if we relinquish one of our three axes we can let the fourth axis take its place, and we can represent what lies in the square, determined by the two axes we retain and the fourth axis.

Let us suppose that we let the *y* axis drop, and that we represent the *w* axis as occupying its direction. We have in fig. 37 a drawing of what we should then see of the cube. The square ABCD remains unchanged, for that is in the play of *xz*, and we still have that plane. But from this plane the cube stretches out in the direction of the *y* axis. Now the *y* axis is gone, and so we have no more of the cube than the face ABCD.

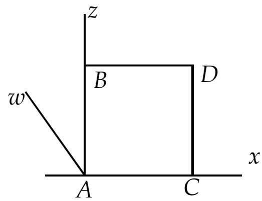


Fig. 37.

Considering now this face ABCD, we see that it is free to turn about the line AB. It can rotate in the *x* to *w* direction about this line. In fig. 38 it is shown on its way, and it can evidently continue this rotation till it lies on the other side of the *z* axis in the plane of *xz*.

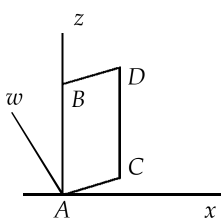


Fig. 38.

We can also take a section parallel to the face ABCD, and then letting drop all our space except the plane of that section, introduce the *w* axis, running in the old *y* direction. This section can be represented by the same drawing, fig. 38, and we see that it can rotate about the line on its left until it swings half way round and runs in the opposite direction to that which it ran in before. These turnings of the different sections are not inconsistent, and taken all together they will bring the cube from the position shown in fig. 36 to that shown in fig. 41.

Since we have three axes at our disposal in our space, we are not obliged to represent the *w* axis by any particular one. We may let any axis we like disappear, and let the fourth axis take its place.

In fig. 36 suppose the *z* axis to go. We have then simply the plane of *xy* and the square base of the cube ACEG, fig. 39, is all that could be seen of it. Let now the *w* axis take the place of the *z* axis and we have, in fig. 39 again, a representation of the space of *xyw*, in which all that exists of the cube is its square base. Now, by a turning of *x* to *w*, this base can rotate around the line AE, it is shown on its way in fig. 40, and finally it will, after half a revolution, lie on the other side of the *y* axis. In a similar way we may rotate sections parallel to the base of the *xw* rotation, and each of them comes to run in the opposite direction from that which they occupied at first.

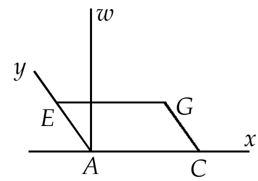


Fig. 39.

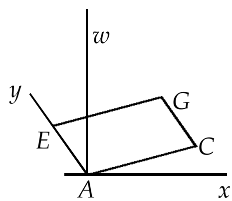


Fig. 40.

Thus again the cube comes from the position of fig. 36 to that of fig. 41. In this *x* to *w* turning, we see that it takes place by the rotations of sections parallel to the front face about lines parallel to AB, or else we may consider it as consisting of the rotations of sections parallel to the base about lines parallel to AE. It is a rotation of the whole cube about the plane ABEF. Two separate sections could not rotate about two separate lines in our space without conflicting, but their motion is consistent when we consider another dimension. Just, then, as a plane being can think of rotation about a line as a rotation about a number of points, these rotations not interfering as they would if they took place in his two-dimensional space, so we can think of a rotation about a

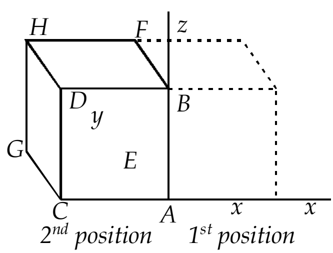


Fig. 41.

plane as the rotation of a number of sections of a body about a number of lines in a plane, these rotations not being inconsistent in a four-dimensional space as they are in three-dimensional space.

We are not limited to any particular direction for the lines in the plane about which we suppose the rotation of the particular sections to take place. Let us draw the section of the cube, fig. 36, through A, F, C, B, forming a sloping plane. Now since the fourth dimension is at right angles to every line in our space it is at right angles to this section also. We can represent our space by drawing an axis at right angles to the plane ACEG, our space is then determined by the plane ACEG, and the perpendicular axis. If we let this axis drop and suppose the fourth axis, w, to take its place, we have a representation of the space which runs off in the fourth dimension from the plane ACEG. In this space we shall see simply the section ACEG of the cube, and nothing else, for one cube does not extend to any distance in the fourth dimension.

If, keeping this plane, we bring in the fourth dimension, we shall have a space in which simply this section of the cube exists and nothing else. This section can turn about the line AF, and parallel sections can turn about parallel lines. Thus in considering the rotation about a plane we can draw any lines we like and consider the rotation as taking place in sections about them.

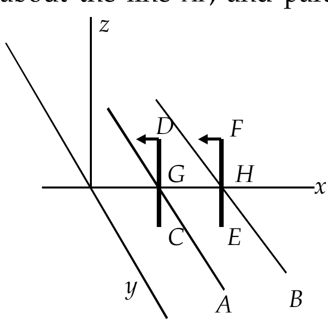


Fig. 42.

To bring out this point more clearly, let us take two parallel lines, A and B, in the space of xyz, and let CD and EF be two rods running above and below the plane of xy, from these lines. If we turn these rods in our space about the lines A and B, as the upper end of one, F, is going down, the lower end of the other, C, will be coming up. They will meet and conflict. But it is quite possible for these two rods each of them to turn about the two lines without altering their relative distances.

To see this suppose the *y* axis to go, and let the *w* axis take its place. We shall see the lines A and B no longer, for they run in the *y* direction from the points G and H.

Fig. 43 is a picture of the two rods seen in the space of *xzw*. If they rotate in the direction shown by the arrows—in the *z* to *w* direction—they move parallel to one another, keeping their relative distances. Each will rotate about its own line, but their rotation will not be inconsistent with their forming part of a rigid body.

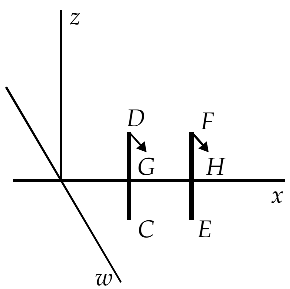


Fig. 43.

Now we have but to suppose a central plane with rods crossing it at every point, like CD and EF cross the plane of *xy*, to have an image of a mass of matter extending equal distances on each side of a diametral plane. As two of these rods can rotate round, so can all, and the whole mass of matter can rotate round its diametral plane.

This rotation round a plane corresponds, in four dimensions, to the rotation round an axis in three dimensions. Rotation of a body round a plane is the analogue of rotation of a rod round an axis.

In a plane we have rotation round a point, in three-space rotation round an axis line, in four-space rotation round an axis plane.

The four-dimensional being's shaft by which he transmits power is a disk rotating round its central plane—

the whole contour corresponds to the ends of an axis of rotation in our space. He can impart the rotation at any point and take it off at any other point on the contour, just as rotation round a line can in three-space be imparted at one end of a rod and taken off at the other end.

A four-dimensional wheel can easily be described from the analogy of the representation which a plane being would form from himself of one of our wheels.

Suppose a wheel to move transverse to a plane, so that the whole disk, which I will consider to be solid and without spokes, can at the same time into contact with the plane. It would appear as a circular portion of plane matter completely enclosing another and smaller portion—the axle.

This appearance would last, supposing the motion of the wheel to continue until it had traversed the plane by the extent of its thickness, when there would remain in the plane only the small disk which is the section of the axle. There would be no means obvious in the plane at first by which the axle could be reached, except by going through the substance of the wheel. But the possibility of reaching it without destroying the substance of the wheel would be shown by the continued existence of the axle section after that of the wheel had disappeared.

In a similar way a four-dimensional wheel moving transverse to our space would appear first as a solid sphere, completely surrounding a smaller solid sphere. The outer sphere would represent the wheel, and would last until the wheel had traversed our space by a distance equal to its thickness. Then the small sphere alone would remain, representing the section of the axle. The large sphere could move round the small one quite freely. Any line in space could be taken as an axis, and round this line the outer sphere could rotate, while the inner sphere remained still. But in all these directions of revolution, there would be in reality one line which remained unaltered, that is the line which stretches away in the fourth dimension, following the axis of the axle. The four-dimensional wheel can rotate in any number of planes, but all these planes are such that there is a line at right angles to them all unaffected by rotation in them.

An objection is sometimes experienced as to this mode of reasoning from a plane world to a higher dimensionality. How artificial, it is argued, this conception of a plane world is. If any real existence confined to a superficies could be shown to exist, there would be an argument for one relative to which our three-dimensional existence is superficial. But, both on the one side and the other of the space we are familiar with, spaces either with less or more than three dimensions are merely arbitrary conceptions.

In reply to this I would remark that a plane being having one less dimension than our three would have one-third of our possibilities of motion, which we have only one-fourth less than those of the higher space. It may very well be that there may be a certain amount of freedom of motion which is demanded as a condition of an organised existence, and that no material existence is possible with a more limited dimensionality than ours. This is well seen if we try to construct the mechanics of a two-dimensional world. No tube could exist, for unless joined together completely at one end two parallel lines would be completely separate. The possibility of an organic structure, subject to conditions such as this, is highly problematical; yet, possibly in the convolutions of the brain there may be a mode of existence to be described as two-dimensional.

We have but to suppose the increase in surface and the diminution in mass carried on to a certain extent to find a region which, though without mobility of the constituents, would have to be described as two-dimensional.

But, however artificial the conception of a plane being may be, it is not the less to be used in passing to the conception of a greater dimensionality than ours, and hence the validity of the first part of the objection altogether disappears directly we find evidence for such a state of being.

The second part of the objection has more weight. How is it possible to conceive that in a four-dimensional space and creatures should be confined to a three-dimensional existence?

In reply I would say that we know as a matter of fact that life is essentially a phenomenon of surface. The amplitude of the movements which we can make is much greater along the surface of the earth than it is up or down.

Now we have but to conceive the extent of a solid surface increased, while the motions possible transverse to it are diminished in the same proportion, to obtain the image of a three-dimensional world in four-dimensional space.

And as our habitat is the meeting of air and earth on the world, so we must think of the meeting place of two as affording the condition for our universe. The meeting of what two? What can that vastness be in the higher space which stretches in such a perfect level that our astronomical observations fail to detect the slightest curvature?

The perfection of the level suggests a liquid—a lake amidst what vast scenery!—whereon the matter of the universe floats speck-like.

But this aspect of the problem is like what are called in mathematics boundary conditions.

We can trace out all the consequences of four-dimensional movements down to their last detail. Then, knowing the mode of action which would be characteristic of the minutest particles, if they were free, we can drawn conclusions from what they actually do of what the constraint of them is. Of the two things, the material conditions and the motion, one is known, and the other can be inferred. If the place of the universe is a meeting of two, there would be a one-sidedness to space. If it lies so that what stretches away in one direction in the unknown is unlike what stretches away in the other, then, as far as the movements which participate in that dimension are concerned, there would be a difference as to which way the motion took place. This would be shown in the dissimilarity of phenomena, which, so far as all three-space movements are concerned, were perfectly symmetrical. To take an instance, merely, for the sake of precising our ideas, not for any inherent probability in it; if it could be shown that the electric current in the positive direction were exactly like the electric current in the negative direction, except for a reversal of the components of motion in three-dimensional space, then the dissimilarity of the discharge from the positive and negative poles would be an indication of a one-sidedness to our space. The only cause of differences in the two discharges would be due to a component in the fourth dimension, which directed in one direction transverse to our space, met with a different resistance to that which it met when directed in the opposite direction.

CHAPTER VII

THE EVIDENCES FOR A FOURTH DIMENSION

THE method necessarily to be employed in the search for the evidences of a fourth dimension, consists primarily in the formation of the conceptions of four-dimensional shapes and motions. When we are in possession of these it is possible to call in the aid of observation, without them we may have been all our lives in the familiar presence of a four-dimensional phenomenon without ever recognising its nature.

To take one of the conceptions we have already formed, the turning of a real thing into its mirror image would be an occurrence which it would be hard to explain, except on the assumption of a fourth dimension.

We know of no such turning. But there exist a multitude of forms which show a certain relation to a plane, a relation of symmetry, which indicates more than an accidental juxtaposition of parts. In organic life the universal type is of right- and left-handed symmetry, there is a plane on each side of which the parts correspond. Now we have seen that in four dimensions a plane takes the place of a line in three dimensions. In our space, rotation about an axis is the type of rotation, and the origin of bodies symmetrical about a line as the earth is symmetrical about an axis can easily be explained. But where there is symmetry about a plane no simple physical motion, such as we are accustomed to, suffices to explain it. In our space a symmetrical object must be built up by equal additions on each side of a central plane. Such additions about such a plane are as little likely as any other increments. The probability against the existence of symmetrical form in inorganic nature is overwhelming in our space, and in organic forms they would be as difficult of production as any other variety of configuration. To illustrate this point we may take the child's amusement of making from dots of ink on a piece of paper a life-like representation of an insect by simply folding the paper over. The dots spread out on a symmetrical line, and give the impression of a segmented form with antennæ and legs.

Now seeing a number of such figures we should naturally infer a folding over. Can, then, a folding over in four-dimensional space account for the symmetry of organic forms? The folding cannot of course be of the bodies we see, but it may be of those minute constituents, the ultimate elements of living matter which, turned in one way or the other, become right- or left-handed, and so produce a corresponding structure.

There is something in life not included in our conceptions of mechanical movement. Is this something a four-dimensional movement?

If we look at it from the broadest point of view, there is something striking in the fact that where life comes in there arises an entirely different set of phenomena to those of the inorganic world.

The interest and values of life as we know it in ourselves, as we know it existing around us in subordinate forms, is entirely and completely different to anything which inorganic nature shows. And in living beings we have a kind of form, a disposition of matter which is entirely different from that shown in inorganic matter.

Right- and left-handed symmetry does not occur in the configurations of dead matter. We have instances of symmetry about an axis, but not about a plane. It can be argued that the occurrence of symmetry in two dimensions involves the existence of a three-dimensional process, as when a stone falls into water and makes rings of ripples, or as when a mass of soft material rotates about an axis. It can be argued that symmetry in any number of dimensions is the evidence of an action in a higher dimensionality. Thus considering living beings, there is an evidence both in their structure, and in their different mode of activity, of a something coming in from without into the inorganic world.

And the objections which will readily occur, such as those derived from the forms of twin crystals and the theoretical structure of chemical molecules, do not invalidate the argument; for in these forms too the presumable seat of the activity producing them lies in that very minute region in which we necessarily place the seat of a four-dimensional movement.

In another respect also the existence of symmetrical forms is noteworthy. It is puzzling to conceive how two shapes exactly equal can exist which are not superposible. Such a pair of symmetrical figures as the two hands, right and left, show either a limitation in our power of movement, by which we cannot superpose the one on the other, or a definite influence and compulsion of space on matter, inflicting limitations which are additional to those of the properties of the parts.

We will, however, put aside the argument to be drawn from the consideration of symmetry as inconclusive, retaining one valuable indication which they afford. If it is in virtue of a four-dimensional motion that symmetry exists, it is only in the very minute particles of bodies that that motion is to be found, for there is no such thing as a bending over in four dimensions of any object of a size which we can observe. The region of the extremely minute is the one, then, which we shall have to investigate. We must look for some phenomenon which, occasioning movements of the kind we know, still is itself inexplicable as any form of motion which we know.

Now in the theories of the actions of the moisture particles of bodies on one another, and in the motions of the ether, mathematicians have tacitly assumed that the mechanical principles are the same as those which prevail in the case of bodies which can be observed, it has been assumed without proof that the conception of motion being three-dimensional, holds beyond the region from observations in which it was formed.

Hence it is not from any phenomena explained by mathematics that we can derive a proof of four dimensions. Every phenomenon that has been explained is explained as three-dimensional. And, moreover, since in the region of the very minute we do not find rigid bodies acting on each other at a distance, but elastic substances and continuous fluids such as ether, we shall have a double task.

We must form the conceptions of the possible movements of elastic and liquid four-dimensional matter, before we can begin to observe. Let us, therefore, take the four-dimensional rotation about a plane, and enquire what it becomes in the case of extensible fluid substances. If four-dimensional movements exist, this kind of rotation must exist, and the finer portions of matter must exhibit it.

Consider for a moment a rod of flexible and extensible material. It can turn about an axis, even if not straight; a ring of india rubber can turn inside out.

What would this be in the case of four dimensions?

Let us consider a sphere of our three-dimensional matter having a definite thickness. To represent this thickness let us suppose that from every point of the sphere in fig. 44 rods project both ways, in and out, like D and F. We can only see the external portion, because the internal parts are hidden by the sphere.

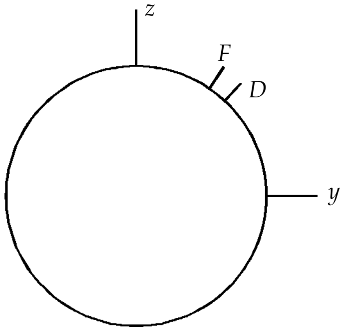


Fig. 44.
Axis of x running towards the observer.

In this sphere the axis of x is supposed to come towards the observer, the axis of z to run up, the axis of y to go to the right.

Now take the section determined by the xy plane. This will be a circle as shown in fig. 45. If we let drop the x axis, this circle is all we have of the sphere. Letting the w axis now run in the plane of the old x axis we have the space yzw and in this space all that we have of the sphere is the circle. Fig 45 then represents all that there is of the sphere in the space of yzw. In this space it is evident that the rods CD and EF can turn round the circumference as an axis. If the matter of the spherical shell is sufficiently extensible to allow the particles C and E to become as widely separated as they would in the positions D and F, then

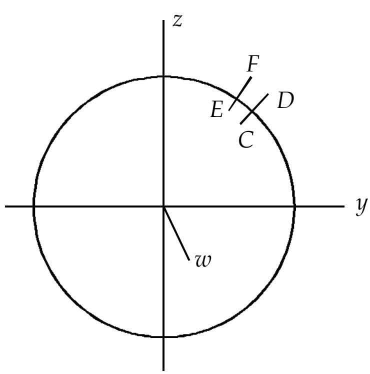


Fig. 45.

the strip of matter represented by CD and EF and a multitude of rods like them can turn round the circular circumference.

Thus this particular section of the sphere can turn inside out, and what holds for any one section holds for all. Hence in four dimensions the whole sphere can, if extensible. turn inside out. Moreover, any part of it--a bowl-shaped portion for instance—can turn inside out, and so on round and round.

This is really no more than we had before in the rotation about a plane, except that we see that the plane can, in the case of an extensible matter, be curved, and still play the part of an axis.

If we suppose the spherical shell to be of four-dimensional matter, our representation will be a little different. Let us suppose there to be a small thickness in the matter in the fourth dimension. This would make no difference in fig. 44, for that merely shows the view in the xyz space. But when the x axis is let drop, and the w axis comes in, then the rods CD and EF which represent the matter of the shell, will have a certain thickness perpendicular to the plane of the paper on which they are drawn. If they have a thickness in the fourth dimension they will show this thickness when looked at from the direction of the w axis.

Supposing these rods, then, to be small slabs strung on the circumference of the circle in fig. 45, we see that there will not be in this case either any obstacle to their turning round the circumference. We can have a shell of extensible matter or of fluid material turning inside out in four dimensions.

And we must remember than in four dimensions there is no such thing as rotation round an axis. If we want to investigate the motion of fluids in four dimensions we must take a movement about an axis in our space, and find the corresponding movement about a plane in four space.

Now, of all the movements which take place in fluids, the most important from a physical point of view is vortex motion.

A vortex is a whirl or eddy—it is shown in the gyrating wreaths of dust seen on a summer day; it is exhibited on a larger scale in the destructive march of a cyclone.

A wheel whirling round will throw off the water on it. But when this circling motion takes place in a liquid itself it is strangely persistent. There is, of course, a certain cohesion between the particles of water by which they mutually impede their motions. But in a liquid devoid of friction, such that every particle is free from lateral cohesion on its path of motion, it can be shown that a vortex or eddy separates from the mass of the fluid a certain portion, which always remains in that vortex.

The shape of the vortex may alter, but it always consists of the same particles of the fluid.

Now, a very remarkable fact about such a vortex is that the ends of the vortex cannot remain suspended and isolated in the fluid. They must always run to the boundary of the fluid. An eddy in water that remains half way down without coming to the top is impossible.

The ends of a vortex must reach the boundary of a fluid—the boundary may be external or internal—a vortex may exist between two objects in the fluid, terminating one end on each object, the objects being internal boundaries of the fluid. Again, a vortex may have its ends linked together, so that it forms a ring. Circular vortex rings of this description are often seen in puffs of smoke, and that the smoke travels on in the ring is a proof that the vortex always consists of the same particles of air.

Let us now enquire what a vortex would be in a four-dimensional fluid.

We must replace the line axis by a plane axis. We should have therefore a portion of fluid rotating round a plane.

We have seen that the contour of this plane corresponds with the ends of the axis line. Hence such a four-dimensional vortex must have its rim on a boundary of the fluid. There would be a region of vorticity with a contour. If such a rotation were started at one part of a circular boundary, its edges would run round the boundary in both directions, till the whole interior region was filled with the vortex sheet.

A vortex in a three-dimensional liquid may consist of a number of vortex filaments lying together producing a tube, or rod of vorticity.

In the same way we can have in four dimensions a number of vortex sheets alongside each other, each of which can be thought of as a bowl-shaped portion of a spherical shell turning inside out. The rotation takes place at any point not in the space occupied by the shell, but from that space to the fourth dimension and round back again.

Is there anything analogous to this within the range of our observation?

An electric current answers this description in every respect. Electricity does not flow through a wire. Its effect travels both ways from the starting point along the wire. The spark which shows its passing midway in its circuit is later than that which occurs at points near its starting point on either side of it.

Moreover, it is known that the action of the current is not in the wire. It is in the region enclosed by the wire, this is the field of force, the locus of the exhibition of the effects of the current.

And the necessity of a conducting circuit for a current is exactly that which we should expect if it were a four-dimensional vortex. According to Maxwell every current forms a closed circuit, and this, from the four-dimensional point of view, is the same as saying a vortex must have its ends on a boundary of the fluid.

Thus, on the hypothesis of a fourth dimension, the rotation of the fluid ether would give the phenomenon of an electric current. We must suppose the ether to be full of movement, for the more we examine the conditions which prevail in the obscurity of the minute, the more we find that an unceasing and perpetual motion reigns. Thus we may say that the conception of a fourth dimension means that there must be a phenomenon which presents the characteristics of electricity.

We know that light is an electro-magnetic action, and that so far from being a special and isolated phenomenon this electric action is universal in the realm of the minute. Hence, may we not conclude that, so far from the fourth dimension being remote and far away, being a thing of symbolic import, a term for the explanation of dubious facts by a more obscure theory, it is really the most important fact within our knowledge. Our three-dimensional world is superficial. These processes, which really lie at the basis of all phenomena of matter, escape our observation by their minuteness, but reveal to our intellect an amplitude of motion surpassing any that we can see. In such shapes and motions there is a realm of the utmost intellectual beauty, and one to which our symbolic methods apply with a better grace than they do to those of three dimensions.

CHAPTER VIII
THE USE OF FOUR DIMENSIONS IN THOUGHT

HAVING held before ourselves this outlines of a conjecture of the world as four-dimensional, having roughly thrown together those facts of movement which we can see apply to our actual experience, let us pass to another branch of our subject.

The engineer uses drawings, graphical constructions, in a variety of manners. He has, for instance, diagrams which represent the expansion of steam, the efficiency of his valves. These exist alongside the actual plans of his machines. They are not the pictures of anything really existing, but enable him to think about the relations which exist in his mechanics.

And so, besides showing us the actual existence of that world which lies beneath the one of visible movements, four-dimensional space enables us to make idea constructions which serve to represent the relations of things, and throw what would otherwise be obscure into a definite and suggestive form.

From amidst the great variety of instances which lies before me I will select two, one dealing with a subject of slight intrinsic interest, which however gives within a limited field a striking example of the method of drawing conclusions and the use of higher space figures.*

The other instance is chosen on account of the bearing it has on our fundamental conceptions. In it I try to discover the real meaning of Kant's theory of experience.

The investigation of the properties of numbers is much facilitated by the fact that relations between numbers are themselves able to be represented as numbers—*e.g.* 12, and 3 are both numbers, and the relation between them is 4, another number. The way is thus opened for a process of constructive theory, without there being any necessity for a recourse to another class of concepts besides that which is given in the phenomena to be studied.

The discipline of number thus created is of great and varied applicability, but it is not solely as quantitative that we learn to understand the phenomena of nature. It is not possible to explain the properties of matter by number simply, but all the activities of matter are energies in space. They are numerically definite and also, we may say, directedly definite, *i.e.* definite in direction.

Is there, then, a body of doctrine about space which, like that of number, is available in science? It is needless to answer: Yes; geometry. But there is a method lying alongside the ordinary methods of geometry, which tacitly used and presenting an analogy to the method of numerical thought deserves to be brought into greater prominence than it usually occupies.

The relation of numbers is a number.

Can we say in the same way that the relation of shapes is a shape?

We can.

* It is suggestive also in another respect, because it shows very clearly that in our processes of thought there are in play faculties other than logical; in it the origin of the idea which proves to be justified is drawn from the consideration of symmetry, a branch of the beautiful.

To take an instance chosen on account of its ready availability. Let us take two right-angled triangles of a given hypothenuse, but having sides of different lengths (fig. 46). These triangles are shapes which have a certain relation to each other. Let us exhibit their relation as a figure.

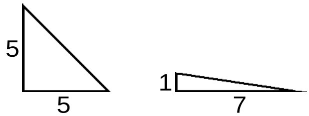


Fig. 46.

Draw two straight lines at right angles to each other, the one HL a horizontal level, the other VL a vertical level (fig. 47). By means of these two co-ordinating lines we can represent a double set of magnitudes; one set as distances to the right of the vertical level, the other as distances above the horizontal level, a suitable unit being chosen.

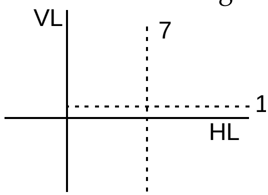


Fig. 47.

Thus the line marked 7 will pick out the assemblage of points whose distance from the vertical level is 7, and the line marked 1 will pick out the point whose distance from the horizontal level is 1. The meeting point of these two lines, 7 and 1, will define a point which with regard to the one set of magnitudes is 7, with regard to the other is 1. Let us take the sides of our triangles as the two sets of magnitudes in question.

Then the point (7, 1) will represent the triangle whose sides are 7 and 1. Similarly, the point (5, 5)—5, that is, to the right of the vertical level and 5 above the horizontal level—will represent the triangle whose sides are 5 and 5 (fig. 48).

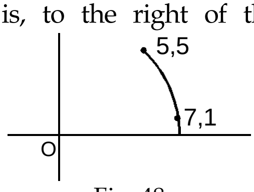


Fig. 48

Thus we have obtained a figure consisting of the two points (7, 1) and (5, 5), representative of our two triangles. But we can go further, and, drawing an arc of a circle about O, the meeting point of the horizontal and vertical levels, which passes through (7, 1) and (5, 5), assert that all the triangles which are right-angled and have a hypothenuse whose square is 50 are represented by the points on this arc.

Thus, each individual of a class being represented by a point, the whole class is represented by an assemblage of points forming a figure. Accepting this representation we can attach a definite and calculable significance to the expression, resemblance, or similarity between two individuals of the class represented, the difference being measure by the length of the line between two representative points. It is needless to multiply examples, or to show how, corresponding to different classes of triangles, we obtain different curves.

A representation of this kind in which an object, a thing in space, is represented as a point, and all its properties are left out, their effect remaining only in the relative position which the representative point bears to the representative points of the other objects, may be called, after the analogy of Sir William Hamilton's hodograph, a "Poiograph."

Representations thus made have the character of natural objects; they have a determined and definite character of their own. Any lack of completeness in them is probably due to a failure in point of completeness of those observations which form the ground of their construction.

Every system of classification is a poiograph. In Mendeléeff's scheme of the elements, for instance, each element is represented as a point, and the relations between the elements are represented by the relations between the points.

So far I have simply brought into prominence processes and considerations with which we are all familiar. But it is worth our while to bring into the full light of our attention our habitual assumptions and processes. It often happens that we find there are two of them which have a bearing on each other, without this dragging into the light, we should have allowed to remain without mutual influence.

There is a fact which it concerns us to take into account in discussing the theory of the poiograph.

With respect to our knowledge of the world we are far from that condition which Laplace imagined when he asserted that an all-knowing mind could determine the future condition of every object, if he knew the co-ordinates of its particles in space, and their velocity at any particular moment.

On the contrary, in the presence of any natural object, we have a great complexity of conditions before us, which we cannot reduce to position in space and date in time.

There is mass, attraction apparently spontaneous, electrical and magnetic properties which must be superadded to spatial configuration. To cut the list short we must say that practically the phenomena of the world present us problems involving many variables, which we must take as independent.

From this it follows that in making poiographs we must be prepared to use space of more than three dimensions. If the symmetry and completeness of our representation is to be of use to us we must be prepared to appreciate and criticise figures of a complexity greater than those in three dimensions. It is impossible to give an example of such a poiograph which will not be merely trivial, without going into details of some kind irrelevant to our subject. I prefer to introduce the irrelevant details rather than treat this part of the subject perfunctorily.

To take an instance of a poiograph which does not lead us into the complexities incident on its application in classificatory science, let us follow Mrs. Alicia Boole Stott in her representation of the syllogism by its means. She will be interested to find that the curious gap she detected has a significance.

A syllogism consists of two statements, the major and the minor premiss, with the conclusion that can be drawn from them. Thus, to take an instance, fig. 49. It is evident, from looking at the successive figures that, if we know that the region M lies altogether within the region P, and also know that the region S lies altogether within the region M, we can conclude that the region S lies altogether within the region P. M is P, major premiss; S is M, minor premiss; S is P, conclusion. Given the first two data we must conclude that S lies in P. The conclusion S is P involves two terms, S and P, which are respectively called the subject and the predicate, the letters S and P being chosen with reference to the parts the notions they designate play in the conclusion. S is the subject of the conclusion, P is the predicate of the conclusion. The major premiss we take to be, that which does not involve S, and here we always write it first.

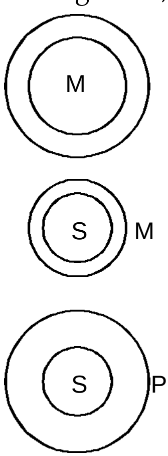


Fig. 49.

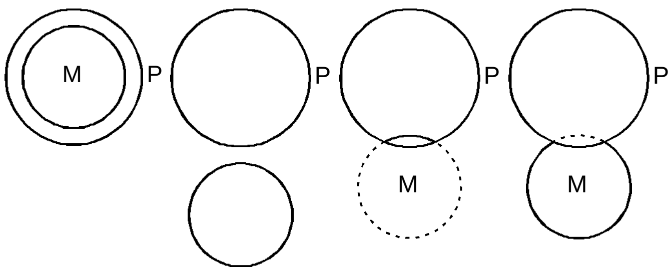
There are several varieties of statement possessing different degrees of universality and manners of assertiveness. These different forms of statement are called the moods.

We will take the major premiss as one variable, as a thing capable of different modifications of the same kind, the minor premiss as another, and the different moods we will consider as defining the variations which these variables undergo.

There are four moods:—

1. The universal affirmative; all M is P, called mood A.
2. The universal negative; no M is P, mood E.
3. The particular affirmative; no M is P, mood I.
4. The particular negative; some M is not P, mood O.

The dotted lines in 3 and 4, fig. 50, denote that it is not known whether or no any objects exist, corresponding



1. Mood A. 2. Mood E. 3. Mood I. 4. Mood O.

Fig. 50.

to the space of which the dotted line forms on delimiting boundary; thus, in mood I we do not know if there are any M's which are not P, we only know some M's are P.

Representing the first premiss in its various moods by regions marked by vertical lines to the right of PQ, we have in fig. 51, running up from the four letters AEIO, four columns, each of which indicates that the major premiss is in the mood denoted by the respective letter. In the first column to the right of PQ is the mood A. Now above the line RS let there be marked off four regions corresponding to the four moods of the minor premiss. Thus, in the first row above us all the region between RS and the first horizontal line above it denotes that the minor premise is in the mood A. The

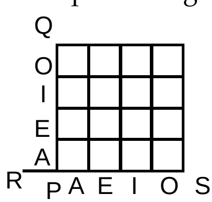


Fig. 51.

letters E, I, O, in the same way show the mood characterising the minor premise in the rows opposite these letters.

We still have to exhibit the conclusion. To do this we must consider the conclusion as a third variable, characterised in its different varieties by four moods—this being the syllogistic classification. The introduction of a third variable involves a change in our type of representation.

Before we started with the regions to the right of a certain line as representing successively the major premiss in its moods; now we must start with the regions to the right of a certain plane. Let LMNR be the plane face of a cube, fig. 52, and let the cube be divided into four parts by vertical sections parallel to LMNR. The variable, the major premiss, is represented by the successive regions which occur to the right of the plane LMNR—that region to which A stands opposite, that slice of the cube, is significative of the mood A. This whole quarter-part of the cube represents that for every part of it the major premiss is in the mood A.

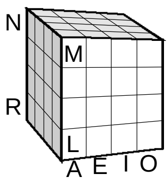


Fig. 52.

In a similar manner the next section, the second with the letter E opposite it, represents that for every one of the sixteen small cubic spaces in it, the major premiss is in the mood E. The third and fourth compartments made by the vertical sections denote the major premise in the moods I and O. But the cube can be divided in other ways by other planes. Let the division, of which four stretch from the front face, correspond to the minor premiss. The first wall of sixteen cubes, facing the observer, has as its characteristic that in each of the small cubes, whatever else may be the case, the minor premiss is in the mood A. The variable—the minor premiss—varies through the phases A, E, I, O, away from the front face of the cube, or the front plane of which the front face is a part.

And now we can represent the third variable in a precisely similar way. We can take the conclusion as the third variable, going through its four phases from the ground plane upwards. Each of the small cubes at the base of the whole cube has this true about it, whatever else may be the case, that the conclusion is, in it, in the mood A. Thus, to recapitulate, the first wall of sixteen small cubes, the first of the four walls which , proceeding from left to right, build up the whole cube, is characterised in each part of it by this, that the major premiss is in the mood A.

The next wall denotes that the major premiss is in the mood E, and so on. Proceeding from the front to the back the first wall presents a region in every part of which the minor premiss is in the mood A. The second wall is a region throughout which the minor premiss is in the mood E, and so on. In the layers, from the bottom upwards, the conclusion goes through its various moods beginning with A in the lowest, E in the second, I in the third, O in the fourth.

In the general case, in which the variables represented in the poiograph pass through a wide range of values, the planes from which we measure their degrees of variation in our experience are taken to be indefinitely extended. In this case, however, all we are concerned with is the finite region.

We have now to represent, by some limitation of the complex we have obtained, the fact that not every combination of premisses justifies any kind of conclusion. This can be simply effected by marking the regions in which, the premisses being such as are defined by the positions, a conclusion which is valid is found.

Taking the conjunction of the major premisses, all M is P, and the minor, all S is M, we conclude that all S is P. Hence, that region must be marked in which we have the conjunction of major premiss in mood A; minor premiss, mood A; conclusion, mood A. This is the cube occupying the lowest left-hand cover of the large cube.

Proceeding in this way, we find that the regions which must be marked are those shown in fig. 53. To discuss the case shown in the marked cube which appears at the top of fig. 53. Here the major premise is in the second wall to the right—it is in the mood E and is of the type no M is P. The minor premiss is in the mood characterised by the third wall from the front. It is of the type some S is M. From these premisses we draw the conclusion that some S is not P, a conclusion in the mood O. Now the mood O of the conclusion is represented in the top layer. Hence we see that the marking is correct in this respect.

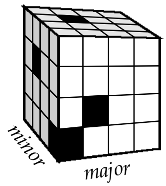


Fig. 53.

It would, of course, be possible to represent the cube on a plane by means of four squares, as in fig. 54, if we consider each square to represent merely the beginning of the region it stands for. Thus the whole cube can be represented by four vertical squares, each standing for a kind of vertical tray, and the markings would be as shown. In No. 1 the major premiss is in mood A for the whole of the region indicated by the vertical square of sixteen divisions; for number 2 it is in the mood E, and so on.

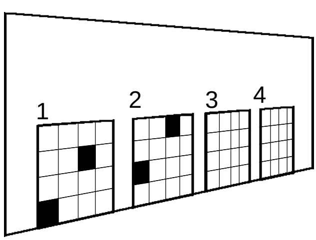


Fig. 54.

A creature confined to a plane would have to adopt some such disjunctive way of representing the whole cube. He would be obliged to represent that which we see as a whole in separate parts, and each part would merely represent, would not be, that solid content which we see.

The view of these four squares which the plane creature would have would not be such as ours. He would not see the interior of the four squares represented above, but each would be entirely contained within its outline, the internal boundaries of the separate small squares he could not see except by removing the outer squares.

We are now ready to introduce the fourth variable involved in the syllogism.

In assigning letters to denote the terms of the syllogism we have taken S and P to represent the subject and predicate in the conclusion, and thus in the conclusion their order is invariable. But in the premisses we have taken arbitrarily the order all M is P, and all S is M. There is no reason why M instead of P should not be the predicate of the major premiss, and so on.

Accordingly we take the order of the terms in the premisses as the fourth variable. Of this order there are four varieties, and these varieties are called figures.

Using the order in which the letters are written to denote that the letter written first is subject, the one written second is predicate, we have the following possibilities:—

	1st Figure.	2nd Figure.	3rd Figure.	4th Figure.
Major	M P	P M	M P	P M
Minor	S M	S M	M S	M S

There are therefore four possibilities with regard to this fourth variable as with regard to the premisses.

We have used up our dimensions of space in representing the phases of the premises and the conclusion in respect of mood, and to represent in an analogous manner the variations in figure we require a fourth dimension.

Now in bringing in this fourth dimension we must make a change in our origins of measurement analogous to that which we made in passing from the plane to the solid.

This fourth dimension is suppose to run at right angles to any of the three space dimensions, as the third space dimension runs at right angles to the two dimensions of a plane, and thus it gives us the opportunity of generating a new kind of volume. If the whole cube moves in this dimension, the solid itself traces out a path, each section of which, made at right angles to the direction in which it moves, is a solid, an exact repetition of the cube itself.

The cube as we see it is the beginning of a solid of such a kind. It represents a kind of tray, as the square face of the cube is a kind of tray against which the cube rests.

Suppose the cube to move in this fourth dimension in four stages, and let the hyper-solid region traced out in the first stages of its progress be characterised by this, that the terms of the syllogism are in the first figure, then we can represent in each of the three subsequent stages the remaining three figures. Thus the whole cube forms the basis from which we measure the variation in figure. The first figure holds good for the cube as we see it, and for that hyper-solid which lies within the first stage; the second figure holds good in the second stage, and so on.

Thus we measure from the whole cube as far as figures are concerned.

But we say that when we measured in the cube itself having three variables, namely, the two premisses and the conclusion, we measured from three planes. The base from which we measured was in every case the same.

Hence, in measuring in this higher space we should have bases of the same kind to measure from, we should have solid bases.

The first solid base is easily seen, it is the cube itself. The other can be found from this consideration.

That solid from which we measure figure is that in which the remaining variables run through their full range of varieties.

Now, if we want to measure in respect of the moods of the major premiss, we must let the minor premiss, the conclusion, run through their range, and also the order of the terms. That is we must take as a basis of measurement in respect to the moods of the major that which represents the variation of the moods of the minor, the conclusion and the variation of the figures.

Now the variation of the moods of the minor and of the conclusion are represented in the square face on the left of the cube. Here are all varieties of the minor premiss and the conclusion. The varieties of the figures are represented by stages in a motion proceeding at right angles to all space directions, at right angles consequently to the face in question, the left-hand face of the cube.

Consequently letting the left-hand face move in this direction we get a cube, and in this cube all the varieties of the minor premiss, the conclusion, and the figure are represented.

Thus another cubic base of measurement is given to the cube, generated by movement of the left-hand square in the fourth dimension.

We find the other bases in a similar manner, one in the cube generated by the front square moved in the fourth dimension so as to generate a cube. From this cube variations in the mood of the minor are measured. The fourth base is that found by moving the bottom square of the cube in the fourth dimension. In this cube the variations of the major, the minor, and the figure are given. Considering this as a basis in the four stages proceeding from it, the variations in the moods of the conclusion are given.

Any one of these cubic bases can be represented in space, and then the higher solid generated from them lies out of our space. It can only be represented by a device analogous to that by which the plane being represents a cube.

He represents the cube shown above, by taking four square sections and placing them arbitrarily at convenient distances the one from the other.

So we must represent this higher solid by four cubes: each cube represents only the beginning of the corresponding higher volume.

It is sufficient for us, then, if we draw four cubes, the first representing that region in which the figure is of the first kind, the second that region in which the figure is of the second kind, and so on. These cubes are the beginnings merely of the respective regions—they are the trays, as it were, against which the real solids must be conceived as resting, from which they start. The first one, as it is the beginning of the region of the first figure, is characterised by the order of the terms in the premisses being that of the first figure. The second similarly has the terms of the premisses in the order of the second figure, and so on.

These cubes are shown below.

For the sake of showing the properties of the method of representation, not for the logical problem, I will make a digression. I will represent in space the moods of the minor and of the conclusion and the different figures, keeping the major always in mood A. Here we have three variables in different stages, the minor, the conclusion, and the figure. Let the square of the left-hand side of the original cube be imagined to be standing by itself, without the solid part of the cube, represented by (2) fig. 55. The A, E, I, O, which run away represent the moods of the minor, the A, E, I, O, which run up represent the moods of the conclusion. The whole square, since it is the beginning of the region in the major premiss, mood A, is to be considered as in major premiss, mood A.

From this square, let it be supposed that that direction in which the figures are represented runs to the left hand. Thus we have a cube (1) running from the square above, in which the square itself is hidden, but the letters A, E, I, O, of the conclusion are seen. In this cube we have the minor premiss and the conclusion in all their moods, and all the figures represented. With regard to the major premiss, since the face (2) belongs to the first wall from the left in the original arrangement, and in this

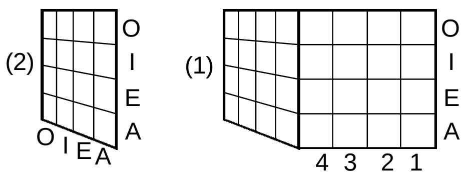


Fig. 55.

arrangement was characterised by the major premiss in the mood A, we may say that the whole of the cube we now have put up represents the mood A of the major premiss.

Hence the small cube at the bottom to the right in 1, nearest to the spectator, is major press, mood A; minor premiss, mood A; conclusion, mood A; and figure the first. The cube next to it, running to the left, is major premiss, mood A; minor premiss, mood A; conclusion, mood A; figure 2.

So in this cube we have the representations of all the combinations which can occur when the major premiss, remaining in the mood A, the minor premiss, the conclusion, and the figures pass through their varieties.

In this case there is no room in space for a natural representation of the moods of the major premiss. To represent them we must suppose as before that there is a fourth dimension, and starting from this cube as base in the fourth direction in four equal stages, all the first volume corresponds to major premiss A, the second to major premiss, mood E, the next to the mood I, and the last to mood O.

The cube we see is as it were merely a tray against which the four-dimensional figure rests. Its section at any stage is a cube. But a transition in this direction being transverse to the whole of our space is represented by no space motion. We can exhibit successive stages of the result of transference of the cube in that direction, but cannot exhibit the product of a transference, however small, in that direction.

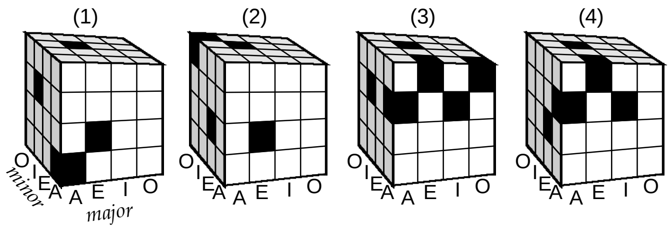


Fig. 56.

To return to the original method of representing our variables, consider fig. 56. These four cubes represent four sections of the figure derived from the first of them by moving it in the fourth dimension. The first portion of the motion, which begins with 1, traces out a more than solid body, which is all in the first figure. The beginning of this body is shown in 1. The next portion of the motion traces out a more than solid body, all of which is in the second figure; the beginning of this body is shown in 2; 3 and 4 follow on in like manner. Here, then, in one four-dimensional figure we have all the combinations of the four variables, major premiss, minor premiss, figure, conclusion, represented, each variable going through its four varieties. The disconnected cubes drawn are our representation in space by means of disconnected sections of this higher body.

Now it is only a limited number of conclusions which are true—their truth depends on the particular combinations of the premisses and figures which they accompany. The total figure thus represented may be called the universe of thought in respect to these four constituents, and out of the universe of possibly existing combinations it is the province of logic to select those which correspond to the results of our reasoning faculties.

We can go over each of the premisses in each of the moods, and find out what conclusion logically follows. But this is done in the works on logic; most simply and clearly I believe in "Jevon's Logic." As we are only concerned with a formal presentation of the results we will make use of the mnemonic lines printed below, in which the words enclosed in brackets refer to the figures, and are not significative:—

Barbara celarent Darii ferio*que* [prioris]
Cesare Camestres Festino Baroco [secundæ].
[Tertia] darapti disamis datisi felapton
Bocardo ferison *habet* [Quarta insuper addit].
Bramantip camenes dimaris fesapo fresinon.

In these lines each significative word has three vowels, the first vowel refers to the major premiss, and gives the mood of that premiss, "a" signifying, for instance, that the major mood is in mood *a*. The second vowel refers to the minor premiss, and gives its mood. The third vowel refers to the conclusion, and gives its moon. Thus (prioris)—of the first figure—the first mnemonic word is "barbara," and this gives major premiss, mood A; minor premiss, mood A; conclusion, mood A. Accordingly in the first of our four cubes we mark the lowest left-hand front cube. To take another instance in the third figure "Tertia," the word "ferison" gives us major premiss mood E—*e.g.*, no M is P, minor premiss mood I; some M is S, conclusion, mood O; some S is not P. The region to be marked then in the third representative cube is the one in the second wall to the right for the major premiss, the third wall from the front for the minor premiss, and the top layer for the conclusion.

It is easily seen that in the diagram this cube is marked, and so with all the valid conclusions. The regions marked in the total region show which combinations of the four variables, major premiss, minor premiss, figure, and conclusion exist.

That is to say, we objectify all possible conclusions, and build up an ideal manifold, containing all possible combinations of them with the premisses, and then out of this we eliminate all that do not satisfy the laws of logic. The residue is the syllogism, considered as a canon of reasoning.

Looking at the shape which represents the totality of the valid conclusions, it does not present any obvious symmetry, or easily characterisable nature. A striking configuration, however, is obtained, if we project the four-dimensional figure obtained into a three-dimensional one; that is, if we take in the base cube all those cubes which have a marked space anywhere in the series of four regions which start from that cube.

This corresponds to making abstraction of the figures, giving all the conclusions which are valid whatever the figure may be.

Proceeding in this way we obtain the arrangement of marked cubes shown in fig. 57. We see that the valid conclusions are arranged almost symmetrically round one cube—the one on the top of the column starting from AAA. There is one breach of continuity however in this scheme. One cube is unmarked, which if marked would give symmetry. It is the one which would be denoted by the

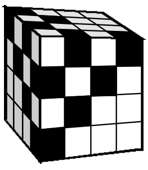
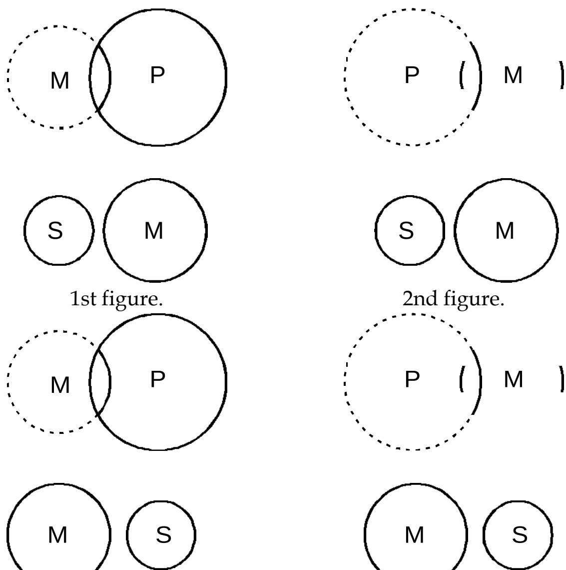


Fig. 57.

letters I, E, O, in the third wall to the right, the second wall away, the topmost layer. Now this combination of premisses in the mood IE, with a conclusion in the mood O, is not noticed in any book on logic with which I am familiar. Let us look at it four ourselves, as it seems that there must be something curious in connection with this break of continuity in the poiograph.



1st figure. 2nd figure. 3rd figure. 4th figure.

The propositions I, E in the various figures are the following, as shown in the accompanying scheme, fig. 58:— First figure: some M is P; no S is M. Second figure; some P is M; no S is M. Third figure: some M is P; no M is S. Fourth figure: some P is M; no M is S.

Examining these figures, we see, taking the first, that if some M is P, and no S is M, we have no conclusion of the form S is P in the various moods. It is quite indeterminate how the circle representing S lies with regard to the circle representing P. It may lie inside, outside, or partly inside P. The same is true in the other figures 2 and 3. But when we come to the fourth figure, since M and S lie completely outside each other, there cannot lie inside S that part of P which lies inside M. Now we know by the major premiss that some of P does lie in M. Hence S cannot contain the whole of P. In words, some P is M, no M is S, therefore S does not contain the whole of P. If we take P as the subject, this gives us a conclusion in the mood O about P. Some P is not S. But it does not give us conclusion about S in any one of the four forms recognised in the syllogism and called its moods. Hence the breach of the continuity in the poiograph has enabled us to detect a lack of completeness in the relations which are considered in the syllogism.

To take an instance:--Some Americans (P) are of African stock (M); no Aryans (S) are of African stock (M); Aryans (S) do not include all of Americans (P).

In order to draw a conclusion about S we have to admit the statement, "S does not contain the whole of P," as a valid logical form—it is a statement about S which can be made. The logic which gives us the form "some P is not S," and which does not allow us to give the exactly equivalent and equally primary form, "S does not contain the whole of P," is artificial.

And I wish to point out that this artificiality leads to an error.

If one trusted to the mnemonic lines given above, one would conclude that no logical conclusion about S can be drawn from the statement, "some P are M, no M are S."

But a conclusion can be drawn: S does not contain the whole of P.

It is not that the result is given expressed in another form. The mnemonic lines deny that any conclusion can be drawn from premises in the moods I, E, respectively.

Thus a simple four-dimensional poiograph has enabled us to detect a mistake in the mnemonic lines which have been handed down unchallenged from mediæval times. To discuss the subject of these lines more fully a logician defending them would probably say that a particular statement cannot be a major premiss; and so deny the existence of the fourth figure in the combination of moods.

To take our instance: some Americans are of African stock; no Aryans are of African stock. He would say that the conclusion is some Americans are not Aryans; and that the second statement is the major. He would refuse to say anything about Aryans, condemning us to an eternal silence about them, as far as these premisses are concerned! But, if there is a statement involving the relation of two classes, it must be expressible as a statement about either of them.

To bar the conclusion, "Aryans do not include the whole of Americans," is purely a makeshift in favour of a false classification.

And the argument drawn from the universality of the major premiss cannot be consistently maintained. It would preclude such combinations as major O, minor A, conclusion O—*i.e.*, such as some mountains (M) are not permanent (P); all mountains (M) are scenery (S); some scenery (S) is not permanent (P).

This is allowed in "Jevon's Logic," and his omission to discuss I, E, O, in the fourth figure is inexplicable. A satisfactory poiograph of the logical scheme can be made by admitting the use of the words some, none, or all about the predicate as well as about the subject. Then we can express the statements, "Aryans do not include the whole of Americans," clumsily, but, when its obscurity is fathomed, correctly, as "Some Aryans are not all

Americans." And this method is what is called the "quantification of the predicate."

The laws of formal logic are coincident with the conclusions which can be drawn about regions of space, which overlap one another in the various possible ways. It is not difficult so to state the relations or to obtain a symmetrical poiograph. But to enter into this branch of geometry is beside our present purpose, which is to show the application of the poiograph in a finite and limited region, without any of these complexities which attend its use in regard to natural objects.

If we take the latter—plants, for instance—and, without assuming fixed directions in space as representative of definite variations, arrange the representative points in such a manner as to correspond to the similarities of the objects, we obtain configurations of singular interest; and perhaps in this way, in the making of shapes of shapes, bodies with bodies omitted, some insight into the structure of the species and genus might be obtained.

CHAPTER IX

APPLICATION TO KANT'S THEORY OF EXPERIENCE

WHEN we observe the heavenly bodies we become aware that they all participate in one universal motion—a diurnal revolution about the polar axis.

In the case of fixed stars this is most unqualifiedly true, but in the case of the sun, and the planets also, the single motion of revolution can be discerned, modified, and slightly altered by other and secondary motions.

Hence the universal characteristic of the celestial bodies is that they move in a diurnal circle.

But we know that this one great fact which is true of them all has in reality nothing to do with them. The diurnal revolution which they visibly perform is the result of the conditions of the observer. It is because the observer is on a rotating earth that a universal statement can be made about the celestial bodies.

The universal statement which is valid about every one of the celestial bodies is that which does not concern them at all, and is but a statement of the condition of the observer.

Now there are universal statements of other kinds which we can make. We can say that all objects of experience are in space and subject to the laws of geometry.

Does this mean that space and all that it means is due to a condition of the observer?

If a universal law in one case means nothing affecting the objects themselves, but only a condition of observation, is this true in every case? There is shown us in astronomy a *vera causa* for the assertion of a universal. Is the same cause to be traced everywhere?

Such is a first approximation to the doctrine of Kant's critique.

It is the apprehension of a relation into which, on the one side and the other, perfectly definite constituents enter—the human observer and the stars—and a transference of this relation to a region in which the constitutents on either side are perfectly unknown.

If spatiality is due to a condition of the observer, the observer cannot be this bodily self of ours—the body, like the objects around it, are equally in space.

This conception Kant applied, not only to the intuitions of sense, but to the concepts of reason—wherever a universal statement is made there is afforded to him an opportunity for the application of his principle. He constructed a system in which one hardly knows which the most to admire, the architectonic skill, or the reticence with regard to things in themselves, and the observer in himself.

His system can be compared to a garden, somewhat formal perhaps, but with the charm of a quality more than intellectual, a *besonnenheit*, an exquisite moderation over all. And from the ground he so carefully prepared with that buried in obscurity, which it is fitting should be obscure, science blossoms and the tree of real knowledge grows.

The critique is a storehouse of ideas of profound interest. The one of which I have given a partial statement leads, as we shall see on studying it in detail, to a theory of mathematics suggestive of enquiries in many directions.

The justification for my treatment will be found amongst other passages in that part of the transcendental analytic, in which Kant speaks of objects of experience subject to the forms of sensibility, not subject to the concepts of reason.

Kant asserts that whenever we think we think of objects in space and time, but he denies that the space and time exist as independent entities. He goes about to explain them, and their universality, not be assuming them, as most other philosophers do, but by postulating their absence. How then does it come to pass that the world is in space and time to us?

Kant takes the same position with regard to what we call nature—a great system subject to law and order. "How do you explain the law and order in nature?" we ask the philosophers. All except Kant reply by assuming law and order somewhere, and then showing how we can recognise it.

In explaining our notions, philosophers from other than the Kantian standpoint, assume the notions as existing outside us, and then it is no difficult task to show how they come to us, either by inspiration or by observation.

We ask "Why do we have an idea of law in nature?" "Because natural processes go according to law," we are answered, "and experience inherited or acquired, gives us this notion."

But when we speak about the law in nature we are speaking about a notion of our own. So all that these expositors do is to explain our notion by an assumption of it.

Kant is very different. He supposes nothing. An experience such as ours is very different from experience in the abstract. Imagine just simply experience, succession of states, of consciousness? Why, there would be no connecting any two together, there would be no personal identity, no memory. It is out of a general experience such as this, which, in respect to anything we call real, is less than a dream, that Kant shows the genesis of an experience such as ours.

Kant takes up the problem of the explanation of space, time, order, and so quite logically does not presuppose them.

But how, when every act of thought is of things in space, and time, and ordered, shall we represent to ourselves that perfectly indefinite somewhat which is Kant's necessary hypothesis—that which is not in space or time and is not ordered. That is our problem, to represent that which Kant assumes not subject to any of our forms of thought, and then show some function which working on that makes it into a "nature" subject to law and order, in space and time. Such a function Kant calls the "Unity of Apperception"; *i.e.*, that which makes our state of consciousness capable of being woven into a system with a self, an outer world, memory, law, cause, and order.

The difficulty that meets us in discussing Kant's hypothesis is that everything we think of is in space and time—how then shall we represent in space an existence not in space, and in time an existence not in time? This difficulty is still more evident when we come to construct a poiograph, for a poiograph is essentially a space structure. But because more evident the difficulty is nearer a solution. If we always think in space, *i.e.* using space concepts, the first condition requisite for adapting them to the representation of non-spatial existence, is to be aware of the limitation of our thought, and so be able to take the proper steps to overcome it. The problem before us, then, is to represent in space an existence not in space.

The solution is an easy one. It is provided by the conception of alternativity.

To get our ideas clear let us go right back behind the distinctions of an inner and an outer world. Both of these, Kant says, are products. Let us take merely states of consciousness, and not ask the question whether they are produced or superinduced—to ask such a question is to have got too far on, to have assumed something of which we have not traced the origin. Of these states let us simply say that they occur. Let us now use the word a "posit" for a phase of consciousness reduced to its last possible stage of evanescence; let a posit be that phase of consciousness of which all that can be said is that it occurs.

Let *a*, *b*, *c*, be three such posits. We cannot represent them in space without placing them in a certain order, as *a*, *b*, *c*. But Kant distinguishes between the forms of sensibility and the concepts of reason. A dream in which everything happens at haphazard would be an experience subject to the form of sensibility and only partially subject to the concepts of reason. It is partially subject to the concepts of reason because, although there is no order of sequence, still at any given time there is order. Perception of a thing as in space is a form of sensibility, the perception of an order is a concept of reason.

We must, therefore, in order to get at that process which Kant supposes to be constitutive of an ordered experience imagine the posits as in space without order.

As we know them they must be in some order, *abc*, *bca*, *cab*, *acb*, *cba*, *bac*, one or another.

To represent them as having no order conceive all these different orders as equally existing. Introduce the conception of alternativity—let us suppose that the order *abc*, and *bac*, for example, exist equally, so that we cannot say about *a* that it comes before or after *b*. This would correspond to a sudden and arbitrary change of *a* into *b* and *b* into *a*, so that, to use Kant's words, it would be possible to call one thing by one name at one time and at another time by another name.

In an experience of this kind we have a kind of chaos, in which no order exists; it is a manifold not subject to the concepts of reason.

Now is there any process by which order can be introduced into such a manifold—is there any function of consciousness in virtue of which an ordered experience could arise?

In the precise condition in which the posits are, as described above, it does not seem to be possible. But if we imagine a duality to exist in the manifold, a function of consciousness can be easily discovered which will produce order out of no order.

Let us imagine each posit, then, as having a dual aspect. Let *a* be 1*a* in which the dual aspect is represented by the combination of symbols. And similarly let *b* be 2*b*, *c* be 3*c*, in which 2 and *b* represent that dual aspects of *b*, 3 and *c* those of *c*.

Since *a* can arbitrarily change into *b*, or into *c*, and so on, the particular combinations written above cannot be kept. We have to assume the equally possible occurrence of form such as 2*a*, 2*b*, and so on; and in order to get a representation of all these combinations out of which any set is alternatively possible, we must take every aspect with every aspect. We must, that is, have every letter with every number.

Let us now apply the method of space representation.

Notes.—At the beginning of the next chapter the same structures as those which follow are exhibited in more detail and a reference to them will remove any obscurity which may be found in the immediately following passages. They are there carried on to a greater multiplicity of dimensions, and the significance of the process here briefly explained becomes more apparent.

Take three mutually rectangular axes in space 1, 2, 3 (fig. 59), and on each mark three points, the common meeting point being the first on each axis. Then by means of these three points on each axis, we define 27 positions, 27 points in a cubical cluster, shown in fig. 60, the same method of co-ordination being used as has been described before. Each of these positions can be named by means of the axes and the points combined.

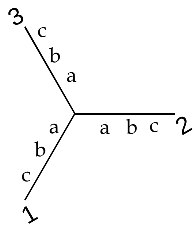


Fig. 59.

Thus, for instance, the one marked by an asterisk can be called 1*c*, 2*b*, 3*c*, because it is opposite to *c* on 1, to *b* on 2, to *c* on 3.

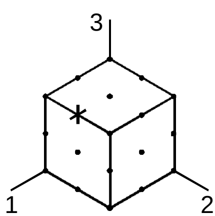


Fig. 60.

Let us now treat of the states of consciousness corresponding to these positions. Each point represents a composite of posits, and the manifold of consciousness corresponding to them is of a certain complexity.

Suppose now the constituents, the points on the axes, to interchange arbitrarily, any one to become any other, and also the axes 1, 2 and 3, to interchange amongst themselves, any one to become any other, and to be subject to no system or law, that is to say, that order does not exist, and that the points which run *abc* on each axis may run *bac*, and so on.

Then any one of the states of consciousness represented by the points in the cluster can become any other. We have a representation of a random consciousness of a certain degree of complexity.

Now let us examine carefully one particular case of arbitrary interchange of the points, *a*, *b*, *c*; as one such case, carefully considered, makes the whole clear.

Consider the points named in the figure 1*c*, 2*a*, 3*c*; 1*c*, 2*c*, 3*a*; 1*a*, 2*c*, 3*c*, and examine the effect on them when a change of order takes place. Let us suppose, for instance, that *a* changes into *b*, and let us call the two sets of points we get, the one before and the one after, their change conjugates.

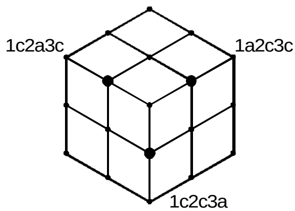


Fig. 61.

Before the change	1*c* 2*a* 3*c*	1*c* 2*c* 3*a*	1*a* 2*c* 3*c*	Conjugates
After the change	1*c* 2*b* 3*c*	1*c* 2*c* 3*b*	1*b* 2*c* 3*c*	

The points surrounded by rings represent the conjugate points.

It is evident that as consciousness, represented first by the first set of points and afterwards by the second set of points, would have nothing in common in its two phases. It would not be capable of giving an account of itself. There would be no identity.

If, however, we can find any set of points in the cubical cluster, which, when any arbitrary change takes place in the points on the axes, or in the axes themselves, repeats itself, is reproduced, then a consciousness represented by those points would have a permanence. It would have a principle of identity. Despite the no law, no order, of the ultimate constituents, it would have an order, it would form a system, the conditions of a personal identity would be fulfilled.

The question comes to this, then. Can we find a system of points which is self-conjugate, which is such that when any point on the axes becomes another other, or when any axis becomes any other, such a set is transformed into itself, its identity is not submerged, but rises superior to the chaos of its constituents?

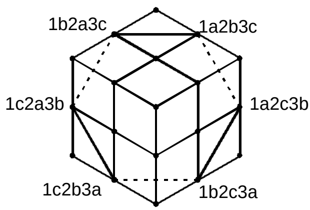


Fig. 62.

Such a set can be found. Consider the set represented in fig. 62, and written down in the first of the two lines—

Self-conjugate.	1*a* 2*b* 3*c*	1*b* 2*a* 3*c*	1*c* 2*a* 3*b*	1*c* 2*b* 3*a*	1*b* 2*c* 3*a*	1*a* 2*c* 3*b*
	1*c* 2*b* 3*a*	1*b* 2*c* 3*a*	1*a* 2*c* 3*b*	1*a* 2*b* 3*c*	1*b* 2*a* 3*c*	1*c* 2*a* 3*b*

If now *a* change into *c* and *c* into *a*, we get the set in the second line, which has the same members as are in the upper line. Looking at the diagram we see that it would correspond simply to the turning of the figure as a whole.* Any arbitrary change of the points on the axes, or of the axes themselves, reproduces the same set.

Thus, a function, by which a random, an unordered consciousness could give an ordered and systematic one, can be represented. It is noteworthy that it is a system of selection. If out of all the alternative forms that only is attended to which is self-conjugate, an ordered consciousness is formed. A selection gives a feature of permanence.

Can we say that the permanent consciousness is this selection?

An analogy between Kant and Darwin comes into light. That which is swings clear of the fleeting, in virtue of its presenting a feature of permanence. There is no need to suppose any function of "attending to." A consciousness capable of giving an account of itself is one which is characterised by this combination. All combinations exist—of this kind is the consciousness which can give an account of itself. And the very duality which

* These figures are described more fully, and extended, in the next chapter.

we have presupposed may be regarded as originated by a process of selection.

Darwin set himself to explain the origin of the fauna and flora of the world. He denied specific tendencies. He assumed an indefinite variability—that is, chance—but a chance confined within narrow limits as regards the magnitude of any consecutive variations. He showed that organisms possessing features of permanence, if they occurred would be preserved. So his account of any structure or organised being was that it possessed features of permanence.

Kant, undertaking not the explanation of any particular phenomena but of that which we call nature as a whole, had an origin of species of his own, an account of the flora and fauna of consciousness. He denied any specific tendency of the elements of consciousness, but taking our own consciousness, pointed out that in which it resembled any consciousness which could survive, which could give an account of itself.

He assumes a chance or random world, and as great and small were not to him any given notions of which he could make use, he did not limit the chance, the randomness, in any way. But any consciousness which is permanent must possess certain features—those attributes namely which give it permanence. Any consciousness like our own is simply a consciousness which possesses those attributes. The main thing is that which he calls the unity of apperception, which we have seen above is simply the statement that a particular set of phases of consciousness on the basis of complete randomness will be self-conjugate, and so permanent.

As with Darwin so with Kant, the reason for existence of any feature comes to this—show that it tends to the permanence of that which possesses it.

We can thus regard Kant as the creator of the first of the modern evolution theories. And, as is so often the case, the first effort was the most stupendous in its scope. Kant does not investigate the origin of any special part of the world, such as its organisms, its chemical elements, its social communities of men. He simply investigates the origin of the whole—of all that is included in consciousness, the origin of that "thought thing" whose progressive realisation is the knowable universe.

This point of view is very different from the ordinary one, in which a man is supposed to be placed in a world like that which he has come to think of it, and then to learn what he has found out from this model which he himself has placed on the scene.

We all know that there are a number of questions in attempting an answer to which such an assumption is not allowable.

Mill, for instance, explains our notion of "law" by an invariable sequence in nature. But what we call nature is something given in thought. So he explains a thought of law and order by a thought of an invariable sequence. He leaves the problem where he found it.

Kant's theory is not unique and alone. It is one of a number of evolution theories. A notion of its import and significance can be obtained by a comparison of it with other theories.

Thus in Darwin's theoretical world of natural selection a certain assumption is made, the assumption of indefinite variability—slight variability it is true, over any appreciable lapse of time, but indefinite in the postulated epochs of transformation—and a whole chain of results is shown to follow.

This element of chance variation is not, however, an ultimate resting place. It is a preliminary stage. This supposing the all is a preliminary step towards finding out what is. If every kind of organism can come into being, those that do survive will possess such and such characteristics. This is the necessary beginning for ascertaining what kinds of organisms do come into existence. And so Kant's hypothesis of a random consciousness is the necessary beginning for the rational investigation of consciousness as it is. His assumption supplies, as it were, the space in which we can observe the phenomena. It gives the general laws constitutive of any experience. If, on the assumption of absolute randomness in the constituents, such and such would be characteristic of the experience, then, whatever the constituents, these characteristics must be universally valid.

We will now proceed to examine more carefully the poiograph, constructed for the purpose of exhibiting an illustration of Kant's theory of apperception.

In order to show the derivation order out of non-order it has been necessary to assume a principle of duality—we have had the axes and the posits on the axes—there are two sets of elements, each non-ordered, and it is in the reciprocal relation of them that the order, the definite system, originates.

Is there anything in our experience of the nature of a duality?

There certainly are objects in our experience which have order and those which are incapable of order. The two roots of a quadratic equation have no order. No one can tell which comes first. If a body rises vertically and then goes at right angles to its former course, no one can assign any priority to the direction of the north or to the east. There is no priority in directions of turning. We associate turnings with no order, progressions in a line with order. But in the axes and points we have assumed above there is no such distinction. It is the same, whether we assume an order among the turnings, and no order among the points on the axes, or, *vice versa*, an order in the points and no order in the turnings. A being with an infinite number of axes mutually at right angles, with a definite sequence between them and no sequence between the points on the axes, would be in a condition formally indistinguishable from that of a creature who, according to an assumption more natural to us, had on each axis an infinite number of ordered points and no order of priority among the axes. A being in such a constituted world would not be able to tell which was turning and which was length along an axis, in order to distinguish between them. Thus to take a pertinent illustration, we may be in a world of an infinite number of dimensions, with three arbitrary points on each—three points whose order is indifferent, or in a world of three axes of arbitrary sequence with in infinite number of ordered points on each. We can't tell which is which, to distinguish it from the other.

Thus it appears the mode of illustration which we have used is not an artificial one. There really exists in nature a duality of the kind which is necessary to explain the origin of order out of no order—the duality, namely, of dimension and position. Let us use the term group for that system of points which remains unchanged, whatever arbitrary change of its constituents takes place. We notice that a group involves a duality, is inconceivable without a duality.

Thus, according to Kant, the primary element of experience is the group, and the theory of groups would be the most fundamental branch of science. Owing to an expression in the critique the authority of Kant is sometimes adduced against the assumption of more than three dimensions to space. It seems to me, however, that the whole tendency of his theory lies in the opposite direction, and points to a perfect duality between dimension and position in a dimension.

If the order and the law we see is due to the conditions of conscious experience, we must conceive nature as spontaneous, free, subject to no predication that we can devise, but, however apprehended, subject to our logic.

And our logic is simply spatiality in the general sense —that resultant of a selection of the permanent from the unpermanent, the ordered from the unordered, by the means of the group and its underlying duality.

We can predicate nothing about nature, only about the way in which we can apprehend nature. All that we can say is that all that which experience gives us will be considered as spatial, subject to our logic. Thus, in exploring the facts of geometry from the simplest logical relations to the properties of space of any number of dimensions, we are merely observing ourselves, becoming aware of the conditions under which we must perceive. Do any phenomena present themselves incapable of explanation under the assumption of the space we are dealing with, then we must habituate ourselves to the conception of a higher space, in order that our logic may be equal to the task before us.

We gain a repetition of the thought that came before, experimentally suggested. If the laws of the intellectual comprehension of nature are those derived from considering her as absolute chance, subject to no law save that derived from a process of selection, then, perhaps, the order of nature requires different faculties from the intellectual to apprehend it. The source and origin of ideas may have to be sought elsewhere than in reasoning.

The total outcome of the critique is to leave the ordinary man just where he is, justified in his practical attitude towards nature, liberated from the fetters of his own mental representations.

The truth of a picture lies in its total effect. It is vain to seek information about the landscape from an examination of the pigments. And in any method of thought it is the complexity of the whole that brings us to a knowledge of nature. Dimensions are artificial enough, but in the multiplicity of them we catch some breath of nature.

We must therefore, and this seems to me the practical conclusion of the whole matter, proceed to form means of intellectual apprehension of a greater and greater degree of complexity, both dimensionally and in extent in any dimension. Such means of representation must always be artificial, but in the multiplicity of the elements with which we deal, however incipiently arbitrary, lies our chance of apprehending nature.

And as a concluding chapter to this part of the book, I will extend the figures, which have been used to represent Kant's theory, two steps, so that the reader may have the opportunity of looking at a four-dimensional figure which can be delineated without any of the special apparatus, to the consideration of which I shall subsequently pass on.

CHAPTER X

A FOUR-DIMENSIONAL FIGURE

THE method used in the preceding chapter to illustrate the problem of Kant's critique, gives a singularly easy and direct mode of constructing a series of important figures in any number of dimensions.

We have seen that to represent our space a plane being must give up one of his axes, and similarly to represent the higher shapes we must give up one amongst our three axes.

But there is another kind of giving up which reduces the construction of higher shapes to a matter of the utmost simplicity.

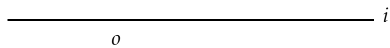
Ordinarily we have on a straight line any number of positions. The wealth of space in position is illimitable, while there are only three dimensions.

I propose to give up this wealth of positions, and to consider the figures obtained by taking just as many positions as dimensions.

In this way I consider dimensions and positions as two "kinds," and applying the simple rule of selecting every one of one kind with every other of every other kind, get a series of figures which are noteworthy because they exactly fill space of any number of dimensions (as the hexagon fills a plane) by equal repetitions of themselves.

The rule will be made more evident by a simple application.

Let us consider one dimension and one position. I will call the axis *i*, and the position *o*.



Here the figure is the position *o* on the line *i*. Take now two dimensions and two positions on each.

We have the two positions *o*; 1 on *i*, and the two positions *o*, 1 on *j*, fig. 63. These give rise to a certain complexity. I will let the two lines *i* and *j* meet in the position I call *o* on each, and I will consider *i* as a direction starting equally from every position on *j*, and *j* as starting equally from every position on *i*. We thus obtain the following figure:--A is both *oi* and *oj*, B is 1 *i* and *oj*, and so on as shown in fig. 63b. The positions on AC are all *oi* positions. They are, if we like to consider them that way, points at no distance in the *i* direction from the line AC. We can call the line AC the *oi* line. Similarly the points on AB are those no distance from AB in the *j* direction, and we can call them *oj* points and the line AB the *oj* line. Again, the line CD can be called the 1*j* line, because the points on it are at a distance 1 in the *j* direction.

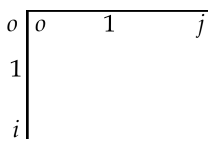


Fig. 63.

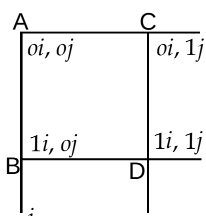


Fig 63b.

We have then four positions or points named as shown, and, considering directions and positions as "kinds," we have the combination of two kinds with two kinds. Now, selecting every one of one kind with every other of every other kind will mean that we take 1 of the kind *i* and with it *o* of the kind *j*; and then, that we take *o* of the kind *i* and with it 1 of the kind *j*.

Thus we get a pair of positions lying in the straight line BC, fig. 64. We call this part 10 and 01 if we adopt the plan of mentally adding an *i* to the first and a *j* to the second of the symbols written thus—01 is a short expression for 0*i*, 1*j*.

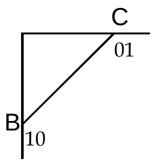


Fig. 64.

Coming now to our space, we have three dimensions, so we take three positions on each. These positions I will suppose to be at equal distances along each

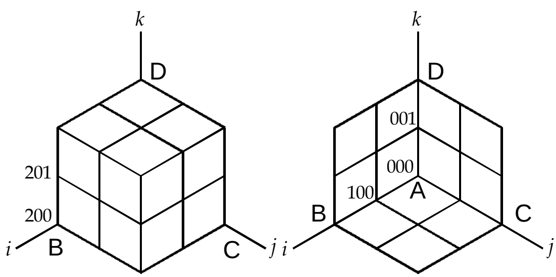


Fig. 65.

axis. The three axes and the three positions on each are shown in the accompanying diagram, fig. 65, of which the first represents a cube with the front faces visible, the second the rear faces of the same cube; the positions I will call 0, 1, 2; the axes, *i*, *j*, *k*. I take the base ABC as the starting place, from which to determine distances in the *k* direction, and hence every point in the base ABC will be an *ok* position, and the base ABC can be called an *ok* plane.

In the same way, measuring the distance from the face ADC, we see that every position in the face ADC is an *oi* position, and the whole plane of the face may be called an *oi* plane. Thus we see that with the introduction of a new dimension the signification of a compound symbol, such as "*oi*," alters. In the plane it meant the line AC. In space it means the whole plane ACD.

Now, it is evident that we have twenty-seven positions, each of them named. If the reader will follow this nomenclature in respect of the positions marked in the figures he will have no difficulty in assigning names to each one of the twenty-seven positions. A is *oi*, *oj*, *ok*. It is at the distance 0 along *i*, 0 along *j*, 0 along *k*, and it can be written in short 000, where the *ijk* symbols are omitted.

The point immediately above is 001, for it is no distance in the *i* direction, and a distance of 1 in the *k* direction. Again, looking at B, it is at a distance of 2 from A, or from the place ADC, in the *i* direction, 0 in the *j* direction from the plane ABD, and 0 in the *k* direction, measured from the plane ABC. Hence it is 200 written for 2*i*, 0*j*, 0*k*.

Now, out of these twenty-seven "things" or compounds of position and dimension, select those which are given by the rule, every one of one kind with every other of every other kind.

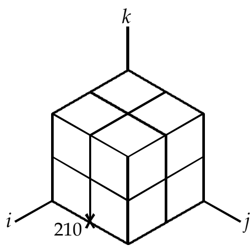


Fig. 66.

Take 2 of the *i* kind. With this we must have a 1 of the *j* kind, and then by the rule we can only have a 0 of the *k* kind, for if we had any other of the *k* kind we should repeat one of the kinds we already had. In 2*i*, 1*j*, 1*k*, for instance, 1 is repeated. The point we obtain is that marked 210, fig. 66.

Proceeding in this way, we pick out the following cluster of points, fig. 67. They are joined by lines, dotted where they are hidden by the body of the cube, and we see that they form a figure—a hexagon which could be taken out of the cube and placed on a plane.

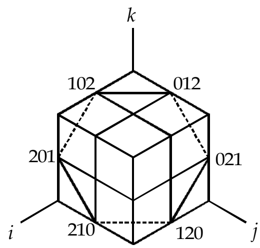


Fig. 67.

It is a figure which will fill a plane by equal repetitions of itself. The plane being representing this construction in his plane would take three squares to represent the cube. Let us suppose that he takes the *ij* axes in his space and *k* represents the axis running out of his space, fig. 68. In each of the three squares shown here as drawn separately he could select the points given by the rule, and he would

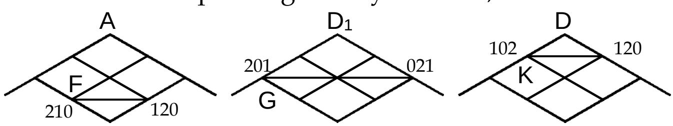


Fig. 68.

then have to try to discover the figure determined by the three lines drawn. The line from 210 to 210 is given in the figure, but the line from 210 to 201 or FG is not given. He can determine FG by making another set of drawings and discovering in them what the relation between these two extremities is.

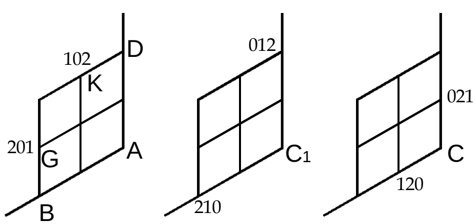


Fig. 69.

Let him draw the *i* and *k* axes in his plane, fig. 69. The *j* axis then runs out and he has the accompanying figure. In the first of these three squares, fig. 69, he can pick out by the rule the two points 201, 102—G, and K. Here they occur in one plane and he can measure the distance between them. In his first representation they occur at G and K in separate figures.

Thus the plane being would find that the ends of each of the lines was distant by the diagonal of a unit square from the corresponding end of the last and he could then place the three lines in their right relative position. Joining them he would have the figure of a hexagon.

We may also notice that the plane being could make a representation of the whole cube simultaneously. The three squares, shown in perspective in fig. 70, all lie in one plane, and on these the plane being could pick out any selection of points just as well as on three separate squares. He would obtain a hexagon by joining the points marked. This hexagon, as drawn, is of the right shape, but it would not be so if actual squares were used instead of perspective, because the relation between the separate squares as they lie in the plane figure is not their real relation. The figure, however, as thus constructed, would give him an idea of the correct figure, and he could determine it accurately by remembering that distances in each square were correct, but in passing from one square to another their distance in the third dimension had to be taken into account.

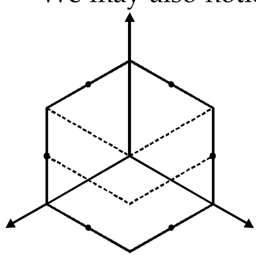


Fig. 70.

Coming now to the figure made by selecting according to our rule from the whole mass of points given by four axes and four positions in each, we must first draw a catalogue figure in which the whole assemblage is shown.

We can represent this assemblage of points by four solid figures. The first giving all those positions which are at a distance 0 from our space in the fourth dimension, the second showing all those that are at a distance 1, and so on.

These figures will each be cubes. The first two are drawn showing the front faces, the second two the rear faces. We will mark the points 0, 1, 2, 3, putting points at those distances along each of these axes, and suppose

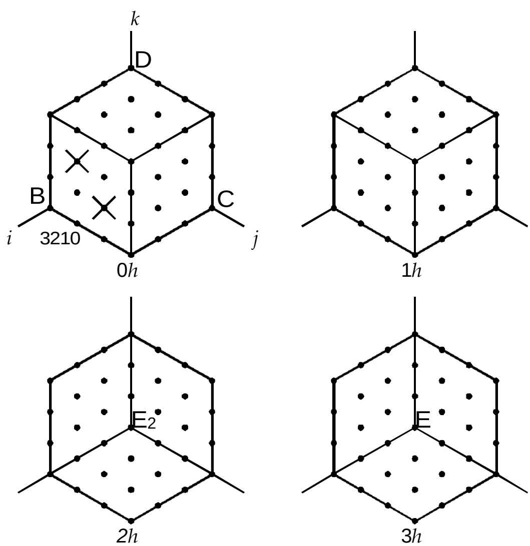


Fig. 71.

all the points thus determined to be contained in solid models of which our drawings in fig. 71 are representations. Here we notice that as on the plane 0*i* meant the whole line from which the distance in the *i* direction was measured, and as in space 0*i* means the whole plane from which distances in the *i* direction are measured, so now 0*h* means the whole space in which the first cube stands—measuring away from that space by a distance of one we come to the second cube represented.

Now selecting according to the rule every one of one kind with every other of every other kind, we must take, for instance, 3*i*, 2*j*, 1*k*, 0*h*. This point is marked 3210 at the lower star in the figure. It is 3 in the *i* direction, 2 in the *j* direction, 1 in the *k* direction, 0 in the *h* direction.

With 3*i* we must also take 1*j*, 2*k*, 0*h*. This point is shown by the second star in the cube 0*h*.

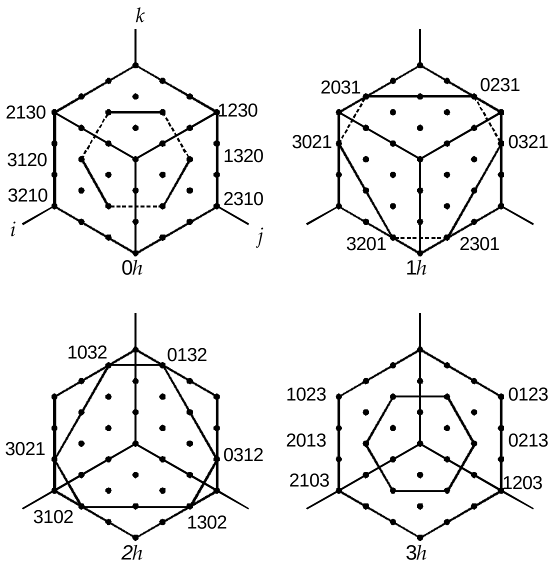


Fig. 72.

In the first cube, since all the points are 0*h* points, we can only have varieties in which *i*, *j*, *k*, are accompanied by 3, 2, 1.

The points determined are marked off in the diagram fig. 72, and lines are drawn joining the adjacent pairs in each figure, the lines being dotted when they pass within the substance of the cube in the first two diagrams.

Opposite each point, on one side or the other of each cube, is written its name. It will be noticed that the figures are symmetrical right and left; and right and left the first two numbers are simply interchanged.

Now this being our selection of points, what figure do they make when all are put together in their proper relative positions?

To determine this we must find the distance between corresponding corners of the separate hexagons.

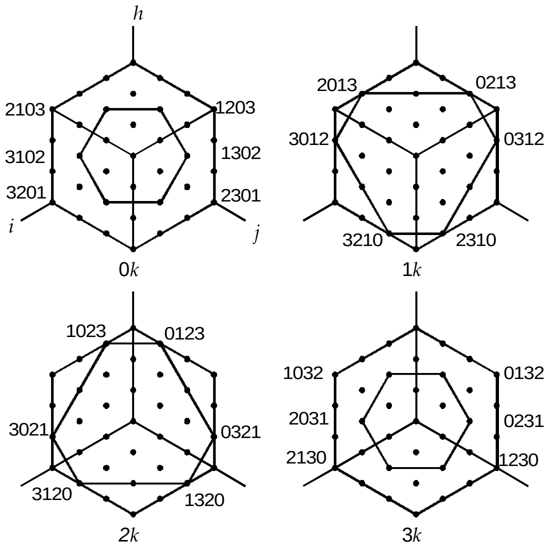


Fig. 73.

To do this let us keep the axes *i*, *j*, in our space, and draw *h* instead of *k*, letting *k* run out in the fourth dimension, fig. 73.

Here we have four cubes again, in the first of which all the points are 0*k* points; that is, points at a distance zero in the *k* direction from the space of the three dimensions *ijh*. We have all the points selected before, and some of the distances, which in the last diagram led from figure to figure are shown here in the same figure, and so capable of measurement. Take for instance the points 3120 to 3021, which in the first diagram (fig. 72) lie in the first and second figures. Their actual relation is shown in fig. 73 in the cube marked 2*k*, where the points in question are marked with a * in fig. 73. We see that the distance in question is the diagonal of a unit square. In like manner we find that the distance between corresponding points of any two hexagonal figures is the diagonal of a unit square. The total figure is now easily constructed. An idea of it may be gained by drawing all the four cubes in the catalogue figure in one (fig. 74). These cubes are exact repetitions of one another, so one drawing will serve as a representation of the whole series, if we take care to remember where we are, whether in a 0*h*, a 1*h*, a 2*h*, or a 3*h* figure, when we pick out the points required. Fig 74 is a representation of all the catalogue cubes put in one. For the sake of clearness the front faces and the back faces of this cube are represented separately.

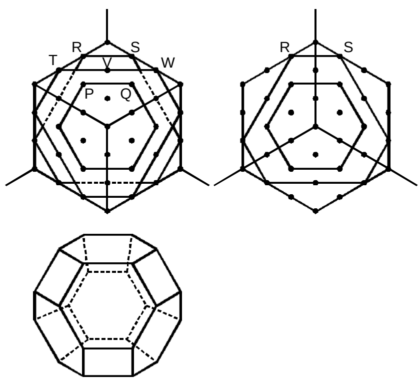


Fig. 74.

The figure determined by the selected points is shown below.

In putting the sections together some of the outlines in them disappear. The line TW for instance is not wanted.

We notice that PQTW and TWRS are each the half of a hexagon. Now QV and VR lie in one straight line.

Hence these two hexagons fit together, forming one hexagon, and the line TW is only wanted when we consider a section of the whole figure, we thus obtain the solid represented in the lower part of fig. 74. Equal repetitions of this figure, called as tetrakaidekagon, will fill up three-dimensional space.

To make the corresponding four-dimensional figure we have to take five axes mutually at right angles with five points on each. A catalogue of the positions determined in five-dimensional space can be found thus.

Take a cube with five points on each of its axes, the fifth point is at a distance of four units of length from the first on any one of the axes. And since the fourth dimension also stretches to a distance of four we shall need to represent the successive sets of points at distances 0, 1, 2, 3, 4, in the fourth dimension, five cubes. Now all of these extend to no distance at all in the fifth dimension. To represent what lies in the fifth dimension we shall have to draw, starting from each of our cubes, five similar cubes to represent the four steps on in the fifth dimension. By this assemblage we get a catalogue of all the points shown in fig. 75, in which *L* represents the fifth dimension.

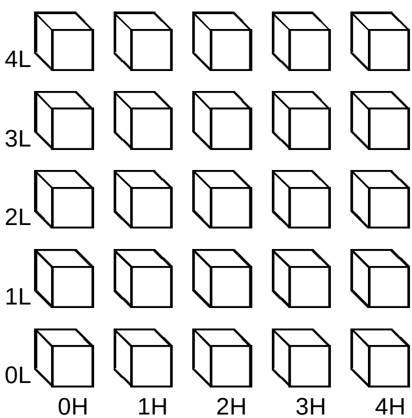


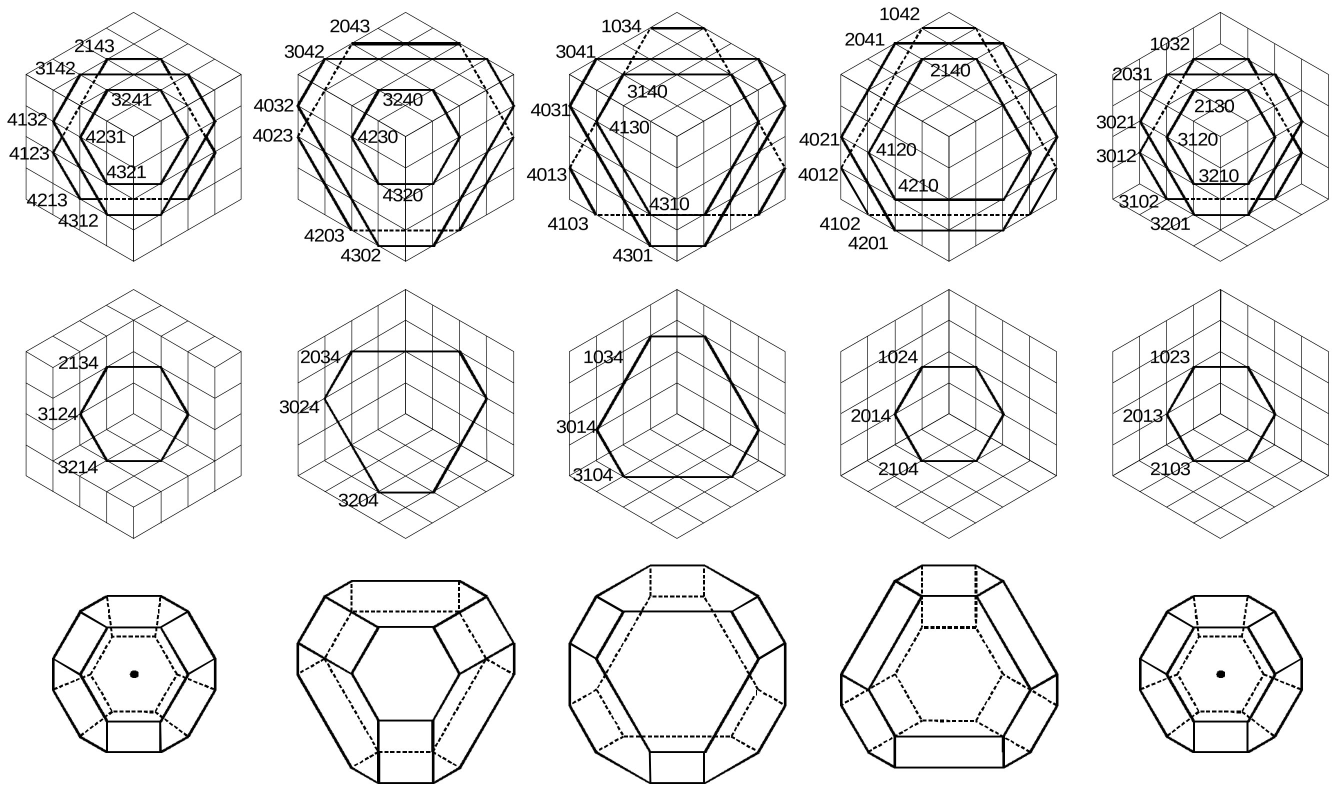
Fig. 75.

Now, as we saw before, there is nothing to prevent us from putting all the cubes representing the different stages in the fourth dimension in one figure, if we take note when we look at it, whether we consider it as a 0*h*, a 1*h*, a 2*h*, etc., cube. Putting then the 0*h*, 1*h*, 2*h*, 3*h*, 4*h* cubes of each row in one, we have five cubes with the sides of each containing five positions, the first of these five cubes represents the 0*l* points, and has in it the *i* points from 0 to 4, the *j* points from 0 to 4, the *k* points from 0 to 4, while we have to specify with regard to any selection we make from it, whether we regard it as a 0*h*, a 1*h*, a 2*h*, a 3*h*, or a 4*h* figure. In fig. 76 each cube is represented by two drawings, one of the front part, the other of the rear part.

Let then our five cubes be arranged before us and our selection be made according to the rule. Take the first figure in which all points are 0*l* points. We cannot have 0 with any other letter. Then, keeping in the first figure, which is that of the 0*l* positions, take first of all that selection which always contains 1*h*. We suppose, therefore, that the cube is a 1*h* cube, and in it we take *i*, *j*, *k* in combination with 4, 3, 2 according to the rule.

The figure we obtain is a hexagon, as shown, the one in front. The points on the right hand have the same figures as those on the left, with the first two numerals interchanged. Next keeping still to the 0*l* figure let us suppose that the cube before us represents a section at a distance 2 in the *h* direction. Let all the points in it be considered as 2*h* points. We then have a 0*l*, 2*h* region, and have the sets *ijk* and 431 left over. We must then pick out in accordance with our rule all such points as 4*i*, 3*j*, 1*k*.

These are shown in the figure and we find that we can draw them without confusion, forming the second hexagon from the front. Going on in this way it will be seen that in each of the five figures a set of hexagons is picked out, which put together form a three-space figure something like the tetrakaidekagon.



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2134
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3204
1034
3014
3104
1024
2014
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1023
2013
2103

Fig. 76

These separate figures are the successive stages in which the four-dimensional figure in which they cohere can be apprehended.

The first figure and the last are tetrakaidekagons. These are two of the solid boundaries of the figure. The other solid boundaries can be traced easily. Some of them are complete from one face in the figure to the corresponding face in the next, as for instance the solid which extends from the hexagonal base of the first figure to the equal hexagonal base of the second figure. This kind of boundary is a hexagonal prism. The hexagonal prism also occurs in another sectional series, as for instance, in the square at the bottom of the first figure, the oblong at the base of the second and the square at the base of the third figure.

Other solid boundaries can be traced through four of the five sectional figures. Thus taking the hexagon at the top of the first figure we find in the next a hexagon also, of which some alternate sides are elongated. The top of the third figure is also a hexagon with the other set of alternate rules elongated, and finally we come in the fourth figure to a regular hexagon.

These four sections are the sections of a tetrakaidekagon as can be recognised from the sections of this figure which we have had previously. Hence the boundaries are of two kinds, hexagonal prisms and tetrakaidekagons.

These four-dimensional figures exactly fill four-dimensional space by equal repetitions of themselves.

CHAPTER XI

NOMENCLATURE AND ANALOGIES PRELIMINARY TO THE STUDY OF FOUR-DIMENSIONAL FIGURES

IN the following pages a method of designating different regions of space by a systematic colour scheme has been adopted. The explanations have been given in such a manner as to involve no reference to models, the diagrams will be found sufficient. But to facilitate the study a description of a set of models is given in an appendix which the reader can either make for himself or obtain. If models are used the diagrams in Chapters XI. and XII. will form a guide sufficient to indicate their use. Cubes of the colours designated by the diagrams should be picked out and used to reinforce the diagrams. The reader, in the following description, should suppose that a board or wall stretches away from him, against which the figures are placed.

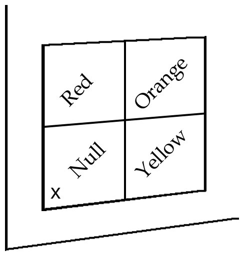


Fig. 77.

Take a square, one of those shown in Fig. 77 and give it a neutral colour, let this colour be called "null," and be such that it makes no appreciable difference to any colour with which it is mixed. If there is no such real colour let us imagine such a colour, and assign to it the properties of the number zero, which makes no difference in any number to which it is added.

Above this square place a red square. Thus we symbolise the going up by adding red to null.

Away from this null square place a yellow square, and represent going away by adding yellow to null.

To complete the figure we need a fourth square. Colour this orange, which is a mixture of red and yellow, and so appropriately represents a going in a direction compounded of up and away. We have thus a colour scheme which will serve to name the set of squares drawn. We have two axes of colours—red and yellow—and they may occupy as in the figure the direction up and away, or they may be turned about; in any case they enable us to name the four squares drawn in their relation to one another.

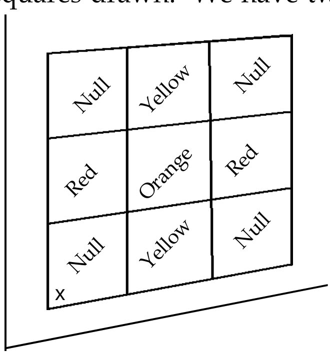


Fig. 78.

Now take, in Fig. 78, nine squares, and suppose that at the end of the going in any direction the colour started with repeats itself.

We obtain a square named as shown.

Let us now, in fig. 79, suppose the number of squares to be increased, keeping still to the principle of colouring already used.

Here the nulls remain four in number. There are three reds between the first null and the null above it, three yellows between the first null and the null beyond it, while the oranges increase in a double way.

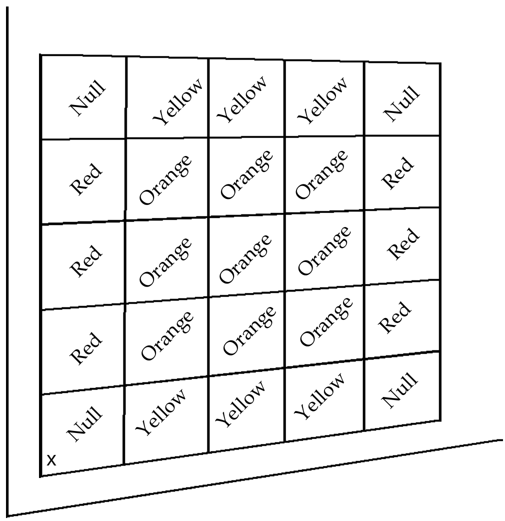


Fig. 79.

Suppose this process of enlarging the number of the squares to be indefinitely pursued and the total figure obtained to be reduced in size, we should obtain a square of which the interior was all orange, which the lines round it were red and yellow, and merely the points null colour, as in fig. 80. Thus all the points, lines, and the area would have a colour.

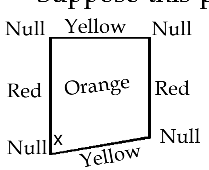


Fig. 80.

We can consider this scheme to originate thus:--Let a small point move in a yellow direction and trace out a yellow line and end in a null point. Then let the whole line thus traced move in a red direction. The null points at the ends of the line will produced red lines, and end in null points. The yellow lines will trace out a yellow and red, or orange square.

Now, turning back to fig. 78, we see that these two ways of naming, the one we started with and the one we arrived at, can be combined.

By its position in the group of four squares, in fig. 77, the null square has a relation to the yellow and to the red directions. We can speak therefore of the red line of the null square without confusion, meaning thereby the line AB, fig. 81, which runs up from the initial null point A in the figure as drawn. The yellow line of the null square is its lower horizontal line AC as it is situated in the figure.

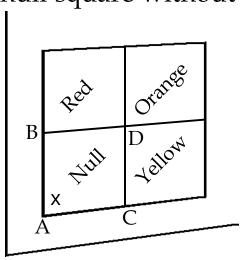


Fig. 81.

If we wish to denote the upper yellow line BD, fig. 81, we can speak of it as the yellow r line, meaning the yellow line which is separated from the primary yellow line by the red movement.

In a similar way each of the other squares has null points, red and yellow lines. Although the yellow square is all yellow its line CD, for instance, can be referred to as its red line.

This nomenclature can be extended.

If the eight cubes drawn in fig. 82 are put close together, as on the right hand of the diagram, they form a cube, and in them, as thus arranged, a going up is represented by adding red to the zero, or null colour, a going away by adding yellow, a going to the right by adding white. White is used as a colour, as a pigment, which produces a colour change in the pigment with which it is mixed. From whatever cube of the lower set we start, a motion up brings us to a cube showing a change to red, thus light yellow becomes light yellow red, or light orange, which is called ochre. And going to the right from the null on the left we have a change involving the introduction of white, while the yellow change runs from front to back. There are three colour axes—the red,

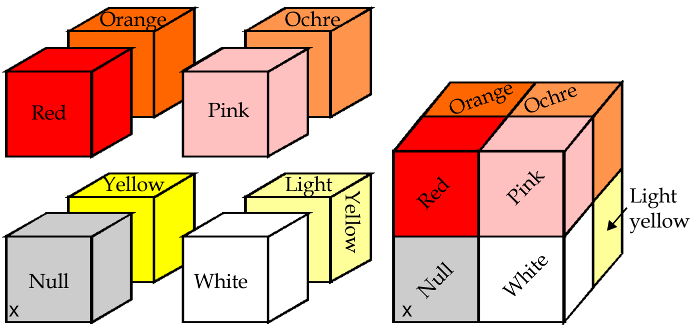


Fig. 82.

the white, the yellow—and these run in the position the cubes occupy in the drawing—up, to the right, away—but they could be turned about to occupy any positions in space.

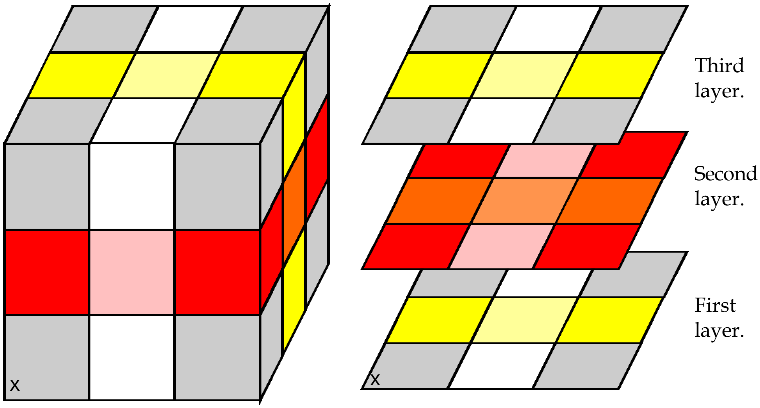


Fig. 83.

We can conveniently represent a block of cubes, by three sets of squares, representing each the base of a cube.

Thus the block, fig. 83, can be represented by the layers on the right. Here, as in the case of the plane, the initial colours repeat themselves at the end of the series.

Proceeding now to increase the number of the cubes, we obtain fig. 84, in which the initial letters of the colours are given instead of their full names.

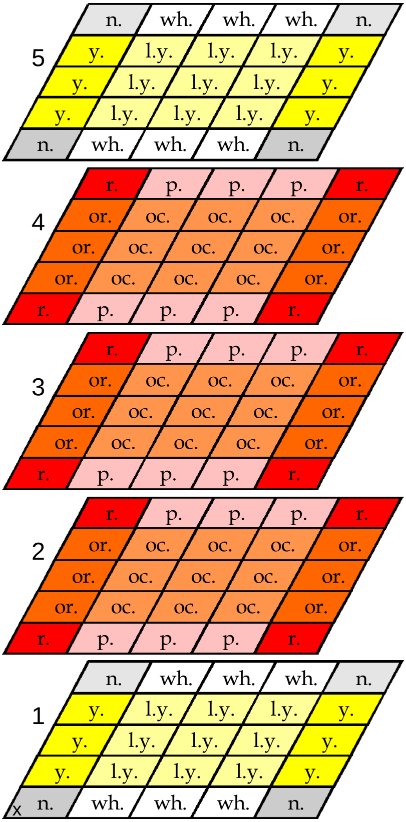


Fig. 84..

Here we see that there are four null cubes as before, but the series which spring from the initial corner will tend to become lines of cubes, as also the sets of cubes parallel to them, starting from other corners. Thus, from the initial null springs a line of red cubes, a line of white cubes, and a line of yellow cubes.

If the number of the cubes is largely increased, and the size of the whole cube is diminished, we get a cube with null points, and the edges coloured with these three colours.

The light yellow cubes increase in two ways, forming ultimately a sheet of cubes, and the same is true of the orange and pink sets. Hence, ultimately the cube thus formed would have red, white, and yellow lines surrounding pink, orange, and light yellow faces. The ochre cubes increase in three ways, and hence ultimately the whole interior of the cube would be coloured ochre.

We have thus a nomenclature for the points, lines, faces, and solid content of a cube, and it can be named as exhibited in fig. 85.

We can consider the cube to be produced in the following way. A null point moves in a direction, to which we attach the colour indication yellow; it generates a yellow line and ends in a null point. The yellow line thus generated moves in a direction to which we give the colour indication red. This lies up in the figure. The yellow line traces out a yellow, red, or orange square, and each of its null points trace out a red line, and ends in a null point.

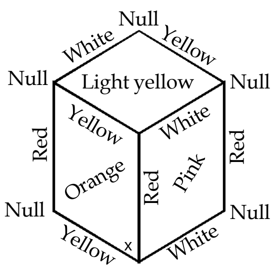


Fig. 85.

This orange square moves in a direction to which we attribute the colour indication white, in this case the direction is the right. The square traces out a cube coloured orange, red, or ochre, the red lines trace out red to white or pink squares, and the yellow lines trace out light yellow squares, each line ending in a line of its own colour. While the points each trace out a null + white, or while line to end in a null point.

Now returning to the first block of eight cubes we can name each point, line, and square in them by reference to the colour scheme, which they determine by their relation to each other.

Thus, in fig. 86, the null cube touches the red cube by a light yellow square; it touches the yellow cube by a pink square, and touches the white cube by an orange square.

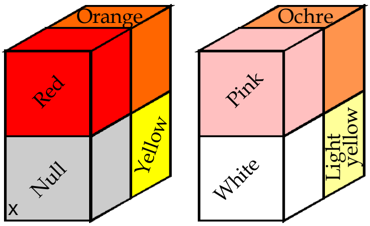


Fig. 86.

There are three axes to which the colours red, yellow, and white are assigned, the faces of each cube are designated by taking these colours in pairs. Taking all the colours together we get a colour name for the solidity of a cube.

Let us now ask ourselves how the cube would be presented to the plane being. Without going into the question of how he could have a real experience of it, let us see how, if we could turn it about and show it to him, he, under his limitations, could get information about it. If the cube were placed with its red and yellow axes against a plane, that is resting against it by its orange

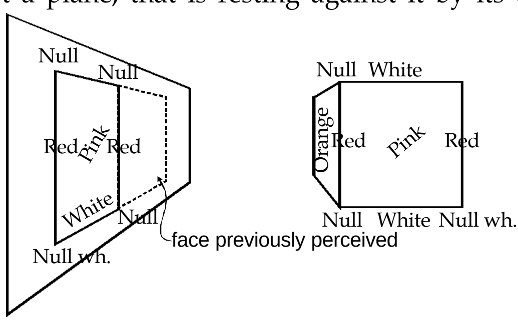


Fig. 87.

face, the plane being would observe a square surrounded by red and yellow lines, and having null points. See the dotted square, fig. 87.

We could turn the cube about the red line so that a different face comes into juxtaposition with the plane.

Suppose the cube turned about the red line. As it is turning from its first position all of it except the red line leaves the plane—goes absolutely out of the range of the plane being's apprehension. But when the yellow line points straight out from the plane then the pink face comes into contact with it. Thus the same red line remaining as he saw it at first, now towards him comes a face surrounded by white and red lines.

If we call the direction to the right the unknown direction, then the line he saw before, the yellow line, goes out into this unknown direction, and the line which before went into the unknown direction, comes in. It comes in in the opposite direction to that in which the yellow line ran before; the interior of the face now against the plane is pink. It is a property of two lines at right angles that, if one turns out of a given direction and stands at right angles to it, then the other of the two lines comes in, but runs the opposite way in that given direction, as in fig. 88.

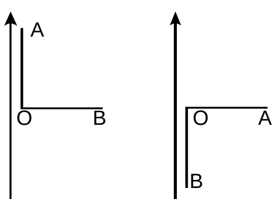


Fig. 88.

Now these two presentations of the cube would seem to the plane creature like perfectly different material bodies, with only that line in common in which they both meet.

Again our cube can be turned about the yellow line. In this case the yellow square would disappear as before, but a new square would come into the plane after the cube had rotated by an angle of 90° about this line. The bottom square of the cube would come in thus in figure 89. The cube supposed in contact with the plane is rotated about the lower yellow line and then the bottom face is in contact with the plane.

Here, as before, the red line going out into the unknown dimension, the white line white before ran in the unknown dimension would come in downwards in the opposite sense to that in which the red line ran before.

Now if we use *i*, *j*, *k*, for the three space directions, *i* left to right, *j* from near away, *k* from below up; then, using the colour names for the axes, we have that first of all white runs *i*, yellow runs *j*, red runs *k*; then after

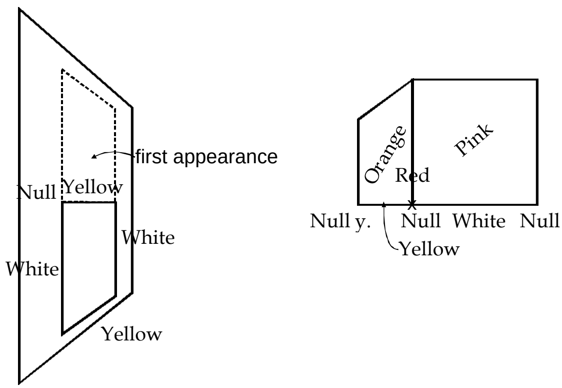


Fig. 89.

the first turning round the *k* axis, white runs negative *j*, yellow runs *i*, red runs *k*; thus we have the table:—

	i	*j*	*k*
1st position	white	yellow	red
2nd position	yellow	white—	red
3rd position	red	yellow	white—

Here white with a negative sign after it in the column under *j* means that white runs in the negative sense of the *j* direction.

We may express the fact in the following way:— In the plane there is room for two axes while the body has three. Therefore in the plane we can represent any two. If we want to keep the axis that goes in the unknown dimension always running in the positive sense, then the axis which originally ran in the unknown dimension (the white axis) must come in the negative sense of that axis which goes out of the plane into the unknown dimension.

It is obvious that the unknown direction, the direction in which the white lines runs at first, is quite distinct from any direction which the plane creature knows. The white line may come in towards him, or running down. If he is looking at a square, which is the face of a cube (looking at it by a line), then any one of the bounding lines remaining unmoved, another face of the cube may come in, any one of the faces, namely, which have the white line in them. And the white line comes sometimes in one of the space directions he knows, sometimes in another.

Now this turning which leaves a line unchanged is something quite unlike any turning he knows in the plane. In the plane a figure turns round a point. The square can turn round the null point in his plane, and the red and yellow lines change places, only of course, as with every rotation of lines at right angles, if red goes where yellow went, yellow comes in negative of red's old direction.

This turning, as the plane creature conceives it, we should call turning about an axis perpendicular to the plane. What he calls turning about the null point we call turning about the white line as it stands out from his plane. There is no such thing as turning about a point, there is always an axis, and really much more turns than the plane being is aware of.

Taking now a different point of view, let us suppose the cubes to be presented to the plane being by being passed transverse to his plane. Let us suppose the sheet of matter over which the plane being and all objects in his world slide, to be of such a nature that objects can pass through it without breaking it. Let us suppose it to be of the same nature as the film of a soap bubble, so that it closes around objects pushed through it, and, however the object alters its shape as it passes through it, let us suppose this film to run up to the contour of the object in every part, maintaining its plane surface unbroken.

Then we can push a cube or any object through the film and the plane being who slips about in the film will know the contour of the cube just and exactly where the film meets it.

Fig. 90 represents a cube passing through a plane film. The plane being now comes into contact with a very thin slice of the cube somewhere between the left and right hand faces. This very thin slice he thinks of as having no thickness, and consequently his idea of it is what we call a section. It is bounded by him by pink lines front and back, coming from the part of the pink face he is in contact with, and above and below, by light yellow lines. Its corners are not null-coloured points, but white points, and its interior is ochre, the colour of the interior of the cube.

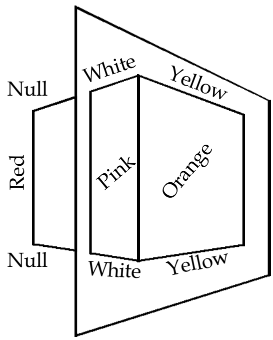


Fig. 90..

If now we suppose the cube to be an inch in each dimension, and to pass across, from right to left, through the plane, then we should explain the appearances presented to the plain being by saying: First of all you have the face of a cube, this lasts only a moment; then you have a figure of the same shape but differently coloured. This, which appears not to move to you in any direction which you know of, is really moving transverse to your plane world. Its appearance is unaltered, but each moment it is something different—a section further on, in the white, the unknown dimension. Finally, at the end of the minute, a face comes in exactly like the face you first saw. This finished up the cube—it is the further face in the unknown dimension.

The white line, which extends in length just like the red or the yellow, you do not see as extensive; you apprehend it simply as an enduring white point. The null point, under the condition of movement of the cube, vanishes in a moment, the lasting white point is really your apprehension of a white line, running in the unknown dimension. In the same way the red line of the face by which the cube is first in contact with the plane lasts only a moment, it is succeeded by the pink line, and this pink line lasts for the inside of a minute. This lasting pink line is your apprehension of a surface, which extends in two dimensions just like the orange surface extends, as you know it, when the cube is at rest.

But the plane creature might answer, "This orange object is substance, solid substance, bounded completely and on every side."

Here, of course, the difficulty comes in. His solid is our surface—his notion of a solid is our notion of an abstract surface with no thickness at all.

We should have to explain to him that, from every point of what he called a solid, a new dimension runs away. From every point a line can be drawn in a direction unknown to him, and there is a solidity of a kind greater than that which he knows. This solidity can only be realised by him by his supposing an unknown direction, by motion in which what he conceives to be solid matter instantly disappears. The higher solid, however, which extends in this dimension as well as in those which he knows, lasts when a motion of that kind takes place, different sections of it come consecutively in the plane of his apprehension, and take the place of the solid which he at first sign conceives to be all. Thus, the higher solid—our solid in contradistinction to his area solid, his two-dimensional solid, must be conceived by him as something which has duration in it, under circumstances in which his matter disappears out of his world.

We may put the matter thus, using the conception of motion.

A null point moving in a direction away generates a yellow line, and the yellow line ends in a null point. We suppose, that is, a point to move and mark out the products of this motion in such a manner. Now suppose this whole line as thus produced to move in an upward direction; it traces out the two-dimensional solid, and the plane being gets an orange square. The null point moves in a red line and ends in a null point, the yellow line moves and generates an orange square and ends in a yellow line, the farther null point generates a red line and ends in a null point. Thus, by movement in two successive directions known to him, he can imagine his two-dimensional solid produced with all its boundaries.

Now we well him: "This whole two-dimensional solid can move in a third or unknown dimension to you. The null point moving in this dimension out of your world generates a white line and ends in a null point. The yellow line moving generates a light yellow two-dimensional solid and ends in a yellow line, and this two-dimensional solid, lying end on to your plane world, is bounded on the far side by the other yellow line. In the same way each of the lines surrounding your square traces out an area, just like the orange area you know. But there is something new produced, something which you had no idea of before; it is that which is produced by the movement of the orange square. That, then which you can imagine nothing more solid, itself moves in a direction open to it and produces a three-dimensional solid. Using the addition of white to symbolise the products of this motion this new kind of solid will be light orange or ochre, and it will be bounded on the far side by the final position of the orange square which traced it out, and this final position we suppose to be coloured like the square in its first position, orange with yellow and red boundaries and null corners."

This product of movement, which is so easy for us to describe, would be difficult for him to conceive. But this difficulty is connected rather with its totality than with any particular part of it.

Any line, or plane of this, to him higher, solid we could show to him, and put in his sensible world.

We have already seen how the pink square could be put in his world by a turning of the cube about the red line. And any section which we can conceive made of the cube could be exhibited to him. You have simply to turn the cube and push it through, so that the plane of his existence is the plane which cuts out the given section of the cube, then the section would appear to him as a solid. In his world he would see the contour, get to any part of it by digging down into it.

The Process by which a Plane Being would gain a Notion of a Solid

If we suppose the plane being to have a general idea of the existence of a higher solid—our solid—we must next trace out in detail the method, the discipline, by which he would acquire a working familiarity with our space existence. The process begins with an adequate realisation of a simple solid figure. For this purpose we will suppose eight cubes forming a larger cube, and first we will suppose each cube to be coloured throughout uniformly. Let the cubes in fig. 91 be the eight making a large cube.

Now, although each cube is supposed to be coloured entirely through with the colour, the name of which is written on it, still we can speak of the faces, edges, and corners of each cube as if the colour scheme we have investigated held for it. Thus, on the null cube we can speak of a null point, a red line, a white line, a pink face, and so on. These colour designations are shown on No. 1 of the views of the tesseract in the plate. Here these colour

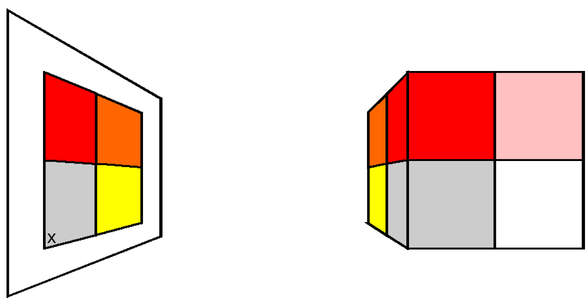


Fig. 91.

names are used simply in their geometrical significance. They denote what the particular line, etc., referred to would have as its colour, if in reference to the particular cube the colour scheme described previously were carried out.

If such a block of cubes were put against the plane and then passed through it from right to left, at the rate of an inch a minute, each cube being an inch each way, the plane being would have the following appearances:—

First of all, four squares null, yellow, red, orange, lasting each a minute; and secondly, taking the exact places of these four squares, four others, coloured white, light yellow, pink, ochre. Thus, to make a catalogue of the solid body, he would have to put side by side in his world two sets of four squares each, as in fig. 92. The first are supposed to last a minute, and then the others to

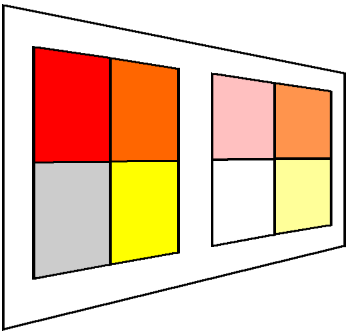


Fig. 92..

come in in place of them and also last a minute.

In speaking of them he would have to denote what part of the respective cube each square represents. Thus, at the beginning he would have null cube orange face, and after the motion had begun he would have null cube ochre section. As he could get the same coloured section whichever way the cube passed through, it would be best for him to call this section white section, meaning that it is transverse to the white axis. These colour-names, of course, are merely used as names, and do not imply in this case that the object is really coloured. Finally, after a minute, as the first cube was passing beyond his plane he would have null cube orange face again.

The same names will hold for each of the other cubes, describing what face or section of them the plane being has before him; and the second wall of cubes will come on, continue, and go out in the same manner. In the area he thus has he can represent any movement which we carry out in the cubes, as long as it does not involve a motion in the direction of the white axis. The relation of parts that succeed one another in the direction of the white axis is realised by him as a consecution of states.

Now, his means of developing his space apprehension lies in this, that that which is represented as a time sequence in one position of the cubes, can become a real co-existence, *if something that has a real co-existence becomes a time sequence*.

We must suppose the cubes turned round each of the axes, the red line, and the yellow line, then something, which was given as time before, will now be given as the plane creature's space; something, which was given as space before, will now be given as a time series as the cube is passed through the plane.

The three positions in which the cubes must be studied are the ones given above and the two following ones. In each case the original null point which was nearest to us at first is marked by an asterisk. In fig. 93 and 94 the

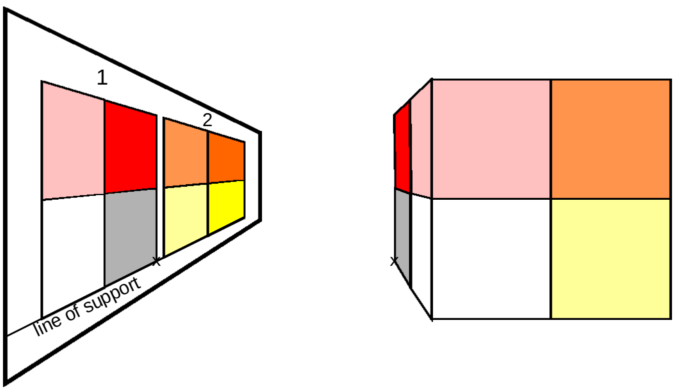


Fig. 93.

The cube swung round the red line, so that the white line points towards us.

point marked with a star is the same in the cubes and in the plane view.

In fig. 93 the cube is swung round the red line so as to point towards us, and consequently the pink face comes next to the plane. As it passes through there are two varieties of appearance designated by the figures 1 and 2 in the plane. These appearances are named in the figure, and are determined by the order in which the cubes come in the motion of the whole block through the plane.

With regard to these squares severally, however, different names must be used, determined by their relations in the block.

Thus, in fig. 93, when the cube first rests against the plane the null cube is in contact by its pink face; as the block passes through we get an ochre section of the null cube, but this is better called a yellow section, as it is made by a plane perpendicular to the yellow line. When

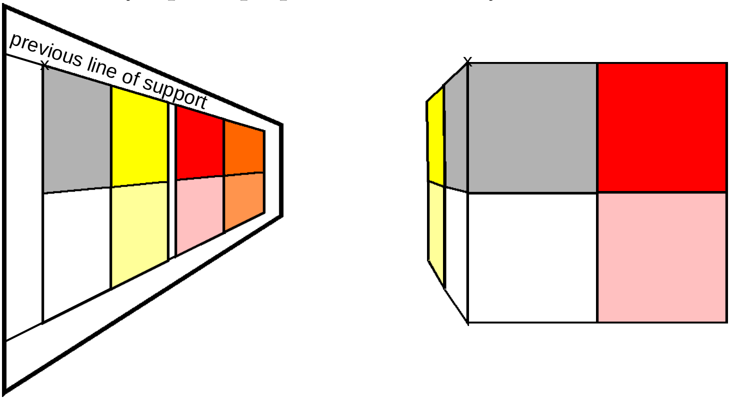


Fig. 94.

The cube swung round yellow line, with red line running from left to right, and white line running down.

the null cube has passed through the plane, as it is leaving it, we get again a pink face.

The same series of changes take place with the cube appearances which follow on those of the null cube. In this motion the yellow cube follows on the null cube, and the square marked yellow in 2 in the plane will be first "yellow pink face," then "yellow yellow section," then "yellow pink face."

In fig. 94, in which the cube is turned about the yellow line, we have a certain difficulty, for the plane being will find that the position his squares are to be placed in will lie below that which they first occupied. They will come where the support was on which he stood his first set of squares. He will get over this difficulty by moving his support.

Then, since the cubes come upon his plane by the light yellow face, he will have, taking the null cube as before for an example, null, light yellow face; null, red section, because the section is perpendicular to the red line; and finally, as the null cube leaves the plane, null, light yellow face. Then, in this case red following on null, he will

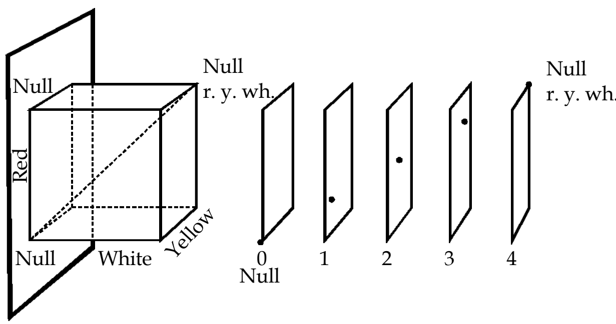


Fig. 95.

have the same series of views of the red as he had of the null cube.

There is another set of considerations which we will briefly allude to.

Suppose there is a hollow cube, and a string extended across it from null to null, *r.*, *y.*, *wh.*, as we may call the far diagonal point, how will this string appear to the plane being as the cube moves transverse to his plane?

Let us represent the cube as a number of sections, say 5, corresponding to 4 equal divisions made along the white line perpendicular to it.

We number these sections 0, 1, 2, 3, 4, corresponding to the distance along the white line at which they are taken, and imagine each section to come in successively, taking the place of the preceding one.

These sections appear to the plane being, counting from the first, to exactly coincide each with the preceding one. But the section of the string occupies a different plane in each to that which it does in the preceding section. The section of the string appears in the position marked by the dots. Hence the slant of the string appears as a motion in the frame work marked out by the cube sides. If we suppose the motion of the cube not be recognised, then the string appears to the plane being as a moving point. Hence extension on the unknown dimension appears as duration. Extension sloping in the unknown direction appears as continuous movement.

CHAPTER XII

THE SIMPLEST FOUR-DIMENSIONAL SOLID

A PLANE being, in learning to apprehend solid existence, must first of all realise that there is a sense of direction altogether wanting to him. That which we call right and left does not exist in his perception. He must assume a movement in a direction, and a distinction of positive and negative in that direction, which has no reality corresponding to it in the movements he can make. This direction, this new dimension, he can only make sensible to himself by bringing in time, and supposing that changes, which take place in time, are due to objects of a definite configuration in three dimensions passing transverse to his plane, and the different sections of it being apprehended as changes of one and the same plane figure.

He must also acquire a distinct notion about his plane world, he must no longer believe that it is the all of space, but that space extends on both side of it. In order, then, to prevent his moving off in this unknown direction, he must assume a sheet, an extended solid sheet, in two dimensions, against which, in contact with which, all his movements take place.

When we come to think of a four-dimensional solid, what are the corresponding assumptions which we must make?

We must suppose a sense which we have not, a sense of direction wanting in us, something which a being in a four-dimensional world has, and which we have not. It is a sense corresponding to a new space direction, a direction which extends positively and negatively from every point of our space, and which goes right away from any space direction we know of. The perpendicular to a plane is perpendicular, not only to two lines in it, but to every line, and so we must conceive this fourth dimension as running perpendicularly to each and every line we can draw in our space.

And as the plane being had to suppose something which prevented his moving off in the third, the unknown dimension to him, so we have to suppose something which prevents us moving off in the direction unknown to us. This something, since we must be in contact with it in every one of our movements, must not be a plane surface, but a solid; it must be a solid, which in every one of our movements we are against, not in. It must be supposed as stretching out in every space dimension that we know; but we are not in it, we are against it, we are next to it, in the fourth dimension.

That is, as the plane being conceives himself as having a very small thickness in the third dimension, of which he is not aware in his new experience, so we must suppose ourselves as having a very small thickness in the fourth dimension, and, being thus four-dimensional beings, to be prevented from realising that we are such beings by a constraint which keeps us always in contact with a vast solid sheet, which stretches on in every direction. We are against that sheet, so that, if we had the power of four-dimensional movement, we should either go away from it or through it; all our space movements as we know them being such that, performing them, we keep in contact with this solid sheet.

Now consider the exposition a plane being would make for himself as to the question of the enclosure of a square, and of a cube.

He would say the square A, in Fig. 96, is completely enclosed by the four squares, A far, A near, A above, A below, or as they are written, A_n, A_f, A_a, A_b,

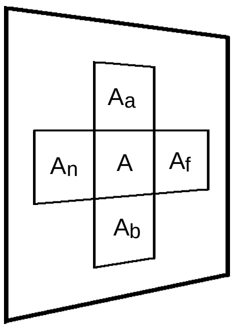
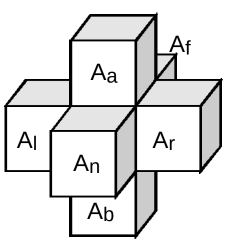


Fig. 96.

If now he conceives the square A to move in the, to him, unknown dimension it will trace out a cube, and the bounding spaces will form cubes. Will these completely surround the cube generated by A? No; there will be two faces of the cube made by A left uncovered; the first, that face which coincident with the square A in its first position; the next, that which coincides with the square A in its final position. Against these two faces cubes must be placed in order to completely enclose the cube A. These may be called the cubes left and right or A_l and A_r. Thus each of the enclosing squares of the square A becomes a cube and two more cubes are wanted to enclose the cube formed by the movement of A in the third dimension.

The plane being could not see the square A with the squares A_n, A_f, etc., placed about it, because they completely hide it from view; and so we, in the analogous case in our three-dimensional world, cannot see a cube surrounded by six other cubes. These cubes we will call A near A_n, A far A_f, A above A_a, A below A_b, A left A_l, A right A_r, shown in fig. 97. If now the cube A moves in the fourth dimension right out of space, it traces out a higher cube—a tesseract, as it may be called.



Fig, 97.

Each of the six surrounding cubes carried on in the same motion will make a tesseract also, and these will be grouped around the tesseract formed by A. But will they enclose it completely?

All the cubes A_n, A_f, etc., lie in our space. But there is nothing between the cube A and that solid sheet in contact with which every particle of matter is. When the cube A moves in the fourth direction it starts from its position, say A_κ, and ends in a final position A_α (using the words "ana" and "kata" for up and down in the fourth dimension). Now the movement in this fourth dimension is not bounded by any of the cubes A_n, A_f, nor by what they form when thus moved. The tesseract which A becomes is bounded in the positive and negative ways in this new direction by the first position of A and the last position of A. Or, if we ask how many tesseracts lie around the tesseract which A forms, there are eight, of which one meets it by the cube A, and another meets it by a cube like A at the end of its motion.

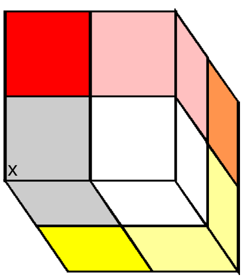
We come here to a very curious thing. The whole solid cube A is to be looked on merely as a boundary of the tesseract.

Yet this is exactly analogous to what the plane being would come to in his study of the solid world. The square A (fig. 96), which the plane being looks on as a solid existence in his plane world, is merely the boundary of the cube which he supposes generated by its motion.

The fact is that we have to recognise that, if there is another dimension of space, our present idea of a solid body, as one which has three dimensions only, does not correspond to anything real, but is the abstract idea of a three-dimensional boundary limiting a four-dimensional solid, which a four-dimensional being would form. The plane being's thought of a square is not the thought of what we should call a possibly existing real square, but the thought of an abstract boundary, the face of a cube.

Let us now take our eight coloured cubes, which form a cube in space, and ask what additions we must make to them to represent the simplest collection of four-dimensional bodies—namely, a group of them of the same extent in every direction. In plane space we have four squares. In solid space we have eight cubes. So we should expect in four-dimensional space to have sixteen four-dimensional bodies—bodies which in four-dimensional space correspond to cubes in three-dimensional space, and these bodies we call tesseracts.

Given then the null, white, red, yellow cubes, and those which make up the block, we notice that we represent perfectly well the extension in three directions (fig. 98). From the null point of the null cube, travelling one inch, we come to the white cube; travelling one inch away we come to the yellow cube; travelling one inch up we come to the red cube. Now, if there is a fourth dimension, then travelling from the same null point for one inch in that direction, we must come to the body lying beyond the null region.



(Orange hidden)
Fig. 98.

I say null region, not cube; for with the introduction of the fourth dimension each of our cubes must become something different from cubes. If they are to have existence in the fourth dimension, they must be "filled up from" in this fourth dimension.

Now we will assume that as we get a transference from null to white going in one way, from null to yellow going in another, so going from null in the fourth direction we have a transference from null to blue, using thus the colours white, yellow, red, blue, to denote transferences in each of the four directions—right, away, up, unknown or fourth dimension.

Hence, as the plane being must represent the solid regions, he would come to by going right, as four squares lying in some position in his plane, arbitrarily chosen, side by side with his original four squares, so we must represent those eight four-dimensional regions, which we should come to by going in the fourth dimension from each of our eight cubes, by eight cubes placed in some arbitrary position relative to our first eight cubes.

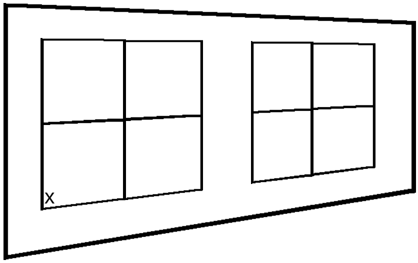
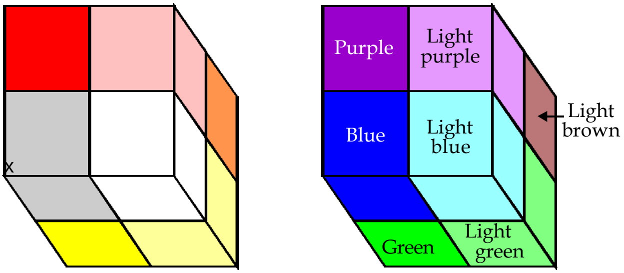


Fig. 99.
A plane being's representation of a block of eight cubes by two sets of four squares.



(1)
Orange hidden

(2)
Brown hidden

Fig. 100.

Our representation of a block of sixteen tesseracts by two blocks of eight cubes.*

Hence, of the two sets of eight cubes, each one will serve

* The eight cubes used here in 2 can be found in the second of the model blocks. They can be taken out and used.

us as a representation of one of the sixteen tesseracts which form one single block in four-dimensional space. Each cube, as we have it, is a tray, as it were, against which the real four-dimensional figure rests—just as each of the squares which the plane being has is a tray, so to speak, against which the cube it represents could rest.

If we suppose the cubes to be one inch each way, then the original eight cubes will give eight tesseracts of the same colours, or the cubes, extending each one inch in the fourth dimension.

But after these there come, going on in the fourth dimension, eight other bodies, eight other tesseracts. They must be there, if we suppose the four-dimensional body we make up to have two divisions, one inch each in each of four directions.

The colour we choose to designate the transference to this second region in the fourth dimension is blue. Thus, starting from the null cube and going in the fourth dimension, we first go through one inch of the null tesseract, then we come to a blue cube, which is the beginning of a blue tesseract. This blue tesseract stretches one inch further on in the fourth dimension.

Thus, beyond each of the eight tesseracts, which are of the same colour as the cubes which are their bases, lie eight tesseracts whose colours are derived from the colours of the first eight by adding blue. Thus—

Null	gives	blue
Yellow	,,	green
Red	,,	purple
Orange	,,	brown
White	,,	light blue
Pink	,,	light purple
Light yellow	,,	light green
Ochre	,,	light brown

The addition to blue of yellow gives green—this is a natural supposition to make. It is also natural to suppose that blue added to red makes purple. Orange and blue can be made to give a brown, by using certain shades and proportions. And ochre and blue can be made to give a light brown.

But the scheme of colours is merely used for getting a definite and realisable set of names and distinctions visible to the eye. Their naturalness is apparent to any one in the habit of using colours, and may be assumed to be justifiable, as the sole purpose is to devise a set of names which are easy to remember, and which will give us a set of colours by which diagrams may be made easy of comprehension. No scientific classification of colours has been attempted.

Starting, then, with these sixteen colour names, we have a catalogue of the sixteen tesseracts, which form a four-dimensional block analogous to the cubic block. But the cube which we can put in space and look at is not one of the constituent tesseracts; it is merely the beginning, the solid face, the side, the aspect, of a tesseract.

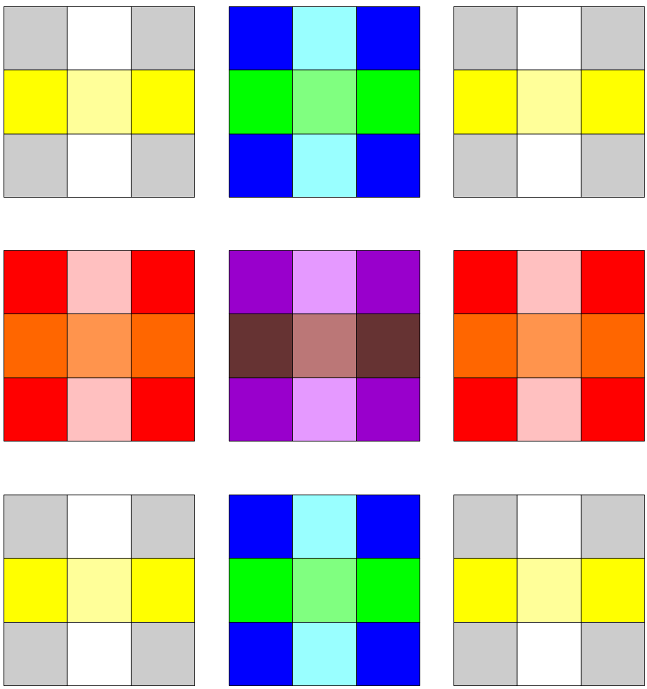
We will now proceed to derive a name for each region, point, edge, plane face, solid and a face of the tesseract.

The system will be clear, if we look at a representation in the plane of a tesseract with three, and one with four divisions in its side.

The tesseract made up of three tesseracts each way corresponds to the cube made up of three cubes each way, and will give us a complete nomenclature.

In this diagram, fig. 101, 1 represents a cube of 27 cubes, each of which is the beginning of a tesseract. These cubes are represented only by their lowest squares, the solid content must be understood. 2 represents the 27 cubes which are the beginnings of the 27 tesseracts one inch on in the fourth dimension. These tesseracts are represented as a block of cubes put side by side with the first block, but in their proper positions they could not be in space with the first set. 3 represents 27 cubes

Fig. 101.



1

Each cube is the beginning of the first tesseract going in the fourth dimension

2

Each cube is the beginning of the second tesseract.

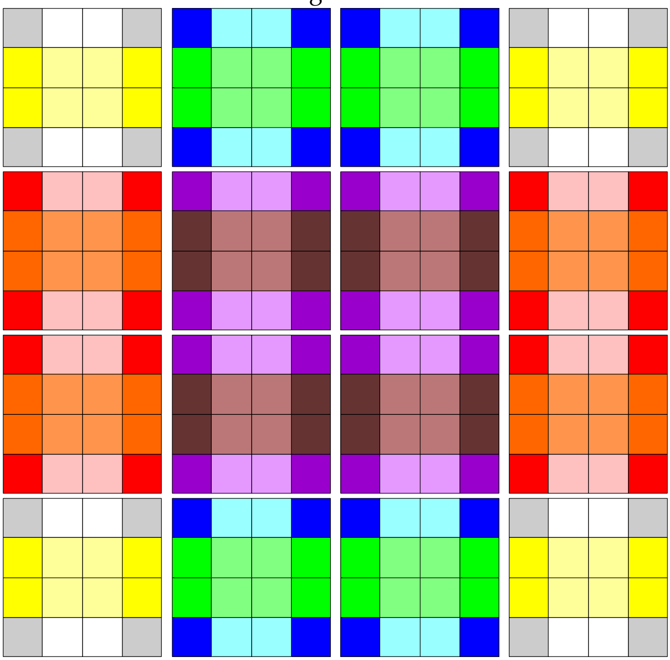
3

Each cube is the beginning of the third tesseract.

(forming a larger cube) which are the beginnings of the tesseracts, which begin two inches in the fourth direction from our space and continue another inch.

In fig. 102, we have the representation of a block of $4 \times 4 \times 4 \times 4$ or 256 tesseracts. They are given in

Fig. 102.*



1

A cube of 64 cubes, each 1 in. × 1 in. × 1 in., the beginning of a tesseract.

2

A cube of 64 cubes, each 1 in. × 1 in. × 1 in., the beginning of tesseracts 1 in. from our space in the 4th dimension.

3

A cube of 64 cubes, each 1 in. × 1 in. × 1 in., the beginning of tesseracts. 2 in. from our space in the 4th dimension.

4

A cube of 64 cubes, each 1 in. × 1 in. × 1 in., the beginning of tesseracts. starting 3 in. from our space in the 4th dimension.

four consecutive sections, each supposed to be taken one inch apart in the fourth dimension, and on giving four

* The coloured plate, figs. 1, 2, 3, shows these relations more conspicuously.

blocks of cubes, 64 in each block. He we see, comparing it with the figure of 81 tesseracts, that the number of the different regions shows a different tendency of increase. But taking five blocks of five divisions each way this would become even more clear.

We see, fig. 102, that starting from the point at any corner, the white coloured regions only extend out in a line. The same is true for the yellow, red, and blue. With regard to the latter is should be noticed that the line of blues does not consist in regions next to each other in the drawing, but in portions which come in in different cubes. The portions which lie next to one another in the fourth dimension must always be represented so, when we have a three-dimensional representation. Again, those regions such as the pink one, go on increasing in two dimensions. About the pink region this is seen without going out of the cube itself, the pink regions increase in length and height, but in no other dimension. In examining these regions it is sufficient to take one as a sample.

The purple increases in the same manner, for it comes in in a succession from below to above in block 2, and in succession from block to block in 2 and 3. Now, a succession from below to above represents a continuous extension upwards, and a succession from block to block represents a continuous extension in the fourth dimension. Thus the purple regions increase in two dimensions, the upward and the fourth, so when we take a very great many divisions, and let each become very small, the purple region forms a two-dimensional extension.

In the same way, looking at the regions coloured in light blue, which starts nearest a corner, we see that the tesseracts occupying it increase in length from left to right, forming a line, and that there are as many lines of light blue tesseracts as there are sections between the first and last section. Hence the light blue tesseracts increase in number in two ways—in the right and left, and in the fourth dimension. They ultimately form what we may call a plane surface.

Now all those regions which contain a mixture of two simple colours, white, yellow, red, blue, increase in two ways. On the other hand, those which contain a mixture of three colours increase in three ways. Take, for instance, the ochre region; this has three colours, white, yellow, red; and in the cube itself it increases in three ways.

Now regard the orange region; if we add blue to this we get a brown. The region of the brown tesseracts extends in two ways on the left of the second block, No. 2 in the figure. It extends also from left to right in succession from one section to another, from section 2 to section 3 in our figure.

Hence the brown tesseracts increase in number in three dimensions upwards, to and fro, fourth dimension. Hence they form a cubic, a three-dimensional region; this region extends up and down, near and far, and in the fourth direction, but is thin in the direction from left to right. It is a cube which, when the complete tesseract is represented in our space, appears as a series of faces on the successive cubic sections of the tesseract. Compare fig. 103 in which the middle block, 2, stands as representing a great number of sections intermediate between 1 and 3.

In a similar way from the pink region by addition of blue we have the light purple region, which can be seen to increase in three ways as the number of divisions becomes greater. The three ways in which this region of tesseracts extends is up and down, right and left, fourth dimension. Finally, therefore, it forms a cubic mass of very small tesseracts, and when the tesseract is given in space sections it appears on the faces containing the upward and the right and left dimensions.

We get then altogether, as three-dimensional regions, ochre, brown, light purple, light green.

Finally, there is the region which corresponds to a mixture of all the colours; there is only one region such as this. It is the one that springs from ochre by the addition of blue—this colour we call light brown.

Looking at the light brown region we see that it increases in four ways. Hence, the tesseracts of which it is composed increase in number in each of four dimensions, and the shape they form does not remain thin in any of the four dimensions. Consequently this region becomes the solid content of the block of tesseracts itself; it is the real four-dimensional solid. All the other regions are then boundaries of this light brown region. If we suppose the process of increasing the number of tesseracts and decreasing their size carried on indefinitely, then the light brown coloured tesseracts become the whole interior mass, the three-coloured tesseracts become three-dimensional boundaries, thin in one dimension, and form the ochre, the brown, the light blue, the light green. The two-coloured tesseracts become two-dimensional boundaries, thin in two dimensions, *e.g.*, the pink, the green, the purple, the orange, the light blue, the light yellow. The one-coloured tesseracts become bounding lines, thin in three dimensions, and the null points become bounding corners, thin in four dimensions. From these thin real boundaries we can pass in thought to the abstractions—points, lines, faces, solids—bounding the four-dimensional solid, which in this case is light brown coloured, and under this supposition the light brown coloured region is the only real one, is the only one which is not an abstraction.

It should be observed that, in taking a square as the representation of a cube on a plane, we only represent one face, or the section between two faces. The squares, as drawn by a plane being, are not the cubes themselves, but represent the faces or the sections of a cube. Thus in the plane being's diagram a cube of twenty-seven cubes "null" represents a cube, but is really, in the normal position, the orange square of a null cube, and may be called null, orange square.

A plane being would save himself confusion if he named his representative squares, not by using the names of the cubes simply, but by adding to the names of the cubes a word to show what part of a cube his representative square was.

Thus a cube null standing against his plane touches it by null orange face, passing through his plane it has in the plane a square as trace, which is null white section, if we use the phrase white section to mean a section drawn perpendicular to the white line. In the same way the cubes which we take as representative of the tesseract are not the tesseract itself, but definite faces or sections of it. In the preceding figures we should say then, not null, but "null tesseract ochre cube," because the cube we actually have is the one determined by the three axes, white, red, yellow.

There is another way in which we can regard the colour nomenclature of the boundaries of a tesseract.

Consider a null point to move tracing out a white line one inch in length, and terminating in a null point, see fig. 103 or in the coloured plate.

Then consider this white line with its terminal points itself to move in a second dimension, each of the points traces out a line, the line itself traces out an area, and give two lines as well, its initial and final position.

Thus, if we call "a region" any element of the figure, such as a point, or a line, etc., every "region" in moving traces out a new kind of region, "a higher region," and give two regions of its own kind, an initial and a final position. The "higher region" means a region with another dimension in it.

Now the square can move and generate a cube. The square light yellow moves and traces out the mass of the cube. Letting the addition of red denote the region made by the motion in the upward direction we get an ochre solid. The light yellow face in its initial and terminal positions give the two square boundaries of the cube above and below. Then each of the four lines of the light yellow square—white, yellow, and the white, yellow opposite them—trace out a bounding square. So there are in all six bounding squares, four of these squares being designated in colour by adding red to the colour of the generating lines. Finally, each point moving in the up direction gives rise to a line coloured null + red, or red, and then there are the initial and terminal positions of the points giving eight points. The number of the lines is evidently twelve, for the four lines of the light yellow square give four lines in their initial, four lines in their final position, while the four points trace out four lines, that is altogether twelve lines.

Now the squares are each of them separate boundaries of the cube, while the lines belong, each of them, to two squares, thus the red line is that which is common to the orange and pink squares.

Now suppose that there is a direction, the fourth dimension, which is perpendicular alike to every one of the space dimensions already used—a dimension perpendicular, for instance, to up and to right hand, so that the pink square moving in this direction traces out a cube.

A dimension, moreover, perpendicular to the up and away directions, so that the orange square moving in this direction also traces out a cube, and the light yellow square, too, moving in this direction traces out a cube.

Under this supposition, the whole cube moving in the unknown dimension, traces out something new—a new kind of volume, a higher volume. This higher volume is a four-dimensional volume, and we designate it in colour by adding blue to the colour of that which by moving generates it.

It is generated by the motion of the ochre solid, and hence it is of the colour we call light brown (white, yellow, red, blue, mixed together). It is represented by a number of sections like 2 in fig. 103.

Now this light brown higher solid has for boundaries: first. the ochre cube in its initial position, second, the same cube in its final position, 1 and 3, fig. 103. Each of the squares which bound the cube, moreover, by movement in this new direction traces out a cube, so we have from the front pink faces of the cube, third, a pink blue or light purple cube, shown as a light purple face on cube 2 in fig. 103, this face standing for any number of intermediate sections; fourth, a similar cube from the opposite pink face; fifth, a cube traced out by the orange face—this is coloured brown and is represented by the brown face of the section cube in fig. 103; sixth, a corresponding brown cube on the right hand; seventh, a cube starting from the light yellow square below; the unknown dimension is at right angles to this also. This cube is coloured light yellow and blue or light green; and finally, eighth, a corresponding cube from the upper light yellow face, shown as the light green square at the top of the section cube.

The tesseract has thus eight cubic boundaries. These completely enclose it, so that it would be invisible to a four-dimensional being. Now, as to the other boundaries, just as the cube has squares, lines, points, as boundaries, so the tesseract has cubes, squares, lines, points, as boundaries.

The number of squares is found thus—round the cube are six squares, these will give six squares in their initial and six in their final positions. Then each of the twelve lines of a cube trace out a square in the motion in the fourth dimension. Hence there will be altogether 12 + 12 = 24 squares.

If we look at any one of these squares we see that it is the meeting surface of two of the cubic sides. Thus, the red line by its movement in the fourth dimension traces out a purple square—this is common to two cubes, one of which is traced out by the pink square moving in the fourth dimension, and the other is traced out by the orange square moving in the same way. To take another square, the light yellow one, this is common to the ochre cube and the light green cube. The ochre cube comes from the light yellow square by moving it in the up direction, the light green cube is made from the light yellow square by moving it in the fourth dimension. The number of lines is thirty-two, for the twelve lines of the cube give twelve lines of the tesseract in their initial position, and twelve in their final position, making twenty-four, while each of the eight points traces out a line, thus forming thirty-two lines altogether.

The lines are each of them common to three cubes, or to three square faces; take, for instance, the red line. This is common to the orange face, the pink face, and that face which is formed by moving the red line in the fourth dimension, namely, the purple face. It is also common to the ochre cube, the pale purple cube, and the brown cube.

The points are common to six square faces and to four cubes; thus, the null point from which we start is common to the three square faces—pink, light yellow, orange, and to the three square faces made by moving the three lines white, yellow, red, in the fourth dimension, namely, the light blue, the light green, the purple faces—that is, to six faces in all. The four cubes which meet in it are the

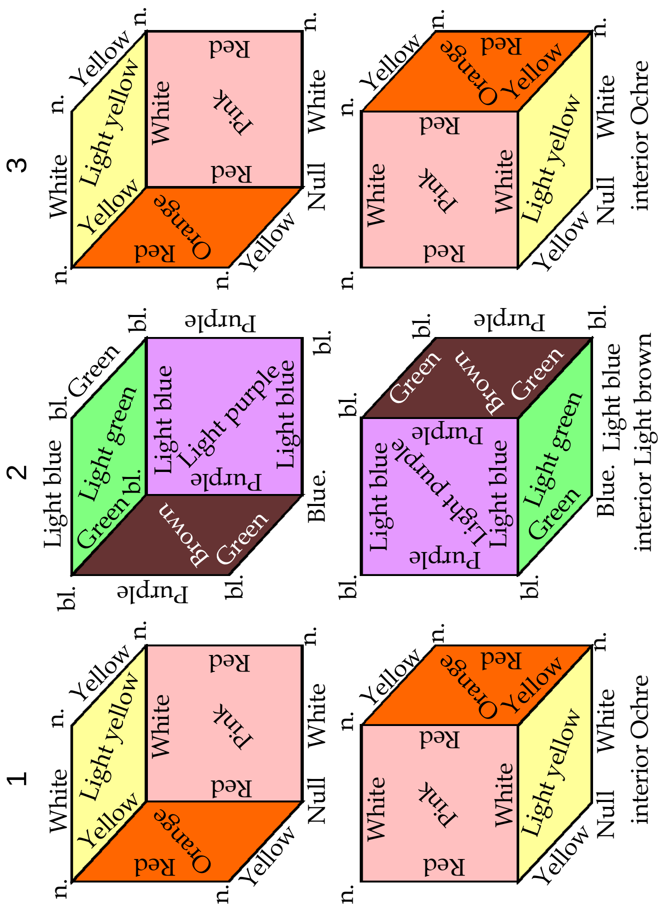


Fig. 103.
The tesseract, red, white, yellow axes in space. In the lower line the three rear faces are shown, the interior being removed.

ochre cube, the light purple cube, the brown cube, and the light green cube.

A complete view of the tesseract in its various space

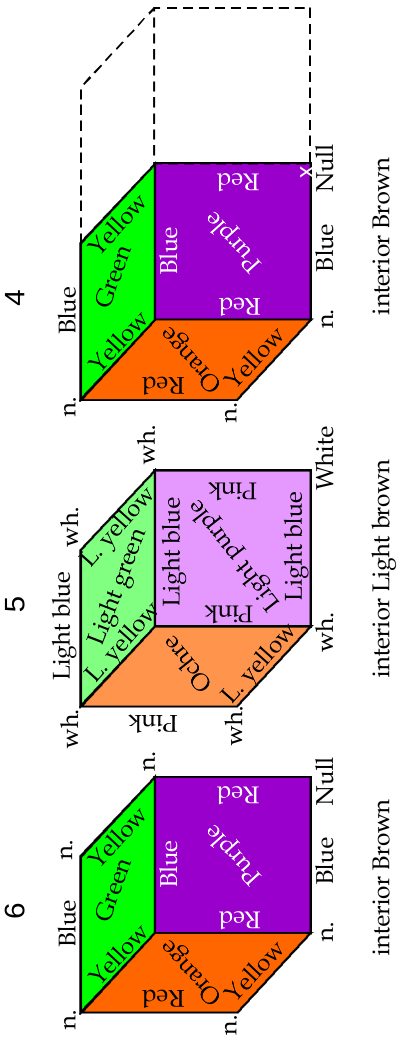


Fig. 104.

The tesseract, red, yellow, blue axes in space, the blue axis running to the left, opposite faces are coloured identically.

presentations is given in the following figures or catalogue cubes, figs. 103-106. The first cube in each figure represents the view of a tesseract coloured as described as it begins to pass transverse to our space. The intermediate figure represents a sectional view when it is partly through,

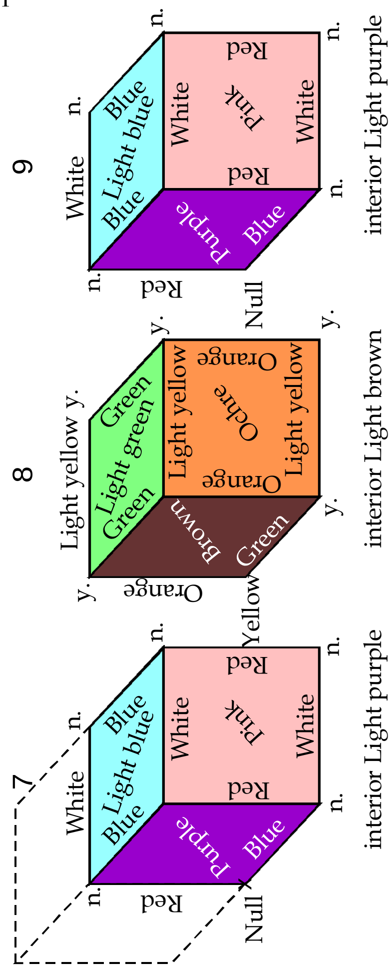


Fig. 105.

The tesseract, with red, white, blue axes in space. Opposite faces are coloured identically.

and the final figure represents the far end as it is just passing out. These figures will be explained in detail in the next chapter.

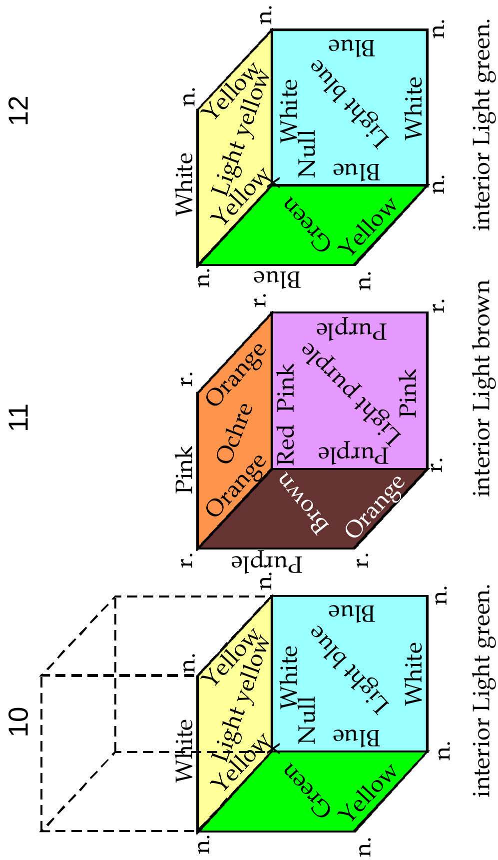


Fig. 106.

The tesseract with blue, white, yellow axes in space. The blue axis runs downward from the face of the ochre cube as it stood originally. Opposite faces are coloured identically.

We have thus obtained a nomenclature for each of the regions of a tesseract; we can speak of any one of the eight bounding cubes, the twenty-four square faces, the thirty-two lines, the sixteen points.

CHAPTER XIII

REMARKS ON THE FIGURES

AN inspection of above figures will give an answer to many questions about the tesseract. If we have a tesseract one inch each way, then it can be represented as a cube—a cube having white, yellow, red axes, and from this cube as a beginning, a volume extending into the fourth dimension. Now suppose the tesseract to pass transverse to our space, the cube of the red, yellow, white axes disappears at once, it is indefinitely thin in the fourth dimension. Its place is occupied by those parts of the tesseract which lie further away from our space in the fourth dimension. Each one of these sections will last only for one moment, but the whole of them will take up some appreciable time in passing. If we take the rate of one inch a minute the sections will take the whole of the minute in their passage across our space, they will take the whole of the minute except the moment which the beginning cube and the end cube occupy in their crossing our space. In each one of the cubes, the section cubes, we can draw lines in all directions except in the direction occupied by the blue line, the fourth dimension; lines in that direction are represented by the transition from one section cube to another. Thus to give ourselves an adequate representation of the tesseract we ought to have a limitless number of section cubes intermediate between the first bounding cube, the ochre cube, and the last bounding cube, the other ochre cube. Practically three intermediate sectional cubes will be found sufficient for most purposes. We will take then a series of five figures—two terminal cubes and three intermediate sections—and show how the different regions appear in space when we take each set of three out of the four axes of the tesseract as lying in our space.

In fig. 107 initial letters are used for the colours. A reference to fig. 103 will show the complete nomenclature, which is merely indicated here.

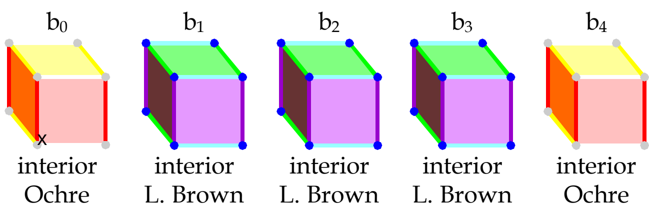


Fig. 107.

In this figure the tesseract is shown in fig stages distant from our face: first, zero; second ¼ in.; third, ½in.; fourth, ¾in.; fifth, 1 in.; which are called b_0, b_1, b_2, b_3, b_4, because they are sections taken at distances 0, 1, 2, 3, 4 quarter inches along the blue line. All the regions can be named from the first cube, the b_0 cube, as before, simply by remembering that transference along the *b* axis gives the addition of blue to the colour of the region in the ochre, the b_0 cube. In the final cube b_4, the colouring of the original b_0 cube is repeated. Thus the red line moved along the blue axis gives a red and blue or purple square. This purple square appears as the three purple lines in the sections b_1, b_2, b_3, taken at ¼, ½ ¾ of an inch in the fourth dimension. If the tesseract moves transverse to our space we have them in this particular region, first of all a red line which lasts for a moment, secondly a purple line which takes its place. This purple line lasts for a minute—that is, all of a minute, except the moment taken by the crossing our space of the initial and final red line. The purple line having lasted for this period is succeeded by a red line, which lasts for a moment; then this goes and the tesseract has passed across our space. The final red line we call red bl., because it is separated from the initial red line by a distance along the axis for which we use the colour blue. Thus a line that lasts represents an area duration; is in this mode of presentation equivalent to a dimension of space. In the same way the white line, during the crossing our space by the tesseract, is succeeded by a light blue line which lasts for the inside of a minute, and as the tesseract leaves our space, having crossed it, the white bl. line appears as the final termination.

Take now the pink face. Moved in the blue direction it traces out a light purple cube. This light purple cube is shown in sections in b_1, b_2, b_3 and the farther face of this cube in the blue direction is shown in b_4-- a pink face, called pink bl. because it is distant from the pink face we began with in the blue direction. Thus the cube which we colour light purple appears as a lasting square. The square face itself, the pink face, vanishes instantly the tesseract begins to move, but the light purple cube appears as a lasting square. Here also duration is the equivalent of a dimension of space—a lasting square is a cube. It is useful to connect these diagrams with the views given in the coloured plate.

Take again the orange face, that determined by the red and yellow axes; from it goes a brown cube in the blue direction, for red and yellow and blue are supposed to make brown. This brown cube is shown in three sections in the faces b1, b2, b3. In b4 is the opposite orange face of the brown cube, the face called orange bl., for it is distant in the blue direction from the orange face. As the tesseract passes transverse to our space, we have then in this region an instantly vanishing orange square, followed by a lasting brown square, and finally an orange face which vanishes instantly.

Now, as any three axes will be in our space, let us send the white axis out into the unknown, the fourth dimension, and take the blue axis into our known space dimension. Since the white and blue axes are perpendicular to each other, if the white axis goes out into the fourth dimension in the positive sense, the blue axis will come into the direction the white axis occupied, in the negative sense.

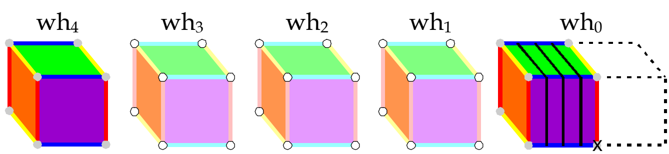


Fig. 108.

Hence, not to complicate matters by having to think of two senses in the unknown direction, let us send the white line into the positive sense of the fourth dimension, and take the blue one as running in the negative sense of that direction which the white line has left; let the blue line, that is, run to the left. We have now the row of figures in fig. 108. The dotted cube shows where we had a cube when the white line ran in our space—now it has turned out of our space, and another solid boundary, another cubic face of the tesseract comes into our space. This cube has red and yellow axes as before; but now, instead of a white axis running to the right, there is a blue axis running to the left. Here we can distinguish the regions by colours in a perfectly systematic way. The red line traces out a purple square in the transference along the blue axis by which this cube is generated from the orange face. This purple square made by the motion of the red line is the same purple face that we saw before as a series of lines in the section b_1, b_2, b_3. Here, since both red and blue axes are in our space, we have no need of duration to represent the area they determine. In the motion of the tesseract across space this purple face would instantly disappear.

From the orange face, which is common to the initial cubes in fig. 107 and fig. 108, there goes in the blue direction a cube coloured brown. This brown cube is now all in our space, because each of its three axes run in space directions, up, away, to the left. It is the same brown cube which appeared as the successive faces on the sections b_1, b_2, b_3. Having all its three axes in our space, it is given in extension; no part of it needs to be represented as a succession. The tesseract is now in a new position with regard to our space, and when it moves across our space the brown cube instantly disappears.

In order to exhibit the other region of the tesseract we must remember that now the white line runs in the unknown dimension. Where shall we put the section at distances along the line? Any arbitrary position in our space will do; there is no way by which we can represent their real position.

However, as the brown cube comes off from the orange face to the left, let us put these successive sections to the left. We can call them wh_0, wh_1, wh_2, wh_3, wh_4, because they are sections along the white axis, which now runs in the unknown dimension.

Running from the purple square in the white direction we find the light purple cube. This is represented in the sections wh_1, wh_2, wh_3, fig. 108. It is the same cube that is represented in the sections b_1, b_2, b_3; in fig. 107 the red and white axes are in our space, the blue out of it; in the other case, the red and blue are in our space, the white out of it. It is evident that the face pink y., opposite the pink face in fig. 107, makes a cube shown in squares in b_1, b_2, b_3 on the opposite side to the light purple squares. Also the light yellow face at the base of the cube b_0, makes a light green cube, shown as a series of base squares.

The same light green cube can be found in fig. 108. The base square in wh_0, is a green square, for it is enclosed by blue and yellow axes. From it goes a cube in the white direction, this is then a light green cube and the same as the one just mentioned as existing in the sections b_0, b_1, b_2, b_3, b_4.

The case is, however, a little different with the brown cube. This cube we have altogether in space in the section wh_0, fig. 108, while it exists as a series of squares, the left-hand ones, in the sections b_0, b_1, b_2, b_3, b_4. The brown cube exists as a solid in our space, as shown in fig. 108. In the mode of representation of the tesseract exhibited in fig. 107, the same brown cube appears as a succession of squares. That is, as the tesseract moves across space, the brown cube would actually be to us a square—it would be merely the lasting boundary of another solid. It would have no thickness at all, only extension in two dimensions, and its duration would show its solidity in three dimensions.

It is obvious that, if there is a four-dimensional space, matter in three dimensions only is a mere abstraction; all material objects must then have a slight four-dimensional thickness. In this case the above statement will undergo modification. The material cube which is used as the model of the boundary of a tesseract will have a slight thickness in the fourth dimension, and when the cube is presented to us in another aspect, it would not be a mere surface. But it is most convenient to regard the cubes we use as having no extension at all in the fourth dimension. This consideration serves to bring out a point alluded to before, that, if there is a fourth dimension, our conception of a solid is the conception of a mere abstraction, and our talking about real three-dimensional objects would seem to a four-dimensional being as incorrect as a two-dimensional being's telling about real squares, real triangles, etc., would seem to us.

The consideration of the two views of the brown cube shows that any section of a cube can be looked at by a presentation of the cube in a different position in four-dimensional space. The brown faces in b_1, b_2, b_3, are the very same brown sections that would be obtained by cutting the brown cube, wh_0, across at the right distances along the blue line, as shown in fig. 108. But as these sections are placed in the brown cube, wh_0, they come behind one another in the blue direction. Now, in the sections wh_1, wh_2, wh_3, we are looking at these sections from the white direction—the blue direction does not exist in these figures. So we see them in a direction at right angles to that in which they occur behind one another in wh_0. They are intermediate views, which would come in the rotation of a tesseract. These brown squares can be looked at from directions intermediate between the white and blue axes. It must be remembered that the fourth dimension is perpendicular equally to all three space axes. Hence we must take the combinations of the blue axis, with each two of our three axes, white, red, yellow, in turn.

In fig. 109 we take red, white, and blue axes in space, sending yellow into the fourth dimension. If it goes into the positive sense of the fourth dimension the blue line will come in the opposite direction to that in which the yellow line ran before. Hence, the cube determined by the white, red, blue axes, will start from the pink plane and run towards us. The dotted cube shows where the ochre cube was. When it is turned out of space, the cube coming towards from its front face is the one which comes into our space in this turning. Since the yellow line now runs in the unknown dimension we call the sections y_0, y_1, y_2, y_3, y_4, as they are made at distances 0, 1, 2, 3, 4, quarter inches along the yellow line. We suppose these cubes arranged in a line coming towards us—not that that is any more natural than any other arbitrary series of positions, but it agrees with the plan previously adopted.

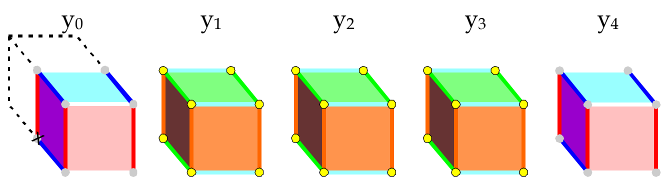


Fig. 109.

The interior of the first cube, y_0, is that derived from pink by adding blue, or, as we call it, light purple. The faces of the cube are light blue, purple, pink. As drawn, we can only see the face nearest to us, which is not the one from which the cube starts—but the face on the opposite side has the same colour name as the face towards us.

The successive sections of the series y_0, y_1, y_2, etc., can be considered as derived from sections of the b_0 cube made at distances along the yellow axis. What is distant a quarter inch from the pink face in the yellow direction? This question is answered by taking a section from a point a quarter inch along the yellow axis in the cube b_0, fig. 107. It is an ochre section with lines orange and light yellow. This section will therefore take the place of the pink face in y_1 when we go on in the yellow direction. Thus, the first section, y_1, will begin from an ochre face with light yellow and orange lines. The colour of the axis which lies in space towards us in blue, hence the regions of this section-cube are determined in nomenclature, they will be found in full in fig. 105.

There remains only one figure to be drawn, and that is the one in which the red axis is replaced by the blue. Here, as before, if the red axis goes out into the positive sense of the fourth dimension, the blue line must come into our space in the negative sense of the direction which the red line has left. Accordingly, the first cube will come in beneath the position of our ochre cube, the one we have been in the habit of starting with.

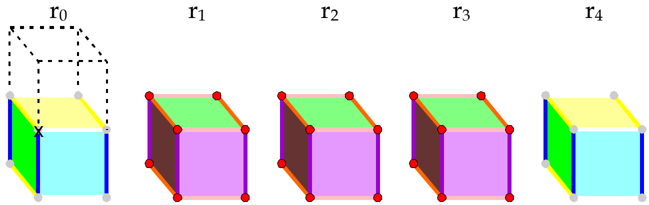


Fig. 110.

To show these figures we must suppose the ochre cube to be on a moveable stand. When the red line swings out into the unknown dimension, and the blue line comes in downwards, a cube appears below the place occupied by the ochre cube. The dotted cube shows where the ochre cube was. That cube has gone and a different cube runs downwards from its base. This cube has white, yellow, and blue axes. Its top is a light yellow square, and hence its interior is light yellow + blue or light green. Its front face is formed by the white line moving along the blue axis, and is therefore light blue, the left-hand side is formed by the yellow line moving along the blue axis, and therefore green.

As the red line now runs in the fourth dimension, the successive sections can be called r_0, r_1, r_2, r_3, r_4, these letters indicating that at distances 0, ¼ ½ ¾ 1 inch along the red axis we take all of the tesseract that can be found in a three-dimensional space, this three-dimensional space extending not at all in the fourth dimension, but up and down, right and left, far and near.

We can see what should replace the light yellow face of r_0, when the section r_1 comes in, by looking at the cube b0, fig 107. What is distant in it one-quarter of an inch from the light yellow face in the red direction? It is an ochre section with orange and pink lines and red points; see also fig. 103.

This square then forms the top square of r_1. Now we can determine the nomenclature of all the regions of r_1 by considering what would be formed by the motion of this square along a blue axis.

But we can adopt another plan. Let us take a horizontal section of r_0 and finding that section in the figures, of fig. 107 or fig. 103, from them determine what will replace it, going in the red direction.

A section of the r_0 cube has green, light blue, green, light blue sides and blue points.

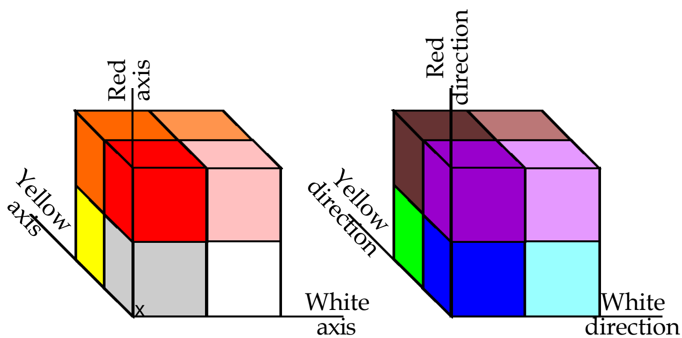
Now this square occurs on the base of each of the section figures, b_1, b_2, etc. In them we see that ¼ inch in the red direction from it lies a section with brown and light purple lines and purple corners, the interior being of light brown. Hence this is the nomenclature of the section which in r_1 replaces the section of r_0 made from a point along the blue axis.

Hence the colouring as given can be derived.

We have thus obtained a perfectly named group of tesseracts. We can take a group of eighty-one of them $3 \times 3 \times 3 \times 3$, in four dimensions, and each tesseract will have its name null, red, white, yellow, blue, etc., and whatever cubic view we take of them we can say exactly what sides of the tesseracts we are handling, and how they touch each other.*

Thus, for instance, if we have the sixteen tesseracts shown below, we can ask how does null touch blue.

In the arrangement given in fig. 111 we have the axes white, red, yellow, in space, blue running in the fourth dimension. Hence we have the ochre cubes as bases. Imagine now the tesseractic group to pass transverse to our space—we have first of all null ochre cube, white



Light yellow hidden
A.b_0

Light green hidden
B.b_1

Fig. 111.

ochre cube, etc.; these instantly vanish, and we get the section shown in the middle cube in fig. 103, and finally, just when the tesseract block has moved one inch transverse to our space, we have null ochre cube, and then immediately afterwards the ochre cube of blue comes in. Hence the tesseract null touches the tesseract blue by its ochre cube, which is in contact, each and every point of it, with the ochre cube of blue.

How does null touch white, we may ask? Looking at the beginning A, fig. 111, where we have the ochre

* At this point the reader will find it advantageous, if he has the models, to go through the manipulations described in Appendix I.

cubes, we see that null ochre touches white ochre by an orange face. Now let us generate the null and white tesseracts by a motion in the blue direction of each of these cubes. Each of them generates the corresponding tesseract, and the plane of contact of the cubes generates the cube by which the tesseracts are in contact. Now an orange plane carried along a blue axis generates a brown cube. Hence null touches white by a brown cube.

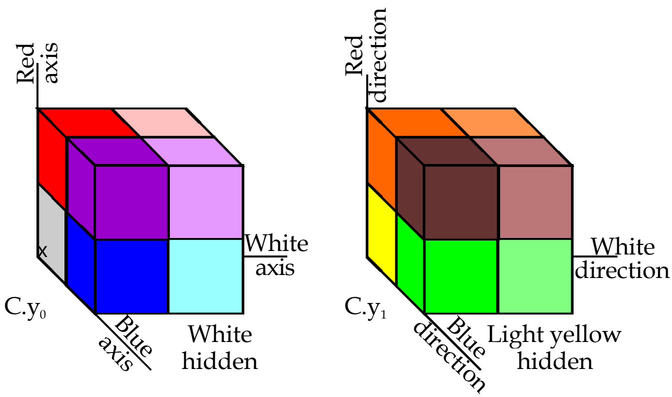


Fig. 112.

If we ask again how red touches light blue tesseract, let us rearrange our group, fig. 112, or rather turn it about so that we have a different space view of it; let the red axis and the white axis run up and right, and let the blue axis come in space towards us, then the yellow axis runs in the fourth dimension. We have then two blocks in which the bounding cubes of the tesseracts are given, differently arranged with regard to us—the arrangement is really the same, but it appears different to us. Starting from the plane of the red and white axes we have the four squares of the null, white, red, pink tesseracts as shown in A, on the red, white plane, unaltered, only from them now comes out towards us the blue axis.

Hence we have null, white, red, pink tesseracts in contact with our space by their cubes which have the red, white, blue axis in them, that is by the light purple cubes. Following on these four tesseracts we have that which comes next to them in the blue direction, that is the four blue, light blue, purple, light purple. These are likewise in contact with our space by their light purple cubes, so we see a block as named in the figure, of which each cube is the one determined by the red, white, blue axes.

The yellow line now runs out of space; accordingly one inch on in the fourth dimension we come to the tesseracts which follow on the eight names in C, fig. 112, in the yellow direction.

These are shown in C.y_0, fig. 112. Between figure C and C.y_1 is that four-dimensional mass which is formed by moving each of the cubes in C one inch in the fourth dimension—that is, along a yellow axis; for the yellow axis now runs in the fourth dimension.

In the block C we observe that red (light purple cube) touches light blue (light purple cube) by a point. Now these two cubes moving together remain in contact during the period in which they trace out the tesseracts red and light blue. This motion is along the yellow axis, consequently red and light blue touch by a yellow line.

We have seen that the pink face moved in a yellow direction traces out a cube; moved in the blue direction it also traces out a cube. Let us ask what the pink face will trace out if it is moved in a direction within the tesseract lying equally between the yellow and blue directions. What section of the tesseract will it make?

We will first consider the red line alone. Let us take a cube with the red line in it and the yellow and blue axes.

The cube with the yellow, red, blue axes is shown in fig. 113. If the red line is moved equally in the yellow and in the blue directions by four equal motions of ¼ inch each, it takes the positions 11, 22, 33, and ends as a red line.

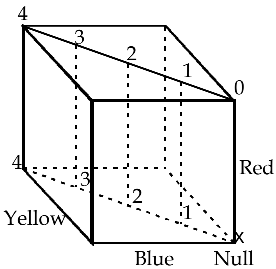


Fig. 113.

Now the whole of this red, yellow, blue, or brown cube appears as a series of faces on the successive sections of the tesseract starting from the ochre cube and letting the blue axis run in the fourth dimension. Hence the plane traced out by the red line appears as a series of lines in the successive sections, in our ordinary way of representing the tesseract; these lines are in different places in each successive section.

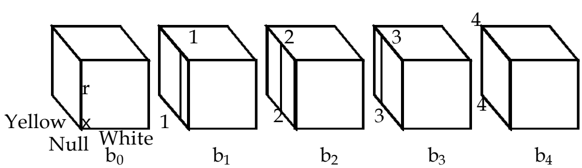


Fig. 114.

Thus drawing our initial cube and the successive sections, calling them b_0, b_1, b_2, b_3, b_4, fig. 115, we have the red line subject to this movement appearing in the positions indicated.

We will now investigate what positions in the tesseract another line in the pink face assumes when it is moved in a similar manner.

Take a sections of the original cube containing a vertical line 4, in the pink plane, fig. 115. We have, in the section, the yellow direction, but not the blue.

From this section a cube goes off in the fourth dimension, which is formed by moving each point of the section in the blue direction.

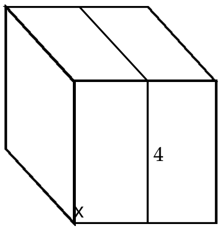


Fig. 115.

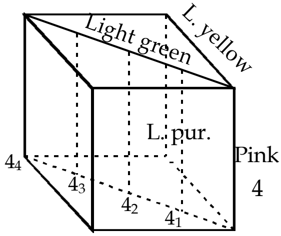


Fig. 116.

Drawing this cube we have fig. 116.

Now this cube occurs as a series of sections in our original representation of the tesseract. Taking four steps as before this cube appears as the sections drawn in b_0, b_1, b_2, b_3, b_4, fig. 117, and if the line 4 is subjected to a movement equal in the blue and yellow directions, it will occupy the positions designated by 4, 4_1, 4_2, 4_3, 4_4.

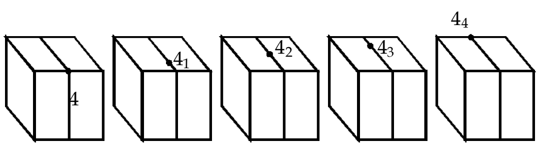


Fig. 117.

Hence, reasoning in a similar manner about every line, it is evident that, moved equally in the blue and yellow directions, the pink plane will trace out a space which is shown by the series of section planes represented in the diagram.

Thus the space traced out by the pink face, if it is moved equally in the yellow and blue directions, is represented by the set of planes delineated in Fig. 118, pink face or 0, then 1, 2, 3, and finally pink face or 4. This solid is a diagonal solid of the tesseract, running from a pink face to a pink face. Its length is the length of the diagonal of a square, its side is a square.

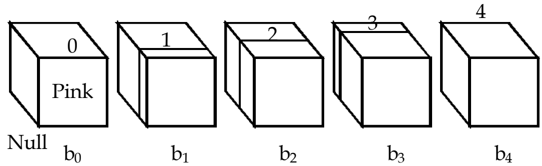


Fig. 118.

Let us now consider the unlimited space which springs from the pink face extended.

This space, if it goes off in the yellow direction, gives us in it the ochre cube of the tesseract. Thus, if we have the pink face given and a point in the ochre cube, we have determined this particular space.

Similarly going off from the pink face in the blue direction is another space, which gives us the light purple cube of the tesseract in it. And any point being taken in the light purple cube, this space going off from the pink face is fixed.

The space we are speaking of can be conceived as swinging round the pink face, and in each of its positions it cuts out a solid figure from the tesseract, one of which we have seen represented in fig. 118.

Each of these solid figures is given by one position of the swinging space, and by one only. Hence in each of them, if one point is taken, the particular one of the slanting space is fixed. Thus we see that given a plane and a point out of it a space is determined.

Now, two points determine a line.

Again, think of a line and a point outside it. Imagine a plane rotating round the line. At some time in its rotation it passes through the point. Thus a line and a point, or three points, determine a plane. And finally, four points determine a space. We have seen that a plane and a point determine a space, and that three points determine a plane; so four points will determine a space.

These four points may be any points, and we can take, for instance, the four points at the extremities of the red, white, yellow, blue axes, in the tesseract. These will determine a space slanting with regard to the section spaces we have been previously considering. This space will cut the tesseract in a certain figure.

One of the simplest sections of a cube by a plan is that in which the plane passes through the extremities of the three edges which meet in a point. We see at once that this plane would cut the cube in a triangle, but we will go through the process by which a plane being would most conveniently treat the problem of the determination of this shape, in order that we may apply the method to the determination of the figure in which a space cuts a tesseract when it passes through the 4 points at unit distance from a corner.

We know that two points determine a line, three points determine a plane, and given any two points in a plane the line between them lies wholly in the plane.

Let now the plane being study the section made by a plane passing through the null r., null wh., and null y. points, fig. 119. Looking at the orange square, which, as usual, we suppose to be initially in his plane, he sees that the line from null r. to null y., which is a line in the section plane, the plane, namely, through the three extremities of the edges meeting in null, cuts the orange

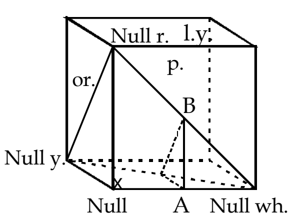


Fig. 119.

face in an orange line with null points. This then is one of the boundaries of the section figure.

Let now the cube be so turned that the pink face comes in his plane. The points null r. and null wh. are now visible. The line between them is pink with null points, and since this line is common to the surface of the cube and the cutting plane, it is a boundary of the figure in which the plane cuts the cube.

Again, suppose the cube turned so that the light yellow face is in contact with the plane being's plane. He sees two points, the null wh. and the null y. The line between these lies in the cutting plane. Hence, since the three cutting lines meet and enclose a portion of the cube between them, he has determined the figure he sought. It is a triangle with orange, pink, and light yellow sides, all equal, and enclosing an ochre area.

Let us now determine in what figure the space, determined by the four points, null r., null y., null wh., null b., cuts the tesseract. We can see three of these points on the primary position of the tesseract resting against our solid sheet by the ochre cube. These three points determine a plane which lies in the space we are considering, and this plane cuts the ochre cube in a triangle, the interior of which is ochre (fig. 119 will serve for this view), with pink, light yellow and orange sides, and null points. Going in the fourth direction, in one sense, from this place we pass into the tesseract, in the other sense we pass away from it. The whole area inside the triangle is common to the cutting plane we see, and a boundary of the tesseract. Hence we conclude that the triangle drawn is common to the tesseract and the cutting space.

Now let the ochre cube turn out and the brown cube come in. The dotted lines show the position the ochre cube has left (fig. 120).

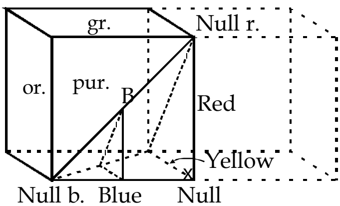


Fig. 120.

Here we see three out of the four points through which the cutting space passes, null r., null y., and null b. The plane they determine lies in the cutting space, and this plane cuts out of the brown cube a triangle with orange, purple and green sides, and null points. The orange line of this figure is the same as the orange line in the last figure.

Now let the light purple cube swing into our space, towards us, fig. 121.

The cutting space which passes through the four points, null r., y., wh., b., passes through the null r., wh., b., and therefore the plane these determine lies in the cutting space.

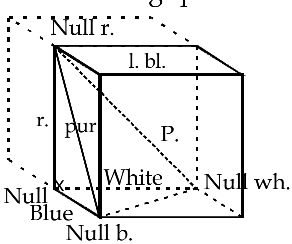


Fig. 121.

This triangle lies before us. It has a light purple interior and pink, light blue, and purple edges with null points.

This, since it is all of the plane that is common to it, and this bounding of the tesseract, gives us one of the bounding faces of our sectional figure. The pink line in it is the same as the pink line we found in the first figure—that of the ochre cube.

Finally, let the tesseract swing around the light yellow plane, so that the light green cube comes into our space. It will point downwards.

The three points, n. y., n. wh., n. b., are in the cutting space, and the triangle they determine is common to the tesseract and the cutting space. Hence this boundary is a triangle having a light yellow line, which is the same as the light yellow line of the first figure, a light blue line and a green line.

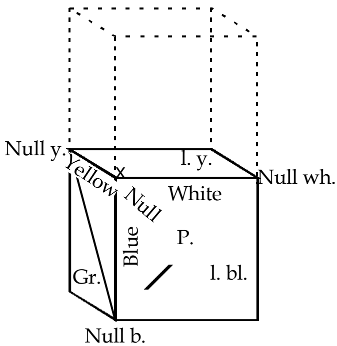


Fig. 122.

We have now traced the cutting space between every set of three that can be made out of the four points in which it cuts the tesseract, and have got four faces which all join on to each other by lines.

The triangles are shown in fig. 123 as they join on to the triangle in the ochre cube. But they join on each to the other in an exactly similar manner; their edges are all identical two and two. They form a closed figure, a tetrahedron, enclosing a light brown portion which is the portion of the cutting space which lies inside the tesseract.

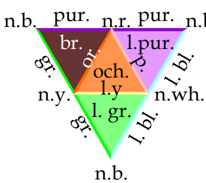


Fig. 123.

We cannot expect to see this light brown portion, any more than a plane being could expect to see the inside of a cube if an angle of it were pushed through his plane. All he can do is to come upon the boundaries of it in a different way to that in which he would if it passed straight through his plane.

Thus in this solid section; the whole interior lies perfectly open in the fourth dimension. Go round it as we may we are simply looking at the boundaries of the tesseract which penetrates through our solid sheet. If the tesseract were not to pass across so far, the triangle would be small; if it were to pass farther, we should have a different figure, the outlines of which can be determined in a similar manner.

The preceding method is open to the objection that it depends rather on our inferring what must be, than our seeing what is. Let us therefore consider our sectional space as consisting of a number of planes, each very close to the last, and observe what is to be found in each place.

The corresponding method in the case of two dimensions is as follows:--the plane being can see that line of the sectional plane through null y., null b., null r., which lies in the orange plane. Let him now suppose the cube and the section plane to pass half way though his plane. Replacing the red and yellow axes are lines parallel to them, sections of the pink and light yellow faces.

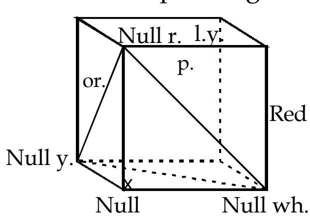


Fig. 124.

Where will the section plane cut these parallels to the red and yellow axes?

Let him suppose the cube, in the position of the drawing, fig. 124, turned so that the pink face lies against his plane. He can see the line from the null r. point to the null wh. point, and can see (compare fig. 119) that it cuts as a parallel to his red axis, drawn at a point half way along the white line, in a point n., half way up. I shall speak of the axis as having the length of an edge of the cube. Similarly, by letting the cube turn so that the light yellow square swings in against his plane, he can see (compare fig. 119) that a parallel to his yellow axis drawn from a point half-way along the white axis, is cut at half its length by the trace of the section plane in the light yellow face.

Hence when this cube had passed half-way through he would have—instead of the orange line with null points, which he had at first—an ochre line of half its length, with pink and light yellow points. Thus, as the cube passed slowly through his plane, he would have a succession of lines gradually diminishing in length and forming an equilateral triangle. The whole interior would be ochre, the line from which it started would be orange. The succession of points at the ends of the succeeding lines would form pink and light yellow lines and the final point would be null. Thus looking at the successive lines in the section plane as it and the cube passed across his plane he would determine the figure cut out bit by bit.

Coming now to the section of the tesseract, let us imagine that the tesseract and its cutting *space* pass slowly across our space; we can examine portions of it, and their relation to portions of the cutting space. Take the section space which passes through the four points, null r., wh., y., b.; we can see in the ochre cube (fig. 119) the plane belonging to this section space, which passes through the three extremities of the red, white, yellow axes.

Now let the tesseract pass half way through out space. Instead of our original axes we have parallels to them, purple, pink and green, each of the same length as the first axes, for the section of the tesseract is of exactly the same shape as its ochre cube.

But the sectional space seen at this stage of the transference would not cut the section of the tesseract in a plane disposed as at first.

To see where the sectional space would cut these parallels to the original axes let the tesseract swing so that, the orange face remaining stationary, the blue line comes in to the left.

Here (fig. 125) we have the null r., y., b. points, and of the sectional space all we see is the plane through these three points in it.

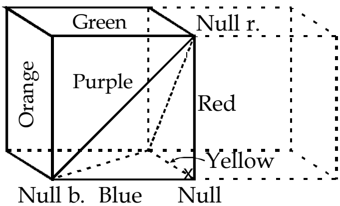


Fig. 125.

In this figure we can draw the parallels to the red and yellow axes and see that, if they started at a point half way along the blue axis, they would each be cut at a point so as to be half of their previous length.

Swinging the tesseract into our space about the pink face of the ochre cube we likewise find that the parallel to the white axis is cut at half its length by the sectional space.

Hence in a section made when the tesseract had passed half across our space the parallels to the red, white, yellow axes, which are now in our space, are cut by the section space, each of them half way along, and for this stage of the traversing motion we should have fig. 126. The section made of this cube by the plane in which the sectional space cuts it, is an equilateral triangle with purple, l. blue, green points, and l. purple, brown, l. green lines.

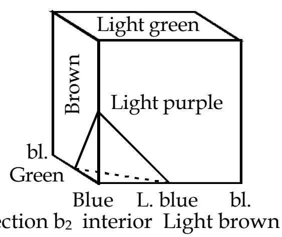
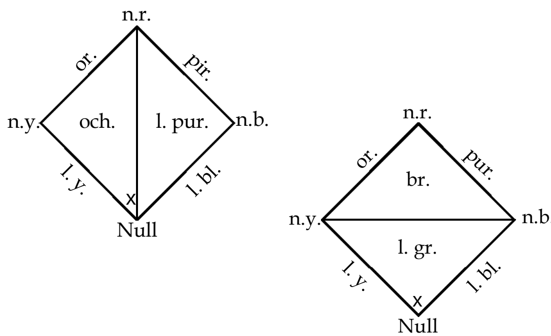


Fig. 126.

Thus the original ochre triangle, with null points and pink, orange, light yellow lines, would be succeeded by a triangle coloured in manner just described.

This triangle would initially be only a very little smaller than the original triangle, it would gradually diminish, until it ended in a point, a null point. Each of its edges would be of the same length. Thus the successive sections of the successive planes into which we analyse the cutting space would be a tetrahedron of the description shown (fig. 123) and the whole interior of the tetrahedron would be light brown.



Front view The rear faces.

Fig. 127.

In fig. 127 the tetrahedron is represented by means of its faces as two triangles which meet in the p. line, and two rear triangles which join on to them, the diagonal of the pink face being supposed to run vertically upward.

We have now reached a natural termination. The reader may pursue the subject in further detail, but will find no essential novelty. I conclude with an indication as to the manner in which figures previously given may be used in determining sections by the method developed above.

Applying this method to the tesseract, as represented in Chapter IX., sections made by a space cutting the axes equidistantly at any distance can be drawn, and also the sections of tesseracts arranged in a block.

If we drawn a plane, cutting all four axes at a point six units distance from null, we have a slanting space. This space cuts the red, white, yellow axes in the points LMN (fig. 128), and so in the region of our space before we go off into the fourth dimension, we have the plane represented by LMN extended. This is what is common to the slanting space and our space.

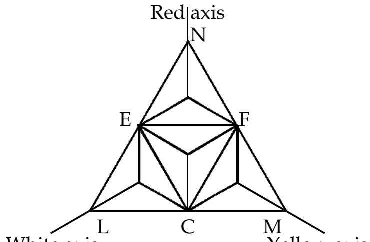


Fig. 128.

This plane cuts the ochre cube in the triangle EFG.

Comparing this with (fig. 72) *oh*, we see that the hexagon there drawn is part of the triangle EFG.

Let us now imagine the tesseract and the slanting space both together to pass transverse to our space, a distance of one unit, we have in 1*h* a section of the tesseract, whose axes are parallels to the previous axes. The slanting space cuts them at a distance of five units along each. Drawing the plane through these points in 1*h* it will be found to cut the cubical section of the tesseract in the hexagonal figure drawn. In 2*h* (fig. 72) the slanting space cuts the parallels to the axes at a distance of four along each, and the hexagonal figure is the section of this section of the tesseract by it. Finally when 3*h* comes in the slanting space cuts the axes at a distance of three along each, and the section is a triangle, of which the hexagon drawn is a truncated portion. After this the tesseract, which extends only three units in each of the four dimensions, has completely passed transverse of our space, and there is no more of it to be cut. Hence, putting the plane sections together in the right relations, we have the section determined by this particular slanting space: namely an octahedron.

CHAPTER XIV*

A RECAPITULATION AND EXTENSION OF THE PHYSICAL ARGUMENT

THERE are two directions of inquiry in which the research for the physical reality of a fourth dimension can be prosecuted. One is the investigation of the infinitely great, the other is the investigation of the infinitely small.

By the measurement of the angles of vast triangles, whose sides are the distances between the stars, astronomers have sought to determine if there is any deviation from the values given by geometrical deduction. If the angles of a celestial triangle do not together equal two right angles, there would be an evidence for the physical reality of a fourth dimension.

This conclusion deserves a word of explanation. If space is really four-dimensional, certain conclusions follow which must be brought clearly into evidence if we are to frame the questions definitely which we put to nature. To account for our limitation let us assume a solid material sheet against which we move. This sheet must stretch alongside every object in every direction in which it

* The contents of this chapter are taken from a paper read before the Philosophical Society of Washington. The mathematical portion of the paper has appeared in part in the Transcations of the Royal Irish Academy under the title, "Cayley's formulæ of orthogonal transformation."

visibly moves. Every material body must slip or slide along this sheet, not deviating from contact with it in any motion which we can observe.

The necessity for this assumption is clearly apparent, if we consider the analogous case of a suppositionary plane world. If there were any creatures whose experiences were confined to a plane, we must account for their limitation. If they were free to move in every space direction, they would have a three-dimensional motion; hence they must be physically limited, and the only way in which we can conceive such a limitation to exist is by means of a material surface against which they slide. The existence of this surface would only be known to them indirectly. It does not lie in any direction from them in which the kinds of motion they know of leads them. If it were perfectly smooth and always in contact with every material object, there would be no difference in their relations to it which would direct their attention to it.

But if this surface were curved—if it were, say, in the form of a vast sphere—the triangles they drew would really be triangles of a sphere, and then these triangles are large enough the angles diverge from the magnitudes they would have for the same lengths of sides if the surface were plane. Hence by the measurement of triangles of very great magnitude a plane being might detect a difference from the laws of a plane world in his physical world, and so be led to the conclusion that there was in reality another dimension to space—a third dimension—as well as the two which his ordinary experience made him familiar with.

Now, astronomers have thought it worth while to examine the measurements of vast triangles drawn from one celestial body to another with a view to determine if there is anything like a curvature in our space—that is to say, they have tried astronomical measurements to find out if the vast solid sheet against which, on the supposition of a fourth dimension, everything slides is curved or not. These results have been negative. The solid sheet, if it exists, is not curved or, being curved, has not a sufficient curvature to cause any observable deviation from the theoretical value of the angles calculated.

Hence the examination of the infinitely great leads to no decisive criterion. It neither proves nor disproves the existence of a fourth dimension.

Coming now to the prosecution of the inquiry in the direction of the infinitely small, we have to state the question thus: Our laws of movement are derived from the examination of bodes which move in three-dimensional space. All our conceptions are founded on the supposition of a space which is represented analytically by three independent axes and variations along them—that is, it is a space in which there are three independent movements. Any motion possible in it can be compounded out of these three movements, which we may call: up, right, away.

To examine the actions of the very small portions of matter with the view of ascertaining if there is any evidence in the phenomena for the supposition of a fourth dimension of space, we must commence by clearly defining what the laws of mechanics would be on the supposition of a fourth dimension. It is of no use asking if the phenomena of the smallest particles of matter are like—we do not know what. We must have a definite conception of what the laws of motion would be on the supposition of the fourth dimension, and then inquire if the phenomena of the activity of the smaller particles of matters resemble the conceptions which we have elaborated.

Now, the task of forming these conceptions is by no means one to be lightly dismissed. Movement in space has many features which differ entirely from movement on a plane; and when we set about to form the conception of motion in four dimensions, we find that there is at least as great a step as from the plane to three-dimensional space.

I do not say that the step is difficult, but I want to point out that it must be taken. When we have formed the conception of four-dimensional motion, we can ask a rational question of Nature. Before we have elaborated our conceptions we are asking if an unknown is like an unknown—a futile inquiry.

As a matter of fact, four-dimensional movements are in every way simple and more easy to calculate than three-dimensional movements, for four-dimensional movements are simply two sets of plane movements put together.

Without the formation of an experience of four-dimensional bodies, their shapes and motions, the subject can be but formal—logically conclusive, not intuitively evident. It is to this logical apprehension that I must appeal.

It is perfectly simple to form an experiential familiarity with the facts of four-dimensional movement. The method is analogous to that which a plane being would have to adopt to form an experiential familiarity with three-dimensional movements, and may be briefly summed up as the formation of a compound sense by means of which duration is regarded as equivalent to extension.

Consider a being confined to a plane. A square enclosed by four lines will be to him a solid, the interior of which can only be examined by breaking through the lines. If such a square were to pass transverse to his plane, it would immediately disappear. It would vanish, going in no direction to which he could point.

If, now, a cube be placed in contact with his plane, its surface of contact would appear like the square which we have just mentioned. But if it were to pass transverse to his plane, breaking through it, it would appear as a lasting square. The three-dimensional matter will give a lasting appearance in circumstances under which two-dimensional matter will at once disappear.

Similarly, a four-dimensional cube, or, as we may call it, a tesseract, which is generated from a cube by a movement of every part of the cube in a fourth dimension at right angles to each of the three visible directions in the cube, if it moved transverse to our space, would appear as a lasting cube.

A cube of three-dimensional matter, since it extends to no distance at all in the fourth dimension, would instantly disappear, if subjected to a motion transverse to our space. It would disappear and be gone, without it being possible to point to any direction in which it had moved.

All attempts to visualise a fourth dimension are futile. It must be connected with a time experience in three space.

The most difficult notion for a plane being to acquire would be that of rotation about a line. Consider a plane being facing a square. If he were told that rotation about a line were possible, he would move his square this way and that. A square in a plane can rotate about a point, but to rotate about a line would seem to the plane being perfectly impossible. How could those parts of his square which were on one side of an edge come to the other side without the edge moving? He could understand their reflection in the edge. He could form an idea of the looking-glass image of his square lying on the opposite side of the line of an edge, but by no motion that he knows of can he make the actual square assume that position. The result of the rotation would be like reflection in the edge, but it would be a physical impossibility to produce it in the plane.

The demonstration of rotation about a line must be to him purely formal. If he conceived the motion of a cube stretching out in an unknown direction away from his plane, then he can see the base of it, his square in the plane, rotating round a point. He can likewise apprehend that every parallel section taken at successive intervals in the unknown direction rotates in like manner round a point. Thus he would come to conclude that the whole body rotates round a line—the line consisting of the succession of points round which the plane sections rotate. Thus, given three axes, x, y, z, if x rotates to take the place of y, and y turns so as to point to negative x, then the third axis remaining unaffected by this turning is the axis about which the rotation takes place. This, then, would have to be his criterion of the axis of a rotation—that which remains unchanged then a rotation of every plane section of a body takes place.

There is another way in which a plane being can think about three-dimensional movements; and, as it affords the type by which we can most conveniently think about four-dimensional movements, it will be no loss of time to consider it in detail.

We can represent the plane being and his object by figures cut out of paper, which slip on a smooth surface. The thickness of these bodies must be taken as so minute that their extension in the third dimension escapes the observation of the plane being, and he thinks about them as if they were mathematical plane figures in a plane instead of being material bodies capable of moving on a plane surface. Let Ax, Ay be any two axes and ABCD a square. As far as movements in the plane are concerned, the square can rotate about a point A, for example. It cannot rotate about a side, such as AC.

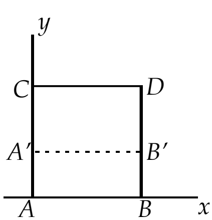


Fig. 1 (129).

But if the plane being is a aware of the existence of a third dimension he can study the movements possible in the ample space, taking his figure portion by portion.

His plane can only hold two axes. But, since it can hold two, he is able to represent a turning into the third dimension if he neglect one of his axes and represent the third axis as lying in his plane. He can make a drawing in his plane of what stands up perpendicularly from his

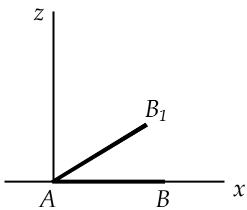


Fig. 2 (130).

plane. Let Ax be the axis, which stands perpendicular to his plane at A. He can draw in his plane two lines to represent the two axes, Ax and Az. Let Fig. 2 be this drawing. Here the x axis has taken the place of the y axis, and the plane of Ax Az is represented in his plane. In this figure all that exists of the square ABCD will be the line AB.

The square extends from this line in the y direction, but none of that direction is represented in Fig. 2. The plane being can study the turning of the line AB in this diagram. It is simple a case of plane turning around the point A. The line AB occupies intermediate portions like AB_1 and after half a revolution will lie on Ax produced through A.

Now, in the same way, the plane being can take another point, A′, and another line, A′B′, in his square. He can make the drawing of the two directions at A′, one along A′B′, the other perpendicular to his plane. He will obtain a figure precisely similar to Fig. 2, and will see that, as AB can turn around A, so A′B′ around A′.

In this turning AB and A′B′ would not interfere with each other, as they would if they moved in the plane around the separate points A and A′.

Hence the plane being would conclude that a rotation round a line was possible. He could see his square as it began to make this turning. He could see it half way round when it came to lie on the opposite side of the line AC. But in intermediate portions he could not see it, for it runs out of the plane.

Coming now to the question of a four-dimensional body, let us conceive of it as a series of cubic sections, the first in our space, the rest at intervals, stretching away from our space in the unknown direction.

We must not think of a four-dimensional body as formed by moving a three-dimensional body in any direction which we can see.

Refer for a moment to Fig. 3. The point A, moving to the right, traces out the line AC. The line AC, moving away in a new direction, traces out the square AEGC at the base of the cube. The square AEGC, moving in a new direction, will trace out the cube ACEGBDHF. The vertical direction of this last motion is not identical with any motion possible in the plane at the base of the cube. It is an entirely new direction, at right angles to every line that can be drawn in the base. To trace out a tesseract the cube must move in a new direction—a direction at right angles to any and every line that can be drawn in the space of the cube.

The cubic sections of the tesseract are related to the cube we see, as the square sections of the cube are related to the square of its base which a plane being sees.

Let us imagine the cube in our space, which is the base of a tesseract, to turn about one of its edges. The rotation will carry the whole body with it, and each of the cubic sections will rotate. The axis we see in our space will remain unchanged, and likewise the series of axes parallel to it about which each of the parallel cubic sections rotate. The assemblage of all of these is a plane.

Hence in four dimensions a body rotates about a plane. There is no such thing as rotation round an axis.

We may regard the rotation from a different point of view. Consider four independent axes each at right angles to all the others, drawn in a four-dimensional body. Of these four axes we can see any three. The fourth extends normal to our space.

Rotation is the turning of one axis into a second, and the second turning to take the place of the negative of the first. It involves two axes. Thus, in this rotation of a four-dimensional body, two axes change and two remain at rest. Four-dimensional rotation is therefore a turning about a plane.

As in the case of a plane being, the result of rotation about a line would appear as the production of a looking-glass image of the original object on the other side of the line, so to us the result of a four-dimensional rotation would appear like the production of a looking-glass image of a body on the other size of a plane. The plane would be the axis of the rotation, and the path of the body between its two appearances would be unimaginable in three-dimensional space.

Let us now apply the method by which a plane being could examine the nature of rotation about a line in our examination of rotation about a plane. Fig. 3 represents a cube in our space, the three axes x, y, z denoting its three dimensions. Let us represent the fourth dimension. Now, since in our space we can represent any three dimensions, we can, if we choose, make a representation of what is in the space determined by the three axes x, z, w. This is a three-dimensional space determined by two of the axes we have drawn, x and z, and in place of y the fourth axis, w. We cannot, keeping x and z, have both y and w in our space;

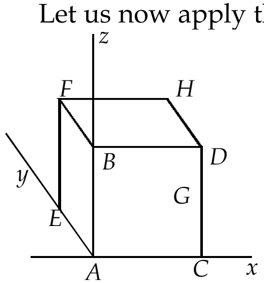


Fig. 3 (131).

so we will let y go and draw w in its place. What will be our view of the cube?

Evidently we shall have simply the square that is in the plane of xz, the square ACDE. The rest of the cube stretches in the y direction, and, as we have none of the space so determined, we have only the face of the cube. This is represented in fig. 4.

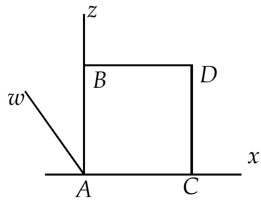


Fig. 4 (132).

Now, suppose the whole cube to be turned from the x to the w direction. Conformably with out method, we will not take the whole of the cube into consideration at once, but will begin with the face ABCD.

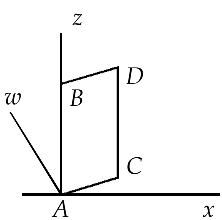


Fig. 5 (133).

Let this face begin to turn. Fig. 5 represents one of the positions it will occupy; the line AB remains on the x axis. The rest of the face extends between the x and the w direction.

Now, since we can take any three axes, let us look at what lies in the space of xyw, and examine the turning there. We must now let the z axis disappear and let the w axis run in the direction in which the z ran.

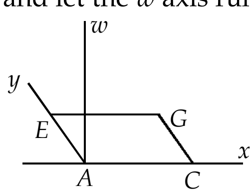


Fig. 6 (134).

Making this representation, what do we see of the cube? Obviously we see only the lower face. The rest of the cube lies in the space of xyz. In the space of xyw we have merely the base of the cube lying in the plane of xy, as shown in fig. 6.

Now let the x to w turning take place. The square ACEG will turn about the line AE. This edge will remain alone the y axis and will be stationary, however far the square turns.

Thus, if the cube be turned by an x to w turning, both the edge AB and the edge AC remain stationary; hence the whole face ABEF in the yz plane remains fixed. The turning has taken place about the face ABEF.

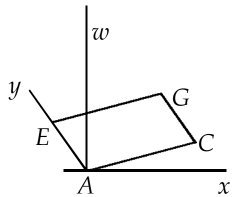


Fig. 7 (135).

Suppose this turning to continue till AC runs to the left from A. The cube will occupy the position shown in fig. 8. This is the looking-glass image of the cube in fig. 3. By no rotation in three-dimensional space can the cube be brought from the position in fig. 3 to that shown in fig. 8.

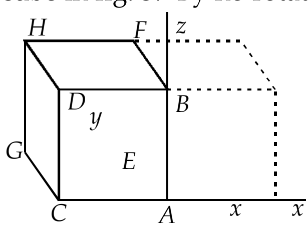


Fig. 8 (136).

We can think of this turning as a turning of the face ABCD about AB, and a turning of each section parallel to ABCD round the vertical line in which it intersects the face ABEF, the space in which the turning takes place being a different one from that in which the cube lies.

One of the conditions, then, of our inquiry in the direction of the infinitely small is that we form the conception of a rotation about a plane. The production of a body in a state in which it presents the appearance of a looking-glass image of its former state is the criterion for a four-dimensional rotation.

There is some evidence for the occurrence of such transformations of bodies in the change of bodies from those which produce a right-handed polarisation of light to those which produce a left-handed polarisation; but this is not a point to which any very great importance can be attached.

Still, in this connection, let me quote a remark from

Prof. John G. McKendrick's address on Physiology before the British Association at Glasgow. Discussing the possibility of the hereditary production of characteristics through the material structure of the ovum, he estimates that in it there exist 12,000,000,000 biophors, or ultimate particles of living matter, a sufficient number to account for hereditary transmission, and observes: "Thus it is conceivable that vital activities may also be determined by the kind of motion that takes place in the molecules of that which we speak of as living matter. It may be different in kind from some of the motions known to physicists, and it is conceivable that life may be the transmission to dead matter, the molecules of which have already a special kind of motion, of a form of motion *sui generis*."

Now, in the realm of organic beings symmetrical structures—those with a right and left symmetry—are everywhere in evidence. Granted that four dimensions exist, the simplest turning produces the image form, and by a folding-over structures could be produced, duplicated right and left, just as in the case of symmetry in a plane.

Thus one very general characteristic of the forms of organisms could be accounted for by the supposition that a four-dimensional motion was involved in the process of life.

But whether four-dimensional motions correspond in other respects to the physiologist's demand for a special kind of motion, or not, I do not know. Our business is with the evidence for their existence in physics. For this purpose it is necessary to examine into the significance of rotation round a plane in the case of extensible and of fluid matter.

Let us dwell a moment longer on the rotation of a rigid body. Looking at the cube in fig. 3, which turns about the face of ABFE, we see that any line in the face can take the place of the vertical and horizontal lines we have examined. Take the diagonal line AF and the section through it to GH. The portions of matter which were on one side of AF in this section in fig. 3 are on the opposite side of it in fig. 8. They have gone round the line AF. Thus the rotation round a face can be considered as a number of rotations of sections round parallel lines in it.

The turning about two different lines is impossible in three-dimensional space. To take another illustration, suppose A and B are two parallel lines in the *xy* plane, and let CD and EF be two rods crossing them. Now, in the space of *xyz* if the rods turn round the lines A and B in the same direction they will make two independent circles.

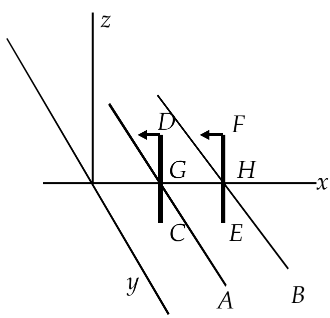


Fig. 9 (137).

When the end F is going down the end C will be coming up. They will meet and conflict.

But if we rotate the rods about the plane of AB by the *z* to *w* rotation these movements will not conflict. Suppose all the figure removed with the exception of the plane *xz*, and from this plane drawn the axis of *w*, so that we are looking at the space of *xzw*.

Here, fig. 10, we cannot see the lines A and B. We see the points G and H, in which A and B intercept the *x* axis, but we cannot see the lines themselves, for they run in the *y* direction, and that is not in our drawing.

Now, if the rods move with the *x* to *w* rotation they will turn in parallel planes, keeping their relative positions. The point B, for instance, will describe a circle. At one time it will be above the line A, at another time below it. Hence it rotates round A.

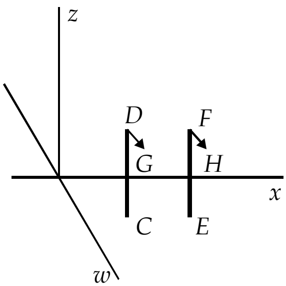


Fig. 10 (138).

Not only two rods but any number of rotes crossing the plane will move round it harmoniously. We can think of this rotation by supposing the rods standing up from one line to move round that line and remembering that it is not inconsistent with this rotation for the rods standing up along another line also to move round it, the relative positions of all the rods being preserved. Now, if the rods are thick together, they may represent a disk of matter, and se see that a disk of matter can rotate round a central plane.

Rotation round a plane is exactly analogous to rotation round an axis in three dimensions. If we want a rod to turn round, the ends must be free; so if we want a disk of matter to turn round its central plane by a four-dimensional turning, all the contour must be free. The whole contour corresponds to the ends of the rod. Each point of the contour can be looked on as the extremity of an axis in the body, round each point of which there is a rotation of the matter in the dis.

If the one end of a rod be clamped, we can twist the rod, but not turn it round; so if any part of the contour of a disk is clamped we can impart a twist to the disk, but not turn it round its central plane. In the case of extensible materials a long, thin rod will twist round its axis, even when the axis is curved, as, for instance, in the case of a ring of India rubber.

In an analogous manner, in four dimensions we can have rotation round a curved plane, if I may use the expression. A sphere can be turned inside out in four dimensions.

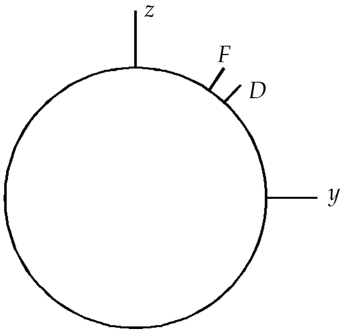


Fig. 11 (139).

Let fig. 11 represent a spherical surface, on each side of which a layer of matter exists. The thickness of the matter is represented by the rods CD and EF, extending equally without and within.

Now, take the section of the sphere by the *yz* plane, we have a circle—fig. 12. Now, let the *w* axis be drawn in place of the *x* axis so that we have the space of *yzw* represented. In this space all that there will be seen of the sphere is the circle drawn.

Here we see that there is no obstacle to prevent the rods turning round. If the matter is so elastic that it will give enough for the particles at E and C to be separated as they are at F and D, they can rotate round to the position D and F, and a similar motion is possible for all other particles. There is no matter or obstacle to prevent them from moving out in the *w* direction, and then on round the circumference as an axis. Now, what will hold for one section will hold for

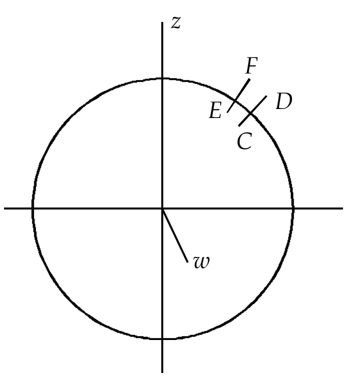


Fig. 12 (140).

all, as the fourth dimension is at right angles to all the sections which can be made of the sphere.

We have supposed the matter of which the sphere is composed to be three-dimensional. If the matter had a small thickness in the fourth dimension, there would be a slight thickness in fig. 12 above the plane of the paper—a thickness equal to the thickness of the matter in the fourth dimension. The rods would have to be replaced by thin slabs. But this would make no difference as to the possibility of the rotation. This motion is discussed by Newcomb in the first volume of the *American Journal of Mathematics*.

Let us now consider, not a merely extensible body, but a liquid one. A mass of rotating liquid, a whirl, eddy, or vortex, has many remarkable properties. On first consideration we should expect the rotating mass of liquid immediately to spread off and lose itself in the surrounding liquid. The water flies off a wheel whirled round, and we should expect the rotating liquid to be dispersed. But see the eddies in a river strangely persistent. The rings that occur in puffs of smoke and last so long are whirls or vortices curved round so that their opposite ends join together. A cyclone will travel over great distances.

Helmholtz was the first to investigate the properties of vortices. He studied them as they would occur in a perfect fluid—that is, one without friction of one moving portion or another. In such a medium vortices would be indestructible. They would go on for ever, altering their shape, but consisting always of the same portion of the fluid. But a straight vortex could not exist surrounded entirely by the fluid. The ends of a vortex must reach to some boundary inside or outside the fluid.

A vortex which is bent round so that its opposite ends join is capable of existing, but no vortex has a free end in the fluid. The fluid round the vortex is always in motion, and one produces a definite movement in another.

Lord Kelvin has proposed the hypothesis that portions of a fluid segregated in vortices account for the origin of matter. The properties of the ether in respect of its capacity of propagating disturbances can be explained by the assumption of vortices in it instead of by a property of rigidity. It is difficult to conceive, however, of any arrangement of the vortex rings and endless vortex filaments in the ether.

Now, the further consideration of four-dimensional rotations shows the existence of a kind of vortex which would make an ether filled with a homogenous vortex motion easily thinkable.

To understand the nature of this vortex, we must go on and take a step by which we accept the full significance of the four-dimensional hypothesis. Granted four-dimensional axes, we have seen that a rotation of one into another leaves two unaltered, and these two form the axial plane about which the rotation takes place. But what about these two? Do they necessarily remain motionless? There is nothing to prevent a rotation of these two, one into the other, taking place concurrently with the first rotation. This possibility of a double rotation deserves the most careful attention, for it is the kind of movement which is distinctly typical of four dimensions.

Rotation round a plane is analogous to rotation round an axis. But in three-dimensional space there is no motion analogous to the double rotation, in which, while axis 1 changes into axis 2, axis 3 changes into axis 4.

Consider a four-dimensional body, with four independent axes, x, y, z, w. A point in it can move in only one direction at a given moment. If the body has a velocity of rotation by which the z axis changes into the y axis and all parallel sections move in a similar manner, then the point will describe a circle. If, now, in addition to the rotation by which the x axis changes into the y axis the body has a rotation by which the z axis turns into the w axis, the point in question will have a double motion in consequence of the two turnings. The motions will compound, and the point will describe a circle, but not the same circle which it would describe in virtue of either rotation separately.

We know that if a body in three-dimensional space is given two movements of rotation they will combine into a single movement of rotation round a definite axis. It is in no different condition from that in which it is subjected to one movement of rotation. The direction of the axis changes; that is all. The same is not true about a four-dimensional body. The two rotations, x to y and z to w, are independent. A body subject to the two is in a totally different condition to that which it is in when subject to one only. When subject to a rotation such as that of x to y, a whole plane in the body, as we have seen, is stationary. When subject to the double rotation no part of the body is stationary except the point common to the two planes of rotation.

If the two rotations are equal in velocity, every point in the body describes a circle. All points equally distant from the stationary point describe circles of equal size.

We can represent a four-dimensional sphere by means of two diagrams, in one of which we take the three axes, x, y, z; in the other the axes x, w, and z. In fig. 13 we have the view of a four-dimensional sphere in the space of xyz. Fig 13. shows all that we can see of the four sphere in the space of xyz, for it represents all the points in that space, which are at an equal distance from the centre.

Let us now take the xz section, and let the axis of w

take the place of the y axis. Here, in fig. 14, we have the space of xzw. In this space we have to take all the points which are at the same distance from the centre, consequently we have another sphere. If we had a three-dimensional sphere, as has been shown before, we should have merely a circle in the xzw space, the xz circle seen in the space of xzw. But now, taking the view in the space of xzw, we have a sphere in that space also. In a similar manner, whichever set of three axes we take, we obtain a sphere.

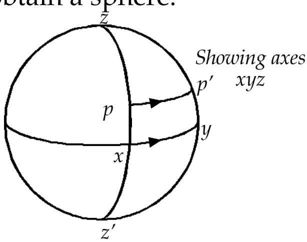


Fig. 13 (141).

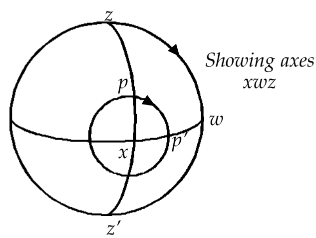


Fig. 14 (142).

In fig. 13, let us imagine the rotation in the direction xy to be taking place. The point x will turn to y, and p to p'. The axis zz' remains stationary, and this axis is all of the plane xw which we can see in the space section exhibited in the figure.

In fig. 14, imagine the rotation from x to w to be taking place. The w axis now occupies the position previously occupied by the y axis. This does not mean that the w axis can coincide with the y axis. It indicates that we are looking at the four-dimensional sphere from a different point of view. Any three-space view will show us three axes, and in fig. 14 we are looking at xzw.

The only part that is identical in the two diagrams is the circle of the x and z axes, which axes are contained in both diagrams. Thus the plane zxz' is the same in both, and the point p represents the same point in both diagrams. Now, in fig. 14 let the zw rotation take place, the x axis will turn toward the point w of the w axis, and the point p will move in a circle about the point x.

Thus in fig. 13 the point p moves in a circle parallel to the xy plane: in fig. 14 it moves in a circle parallel to the zw plane, indicated by the arrow.

Now, suppose both of these independent rotations compounded, the point p will move in a circle, but this circle will coincide with neither of the circles in which either one of the rotations will take it. The circle the point p will move in will depend on its position on the surface of the four sphere.

In this double rotation, possible in four-dimensional space, there is a kind of movement totally unlike any with which we are familiar in three-dimensional space. It is a requisite preliminary to the discussion of the behaviour of the small particles of matter, with a view to determining whether they show the characteristics of four-dimensional movements, to become familiar with the main characteristics of this double rotation. And here I must rely on a formal and logical assent rather than on the intuitive apprehension, which can only be obtained by a more detailed study.

In the first place this double rotation consists in two varieties or kinds, which we will call the A and B kinds. Consider four axes, x, y, z, w. The rotation of x to y can be accompanied with the rotation of z to w. Call this the A kind.

But also the rotation of x to y can be accompanied by the rotation, of not z to w, but w to z. Call this the B kind.

They differ in only one of the component rotations. One is not the negative of the other. It is the semi-negative. The opposite of an x to y, z to w rotation would be y to x, w to z. The semi-negative is x to y and w to z.

If four dimensions exist and we cannot perceive them, because the extension of matter is so small in the fourth dimension that all movements are withheld from direct observation except those which are three-dimensional, we should not observe these double rotations, but only the effects of them in three-dimensional movements of the type with which we are familiar.

If matter in its small particles is four-dimensional, we should expect this double rotation to be a universal characteristic of the atoms and molecules, for no portion of matter is at rest. The consequences of this corpuscular motion can be perceived, but only under the form of ordinary rotation or displacement. Thus, if the theory of four dimensions is true, we have in the corpuscles of matter a whole world of movement, which we can never study directly, but only by means of inference.

The rotation A, as I have defined it, consists of two equal rotations—one about the plane of *zw*, the other about the plane of *xy*. It is evident that these rotations are not necessarily equal. A body may be moving with a double rotation, in which these two independent components are not equal; but in such a case we can consider the body to be moving with a composite rotation—a rotation of the A or B kind and, in addition, a rotation about a plane.

If we combine an A and a B movement, we obtain a rotation about a plane; for the first being x to y and x to w, and the second being x to y and w to z, when they are put together the z to w and w to z rotations neutralise each other, and we obtain an x to y rotation only, which is a rotation about the plane of zw. Similarly, if we take a B rotation, y to x and z to w, we get, on combining this with the A rotation, a rotation of z to w about the xy plane. In this case the plane of rotation is in the three-dimensional space of xyz, and we have—what has been described before—a twisting about a plane in our space.

Consider now a portion of a perfect liquid having an A motion. It can be proved that it possesses the properties of a vortex. It forms a permanent individuality—a separated-out portion of the liquid—accompanied by a motion of the surrounding liquid. It has properties analogous to those of a vortex filament. But it is not necessary for its existence that its ends should reach the boundary of the liquid. It is self-contained and, unless disturbed, is circular in every section.

If we suppose the ether to have its properties of transmitting vibration given it by such vortices, we must inquire how they lie together in four-dimensional space. Placing a circular disk on a plane and surrounding it by six others, we find that if the central one is given a motion of rotation, it imparts to the others a rotation which is antagonistic in every two adjacent ones. If A goes round, as shown by the arrow, B and C will be moving in opposite ways, and each tends to destroy the motion of the other.

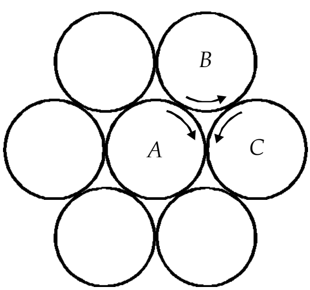


Fig. 15 (143).

Now, if we suppose spheres to be arranged in a corresponding manner in three-dimensional space, they will be grouped in figures which are for three-dimensional space what hexagons are for plane space. If a number of spheres of soft clay be pressed together, so as to fill up the interstices, each will assume the form of a fourteen-sided figure called a tetrakaidekagon.

Now, assuming space to be filled with such tetrakaidekagons, and placing a sphere in each, it will be found that one sphere is touched by six others. The remaining eight spheres of the fourteen which surround the central one will not touch it, but will touch three of those in contact with it. Hence, if the central sphere rotates, it will not necessarily drive those around it so that their motions will be antagonistic to each other, but the velocities will not arrange themselves in a systematic manner.

In four-dimensional space the figure which forms the next term of the series hexagon, tetrakaidekagon, is a thirty-sided figure. It has for its faces ten solid tetrakaidekagons and twenty hexagonal prisms. Such figures will exactly fill four-dimensional space, five of them meeting at every point. If, now, in each of these figures we suppose a solid four-dimensional sphere to be placed, any one sphere is surrounded by thirty others. Of these it touches ten, and, if it rotates, it drives the rest by means of these. Now, if we imagine the central sphere to be given an A or B rotation, it will turn the whole mass of sphere round in a systematic manner. Suppose four-dimensional space to be filled with such spheres, each rotating with a double rotation, the whole mass would form one consistent system of rotation, in which each one drove every other one, with no friction or lagging behind.

Every sphere would have the same kind of rotation. In three-dimensional space, if one body drives another round the second body rotates with the opposite kind of rotation; but in four-dimensional space these four-dimensional spheres would each have the double negative of the rotation of the one next it, and we have seen that the double negative of an A or B rotation is still an A or B rotation. Thus four-dimensional space could be filled with a system of self-preservative living energy. If we imagine the four-dimensional spheres to be of liquid and not of solid matter, then, even if the liquid were not quite perfect and there were a slight retarding effect of one vortex on another, the system would still maintain itself.

In this hypothesis we must look on the ether as possessing energy, and its transmission of vibrations, not as the conveying of a motion imparted from without, but as a modification of its own motion.

We are now in possession of some of the conceptions of four-dimensional mechanics, and will turn aside from the line of their development to inquire if there is any evidence of their applicability to the processes of nature.

Is there any mode of motion in the region of the minute which, giving three-dimensional movements for its effect, still in itself escapes the grasp of our mechanical theories? I would point to electricity. Through the labours of Faraday and Maxwell we are convinced that the phenomena of electricity are of the nature of the stress and strain of a medium; but there is still a gap to be bridged over in their explanation—the laws of elasticity, which Maxwell assumes are not those of ordinary matter. And, to take another instance: a magnetic pole in the neighbourhood of a current tends to move. Maxwell has shown that the pressures on it are analogous to the velocities in a liquid which would exist if a vortex took the place of the electric current: but we cannot point out the definite mechanical explanation of these pressures. There must be some mode of motion of a body or of the medium in virtue of which a body is said to be electrified.

Take the ions which convey charges of electricity 500 times greater in proportion to their mass than are carried by the molecules of hydrogen in electrolysis. In respect of what motion can these ions be said to be electrified? It can be shown that the energy they possess is not energy of rotation. Think of a short rod rotating. If it is turned over it is found to be rotating in the opposite direction. Now, if rotation in one direction corresponds to positive electricity, rotation in the opposite direction corresponds to negative electricity, and the smallest electrified particles would have their charges reversed by being turned over—an absurd supposition.

If we fix on a mode of motion as a definition of electricity, we must have two varieties of it, one for positive and one for negative; and a body possessing the one kind must not become possessed of the other by any change in its position.

All three-dimensional motions are compounded of rotations and translations, and note of them satisfy this first condition for serving as a definition of electricity.

But consider the double rotation of the A and B kinds. A body rotating with the A motion cannot have its motion transformed into the B kind by being turned over in any way. Suppose a body has the rotation x to y and z to w. Turning it about the xy plane, we reverse the direction of the motion x to y. But we also reverse the z to w motion, for the point at the extremity of the positive z axis is now at the extremity of the negative z axis, and since we have not interfered with its motion it goes in the direction of position w. Hence we have y to x and w to z, which is the same as x to y and z to w. Thus both components are reversed, and there is the A motion over again. The B kind is the semi-negative, with only one component reversed.

Hence a system of molecules with the A motion would not destroy it in one another, and would impart it to a body in contact with them. Thus A and B motions possess the first requisite which must be demanded in any mode of motion representative of electricity.

Let us trace out the consequences of defining positive electricity as an A motion and negative electricity as a B motion. The combination of positive and negative electricity produces a current. Imagine a vortex in the ether of the A kind and unite with the one of the B kind. An A motion and B motion produce rotation round a plane, which is in the ether a vortex round an axial surface. It is a vortex of the kind we represent as part of a sphere turning inside out. Now such a vortex must have its rim on a boundary of the ether—on a body in the ether.

Let us suppose that a conductor is a body which has the property of serving as the terminal abutment of such a vortex. Then the conception we must form of a closed current is of a vortex sheet having its edge along the circuit of the conducting wire. The whole wire will then be like the centres on which a spindle turns in three-dimensional space, and any interruption of the continuity of the wire will produce a tension in place of a continuous revolution.

As the direction of the rotation of the vortex is from a three-space direction into the fourth dimension and back again, there will be no direction of flow to the current; but it will have two sides, according to whether z goes to w or z goes to negative w.

We can draw any line from one part of the circuit to another; then the ether along that line is rotating round its points.

This geometric image corresponds to the definition of an electric circuit. It is known that the action does not lie in the wire, but in the medium, and it is known that there is no direction of flow in the wire.

No explanation has been offered in three-dimensional mechanics of how an action can be impressed throughout a region and yet necessarily run itself out along a closed boundary, as is the case in an electric current. But this phenomenon corresponds exactly to the definition of a four-dimensional vortex.

If we take a very long magnet, so long that one of its poles is practically isolated, and put this pole in the vicinity of an electric circuit, we find that it moves.

Now, assuming for the sake of simplicity that the wire which determines the current is in the form of a circle, if we take a number of small magnets and place them all pointing in the same direction normal to the plane of the circle, so that they fill it and the wire binds them round, we find that this sheet of magnets has the same effect on the magnetic pole that the current has. The sheet of magnets may be curved, but the edge of it must coincide with the wire. The collection of magnets is then equivalent to the vortex sheet, and an elementary magnet to a part of it. Thus, we must think of a magnet as conditioning a rotation in the ether round the plane which bisects at right angles the line joining its poles.

If a current is started in a circuit, we must imagine vortices like bowls turning themselves inside out, starting from the contour. In reaching a parallel circuit, if the vortex sheet were interrupted and joined momentarily to the second circuit by a free rim, the axis plane would lie between the two circuits, and a point on the second circuit opposite a point on the first would correspond to a point opposite to it on the first; hence we should expect a current in the opposite direction in the second circuit. Thus the phenomena of induction are not inconsistent with the hypothesis of a vortex about an axial plane.

In four-dimensional space, in which all four dimensions were commensurable, the intensity of the action transmitted by the medium would vary inversely as the cube of the distance. Now, the action of a current on a magnetic pole varies inversely as the square of the distance; hence, over measurable distances the extension of the ether in the fourth dimension cannot be assumed as other than small in comparison with these distances.

If we suppose the ether to be filled with vortices in the shape of four-dimensional spheres rotating with the A motion, the B motion would correspond to electricity in the one-fluid theory. There would thus be a possibility of electricity existing in two forms, statically, by itself, and combined with the universal motion, in the form of a current.

To arrive at a definite conclusion it will be necessary to investigate the resultant pressures which accompany the collocation of solid vortices with surface ones.

To recapitulate:

The movements and mechanics of four-dimensional space are definite and intelligible. A vortex with a surface as its axis affords a geometric image of a closed circuit, and there are rotations which by their polarity afford a possible definition of statical electricity.

APPENDIX I
THE MODELS

IN Chapter XI. a description has been given which will enable any one to make a set of models illustrative of the tesseract and its properties. The set here supposed to be employed consists of:—

1. Three sets of twenty-seven cubes each.
2. Twenty-seven slabs.
3. Twelve cubes with points, lines, faces, distinguished by colours, which will be called the catalogue cubes.

The preparation of the twelve catalogue cubes involves the expenditure of a considerable amount of time. It is advantageous to use them, but they can be replaced by the drawing of the views of the tesseract or by a reference to figs. 103, 104, 105, 106 of the text.

The slabs are coloured like the twenty-seven cubes of the first cubic block in fig. 101, the one with red, white, yellow axes.

The colours of the three sets of twenty-seven cubes are those of the cubes shown in fig. 101.

The slabs are used to form the representation of a cube in a plane, and can well be dispensed with by any one who is accustomed to deal with solid figures. But the whole theory depends on a careful observation of how the cube would be represented by these slabs.

In the first step, that of forming a clear idea how a plane being would represent three-dimensional space, only one of the catalogue cubes and one of the three blocks is needed.

APPLICATION TO THE STEP FROM PLANE TO SOLID.

Look at fig. 1 of the view of the tesseract, or, what comes to the same thing, take catalogue cube No. 1 and place it before you with the red line running up, the white line running to the right, the yellow line running away. The three dimensions of space are then marked out by these lines or axes. Now take a piece of card-board, or a book, and place it so that it forms a wall extending up and down, not opposite to you, but running away parallel to the wall of the room on your left hand.

Placing the catalogue cube against this wall we see that it comes into contact with it by the red and yellow lines, and by the included orange face.

In the plane being's world the aspect he has of the cube would be a square surrounded by red and yellow lines with grey points.

Now, keeping the red line fixed, turn the cube about it so that the yellow line goes out to the right, and the white line comes into contact with the plane.

In this case a different aspect is presented to the plane being, a square, namely, surrounded by red and white lines, and grey points. You should particularly notice that when the yellow line goes out, at right angles to the plane, and the white comes in, the latter does not run in the same sense that the yellow did.

From the fixed grey point at the base of the red line the yellow line ran away from you. The white line now runs towards you. This turning at right angles makes the line which was out of the plane before, come into it in an opposite sense to that in which the line ran which has just left the plane. If the cube does not break through the plane this is always the rule.

Again turn the cube back to the normal position with red running up, white to the right, and yellow away, and try another turning.

You can keep the yellow line fixed, and turn the cube about it. In this case, the red line going out to the right, the white line will come in pointing downwards.

You will be obliged to elevate the cube form the table in order to carry out this turning. It is always necessary when a vertical axis goes out of a space to imagine a movable support which will allow the line which ran out before to come in below.

Having looked at the three ways of turning the cube so as to present different faces to the plane, examine what would be the appearance if a square hole were cut in the piece of cardboard, and the cube were to pass through it. A hole can be actually cut, and it will be seen that in the normal position, with red axis running up, yellow away, and white to the right, the square first perceived by the plane being—the one contained by red and yellow lines—would be replaced by another square of which the line towards you is pink—the section line of the pink face. The line above is light yellow, below is light yellow and on the opposite side away from you is pink.

In the same way the cube can be pushed through a square opening in the plane from any of the positions which you have already turned it into. In each case the plane being will perceive a different set of contour lines.

Having observed these facts about the catalogue cube, turn now to the first block of twenty-seven cubes.

You notice that the colour scheme on the catalogue cube and that of this set of blocks is the same.

Place them before you, a grey or null cube on the table, above it a red cube, and on the top a null cube again. Then away from you place a yellow cube, and beyond it a null cube. Then to the right place a white cube and beyond it another null. Then complete the block, according to the scheme of the catalogue cube, putting in the centre of all an ochre cube.

You now have a cube like that which is described in the text. For the sake of simplicity, in some cases, this cubic block can be reduced to one of eight cubes, by leaving out the terminations in each direction. Thus, instead of null, red, null, three cubes, you can take null, red, two cubes, and so on.

It is useful, however, to practise the representation in a plane of a block of twenty-seven cubes. For this purpose take the slabs, and build them up against the piece of cardboard, or the book, in such a way as to represent the different aspects of the cube.

Proceed as follows:—

First, cube in normal position.

Place nine slabs against the cardboard to represent the nine cubes in the wall of the red and yellow axes, facing the cardboard; these represent the aspect of the cube as it touches the plane.

Now push these along the cardboard and make a different set of nine slabs to represent the appearance which the cube would present to a plane being if it were to pass half way through the plane.

There would be a white slab, above it a pink one, above that another white one, and six others, representing what would be the nature of a section across the middle of the block of cubes. The section can be thought of as a thin slice cut out by two parallel cuts across the cube. Having arranged these nine slabs, push them along the plane, and make another set of nine to represent what would be the appearance of the cube when it had almost completely gone through. This set of nine will be the same as the first set of nine.

Now we have in the plane three sets of nine slabs each, which represent three sections of the twenty-seven block.

They are put alongside one another. We see that it does not matter in what order the three sets of nine are put. As the cube passes through the plane they represent appearances which follow the one after the other. If they were what they represented, they could not exist in the same plane together.

This is a rather important point, namely, to notice that they should not co-exist on the plane, and that the order in which they are placed is indifferent. When we represent a four-dimensional body our solid cubes are to us in the same position that the slabs are to the plane being. You should also notice that each of these slabs represents only the very thinnest slice of a cube. The set of nine slabs first set up represents the side surface of the block. It is, as it were, a kind of tray—a beginning from which the solid cube goes off. The slabs as we use them have thickness, but this thickness is a necessity of construction. They are to be thought of as merely of the thickness of a line.

If now the block of cubes passed through the plane at the rate of an inch a minute the appearance to a plane being would be represented by:—

1. The first set of nine slabs lasting for one minute.
2. The second set of nine slabs lasting for one minute.
3. The third set of nine slabs lasting for one minute.

Now the appearances which the cubes would present to the plane being in other positions can be shown by means of these slabs. The use of such slabs would be the means by which a plane being could acquire a familiarity with out cube. Turn the catalogue cube (or imagine the coloured figure turned) so that the red line runs up, the yellow line out to the right, and the white line towards you. Then turn the block of cubes to occupy a similar position.

The block has now a different wall in contact with the plane. Its appearance to a plane being will not be the same as before. He has, however, enough slabs to represent this new set of appearances. But he must remodel his former arrangement of them.

He must take a null, a red, and a null slab from the first of his sets of slabs, then a white, a pink, and a white from the second, and then a null, a red, and a null from the third set of slabs.

He takes the first column from the first set, the first column from the second set, and the first column from the third set.

To represent the half-way-through appearance, which is as if a very thin slice were cut out half way through the block, he must take the second column of each of his sets of slabs, and to represent the final appearance, the third column of each set.

Now turn the catalogue cube back to the normal position, and also the block of cubes.

There is another turning—a turning about the yellow line, in which the white axis comes below the support.

You cannot break through the surface of the table, so you must imagine the old support to be raised. Then the top of the block of cubes in its new position is at the level at which the base of it was before.

Now representing the appearance on the plane, we must draw a horizontal line to represent the old base. The line should be drawn three inches high on the cardboard.

Below this the representative slabs can be arranged.

It is easy to see what they are. The old arrangements have to be broken up, and the layers taken in order, the first layer of each for the representation of the aspect of the block as it touches the plane.

Then the second layers will represent the appearance half way through, and the third layers will represent the final appearance.

It is evident that the slabs individually do not represent the same portion of the cube in these different presentations.

In the first case each slab represents a section or a face perpendicular to the white axis, in the second case a face or a section which runs perpendicularly to the yellow axis, and in the third case a section or a face perpendicular to the red axis.

But by means of these nine slabs the plane being can represent the whole of the cubic block. He can touch and handle each portion of the cubic block, there is no part of it which he cannot observe. Taking it bit by bit, two axes at a time, he can examine the whole of it.

OUR REPRESENTATION OF A BLOCK OF TESSERACTS.

Look at the views of the tesseract 1, 2, 3, or take the catalogue cubes 1, 2, 3, and place them in front of you, in any order, say running from left to right, placing 1 in the normal position, the red axis running up, the white to the right, and yellow away.

Now notice that in catalogue cube 2 the colours of each region are derived from those of the corresponding region of cube 1 by the addition of blue. Thus null + blue = blue, and the corners of number 2 are blue. Again, red + blue = purple, and the vertical lines of 2 are purple. Blue + yellow = green, and the line which runs away is coloured green.

By means of these observations you may be sure that catalogue cube 2 is rightly placed. Catalogue cube 3 is just like number 1.

Having these cubes in what we may call their normal position, proceed to build up the three sets of blocks.

This is easily done in accordance with the colour scheme on the catalogue cubes.

The first block we already know. Build up the second block, beginning with a blue corner cube, placing a purple on it, and so on.

Having these three blocks we have the means of representing the appearances of a group of eighty-one tesseracts.

Let us consider a moment what the analogy in the case of the plane being is.

He has his three sets of nine slabs each. We have our three sets of twenty-seven cubes each.

Our cubes are like his slabs. As his slabs are not the things which they represent to him, so our cubes are not the things they represent to us.

The plane being's slabs are to him the faces of cubes.

Our cubes then are the faces of tesseracts, the cubes by which they are in contact with our space.

As each set of slabs in the case of the plane being might be considered as a sort of tray from which the solid contents of the cubes came out, so our three blocks of cubes may be considered as three-space trays, each of which is the beginning of an inch of the solid contents of the four-dimensional solids starting from them.

We want now to use the names null, red, white, etc., for tesseracts. The cubes we use are only tesseract faces. Let us denote that fact by calling the cube of null colour, null face; or, shortly, null f., meaning that it is the face of a tesseract.

To determine which face it is let us look at the catalogue cube 1 or the first of the views of the tesseract, which can be used instead of the models. It has three axes, red, white, yellow, in our space. Hence the cube determined by these axes is the face of the tesseract which we now have before us. It is the ochre face. It is enough, however, simply to say null f., red f. for the cubes which we use.

To impress this in your mind, imagine that tesseracts do actually run from each cube. Then, when you move the cubes about, you move the tesseracts about with them. You move the face but the tesseract follows with it, as the cube follows when its face is shifted in a plane.

The cube null in the normal position is the cube which has in it the red, yellow, white axes. It is the face having these, but wanting the blue. In this way you can define which face it is you are handling. I will write an "f." after the name of each tesseract just as the plane being might call each of his slabs null slab, yellow slab, etc., to denote that they were representations.

We have then in the first block of twenty-seven cubes, the following—null f., red f., null f., going up; white f., null f., lying to the right, and so on. Starting from the null point and travelling up one inch we are in the null region, the same for the away and the right-hand directions. And if we were to travel in the fourth dimension for an inch we should still be in a null region. The tesseract stretches equally all four ways. Hence the appearance we have in this first block would do equally well if the tesseract block were to move across our space for a certain distance. For anything less than an inch of their transverse motion we would still have the same appearance. You must notice, however, that we should not have null face after the motion had begun.

When the tesseract, null for instance, had moved ever so little we should not have a face of null but a section of null in our space. Hence, when we think of the motion across our space we must call our cubes tesseract sections. Thus on null passing across we should see first null f., then null s., and then, finally, null f. again.

Imagine now the whole first block of twenty-seven tesseracts to have moved transverse to our space a distance of one inch. Then the second set of tesseracts, which originally were an inch distant from our space, would be ready to come in.

Their colours are shown in the second block of twenty-seven cubes which you have before you. These represent the tesseract faces of the set of tesseracts that lay before an inch away from our space. They are ready now to come in, and we can observe their colours. In the place which null f. occupied before we have blue f., in place of red f. we have purple f., and so on. Each tesseract is coloured like the one whose place it takes in this motion with the addition of blue.

Now if the tesseract block goes on moving at the rate of an inch a minute, this next set of tesseracts will occupy a minute in passing across. We shall see, to take the null one for instance, first of all null face, then null section, then null fact again.

At the end of the second minute the second set of tesseracts has gone through, and the third set comes in. This, as you see, is coloured just like the first. Altogether, these three sets extend three inches in the fourth dimension, making the tesseract block of equal magnitude in all dimensions.

We have now before us a complete catalogue of all the tesseracts in our group. We have seen them all, and we shall refer to this arrangement of the blocks as the "normal position." We have seen as much of each tesseract at as time as could be done in a three-dimensional space. Each part of each tesseract has been in our space, and we could have touched it.

The fourth dimension appeared to us as the duration of the block.

If a bit of our matter were to be subjected to the same motion it would be instantly removed out of our space. Being thin in the fourth dimension it is at once taken out of our space by a motion in the fourth dimension.

But the tesseract block we represent, having length in the fourth dimension remains steadily before our eyes for three minutes, when it is subjected to this transverse motion.

We have now to form representations of the other views of the same tesseract group which are possible in our space.

Let us then turn the block of tesseracts so that another face of it comes into contact with our space, and then by observing what we have, and what changes come when the block traverses our space, we shall have another view of it. The dimension which appeared as duration before will become extension in one of our known dimensions, and a dimension which coincides with one of our space dimensions will appear as duration.

Leaving catalogue cube 1 in the normal position, remove the other two, or suppose them removed. We have in space the red, the yellow, and the white axes. Let the white axis go out into the unknown, and occupy the position the blue axis holds. Then the blue axis, which runs in that direction now will come into space. But it will not come in pointing in the same way that the white axis does now. It will point in the opposite sense. It will come in running to the left instead of running to the right as the white axis does now.

When this turning takes place every part of the cube 1 will disappear except the left-hand face—the orange face.

And the new cube that appears in our space will run to the left from this orange face, having axes, red, yellow, blue.

Take models 4, 5, 6. Place 4, or suppose No. 4 of the tesseract views place, with its orange face coincident with the orange face of 1, red line to red line, and yellow line to yellow line, with the blue line pointing to the left. Then remove cube 1 and we have the tesseract face which comes in when the white axis runs in the positive unknown, and the blue axis comes into our space.

Now place catalogue cube 5 in some position, it does not matter which, say to the left; and place it so that there is a correspondence of colour corresponding to the colour of the line that runs out of space. The line that runs out of space is white, hence, every part of this cube 5 should differ from the corresponding part of 4 by an alteration in the direction of white.

Thus we have white points in 5 corresponding to the null points in 4. We have a pink line corresponding to a red line, a light yellow line corresponding to a yellow line, an ochre face corresponding to an orange face. This cube section is completely named in Chapter XI. Finally cube 6 is a replica of 4.

These catalogue cubes will enable us to set up our models of the block of tesseracts.

First of all for the set of tesseracts, which beginning in our space reach out one inch in the unknown, we have the pattern of catalogue cube 4.

We see that we can build up a block of twenty-seven tesseract faces after the colour scheme of cube 4, by taking the left-hand wall of block 1, then the left-hand wall of block 2, and finally that of block 3. We take, that is, the first three walls of our previous arrangement to form the first cubic block of this new one.

This will represent the cubic faces by which the group of tesseracts in its new position touches our space. We have running up, null f., red f., null f. In the next vertical line, on the side remote from us, we have yellow f., orange f., yellow f., and then the first colours over again. Then the three following columns are, blue f., purple f., blue f.; green f., brown f., green f.; blue f., purple f., blue f. The last three columns are like the first.

These tesseracts touch our space, and none of them are by any part of them distant more than an inch from it. What lies beyond them in the unknown?

This can be told be looking at catalogue cube 5. According to its scheme of colour we see that the second wall of each of our old arrangements must be taken. Putting them together we have, as the corner, white f. above it, pink f. above it, white f. The column next to this remote from us is as follows:—light yellow f., ochre f., light yellow f., and beyond this a column like the first. Then for the middle of the block, light blue f., above it light purple, then light blue. The centre column has, at the bottom, light green f., light brown f. in the centre and at the top light green f. The last wall is like the first.

The third block is made by taking the third walls of our previous arrangement, which we called the normal one.

You may ask what faces and what sections our cubes represent. To answer this question look at what axes you have in our space. You have red, yellow, blue. Now these determine brown. The colours red, yellow, blue are supposed by us when mixed to produce a brown colour. And that cube which is determined by the red, yellow, blue axes we call the brown cube.

When the tesseract block in its new position begins to move across our space each tesseract in it gives a section in our space. This section is transverse to the white axis, which now runs in the unknown.

As the tesseract in its present position passes across our space, we should see first of all the first of the block of cubic faces we have put up—these would last for a minute, then would come the second block and then the third. At first we should have a cube of tesseract faces, each of which would be brown. Directly the movement began, we should have tesseract sections transverse to the white line.

There are two more analogous positions in which the block of tesseracts can be placed. To find the third position, restore the blocks to the normal arrangement.

Let us make the yellow axis go out into the positive unknown, and let the blue axis, consequently, come in running towards us. The yellow ran away, so the blue will come in running towards us.

Put catalogue cube 1 in its normal position. Take catalogue cube 7 and place it so that its pink face coincides with the pink face of cube 1, making also its red axis coincide with the red axis of 1 and its white with the white. Moreover, make cube 7 come towards us from cube 1. Looking at it we see in our space, red, white, and blue axes. The yellow runs out. Place catalogue cube 8 in the neighbourhood of 7—observe that every region in 8 has a change in the direction of yellow from the corresponding region in 7. This is because it represents what you come to now in going in the unknown, when the yellow axis runs out of our space. Finally catalogue cube 9, which is like number 7, shows the colours of the third set of tesseracts. Now evidently, starting from the normal position, to make up our three blocks of tesseract faces we have to take the near wall from the first block, the near wall from the second, and then the near wall from the third block. This gives us the cubic block formed by the faces of the twenty-seven tesseracts which are now immediately touching our space.

Following the colour scheme of catalogue cube 8, we make the next set of twenty-seven tesseract faces, representing the tesseracts, each of which begins one inch off from our space, by putting the second walls of our previous arrangement together, and the representation of the third set of tesseracts is the cubic block formed of the remaining three walls.

Since we have red, white, blue axes in our space to begin with, the cubes we see at first are light purple tesseract faces, and after the transverse motion begins we have cubic sections transverse to the yellow line.

Restore the blocks to the normal position, there remains the case in which the red axis turns out of space. In this case the blue axis will come in downwards, opposite to the sense in which the red axis ran.

In this case take catalogue cubes 10, 11, 12. Lift up catalogue cube 1 and put 10 underneath it, imagining that it goes down from the previous position of 1.

We have to keep in space the white and the yellow axes, and let the red go out, the blue come in.

Now, you will find on cube 10 a light yellow face; this should coincide with the base of 1, and the white and yellow lines on the two cubes should coincide. Then, the blue axis running down, you have the catalogue cube correctly placed, and it forms a guide for putting up the first representative block.

Catalogue cube 11 will represent what lies in the fourth dimension—now the red line runs in the fourth dimension. Thus the change from 10 to 11 should be towards red; corresponding to a null point is a red point, to a white line is a pink line, to a yellow line an orange line, and so on.

Catalogue cube 12 is like 10. Hence we see that to build up our blocks of tesseract faces we must take the bottom layer of the first block, hold that up in the air, underneath it place the bottom layer of the second block, and finally underneath this last the bottom layer of the last of our normal blocks.

Similarly we make the second representative group by taking the middle courses of our three blocks. The last is made by taking the three topmost layers. The three axes in our space before the transverse motion begins are blue, white, yellow, so we have light green tesseract faces, and after the motion begins sections transverse to the red light.

These three blocks represent the appearances as the tesseract group in its new position passes across our space. The cubes of contact in this case are those determined by the three axes in our space, namely, the white, the yellow, the blue. Hence they are light green.

It follows from this that light green is the interior cube of the first block of representative cubic faces.

Practice in the manipulations described, with a realisation in each case of the face or section which is in our space, is one of the best means of a thorough comprehension of the subject.

We have to learn how to get any part of these four-dimensional figures into space, so that we can look at them. We must first learn to swing a tesseract, and a group of tesseracts about in any way.

When these operations have been repeated and the method of arrangement of the set of blocks has become familiar, it is a good plan to rotate the axes of the normal cube 1 about a diagonal, and then repeat the whole series of turnings.

Thus, in the normal position, red goes up, white to the right, yellow away. Make white go up, yellow to the right, and red away. Learn the cube in this position by putting up the set of blocks of the normal cube, over and over again till it becomes as familiar to you as in the normal position. Then when this is learned, and the corresponding changes to the arrangements of the tesseract groups are made, another change should be made: let, in the normal cube, yellow go up, red to the right, and white away.

Learn the normal block of cubes in this new position by arranging them and re-arranging them till you know without thought where each one goes. Then carry out all the tesseract arrangements and turnings.

If you want to understand the subject, but do not see your way clearly, if it does not seem natural and easy to you, practise these turnings. Practise, first of all, the turning of a block of cubes round, so that you know it in every position as well as in the normal one. Practice by gradually putting up the set of cubes in their new arrangements. Then put up the tesseract blocks in their arrangements. This will give you a working conception of higher space, you will gain the feeling of it, whether you take up the mathematical treatment of it or not.

APPENDIX II
A LANGUAGE OF SPACE

THE mere naming the parts of the figures we consider involves a certain amount of time and attention. This time and attention leads to no result, for with each new figure the nomenclature applied is completely changed, every letter or symbol is used in a different significance.

Surely it must be possible in some way to utilise the labour thus at present wasted!

Why should we not make a language for space itself, so that every position we want to refer to would have its own name? Then every time we named a figure in order to demonstrate its properties we should be exercising ourselves in the vocabulary of place.

If we use a definite system of names, and always refer to the same space position by the same name, we create as it were a multisided of little hands, each prepared to grasp a special point, position, or element, and hold it for us in its proper relations.

We make, to use another analogy, a kind of mental paper, which has somewhat of the properties of a sensitive plate, in that it will register, without effort, complex, visual, or tactual impressions.

But of far more importance than the application of a space language to the plane and solid space is the facilitation it brings with it to the study of four-dimensional space.

I have delayed introducing a space language because all the systems I made turned out, after giving them a fair trial, to be intolerable. I have now come upon one which seems to present features of permanence, and I will here give an outline of it, so that it can be applied to the subject of the text, and in order that it may be subjected to criticism.

The principle on which the language is constructed is to sacrifice every other consideration for brevity.

It is indeed curious that we are able to talk and converse on every subject of thought except the fundamental one of space. The only way of speaking about the spatial configurations that underlie every subject of discursive thought is a co-ordinate system of numbers. This is so awkward and incommodious that it is never used. In thinking also, in realising shapes, we do not use it; we confine ourselves to a direct visualisation.

Now, the use of words corresponds to the storing up of our experience in a definite brain structure. A child, in the endless tactual, visual, mental manipulations it makes for itself, is best left to itself, but in the course of instruction the introduction of space names would make the teachers' work more cumulative, and the child's knowledge more social.

Their full use can only be appreciated if they are introduced early in the course of education; but in a minor degree any one can convince himself of their utility, especially in our immediate subject of handling four-dimensional shapes. The sum total of the results obtained in the preceding pages can be compendiously and accurately expressed in nine words of the Space Language.

In one of Plato's dialogues Socrates makes an experiment on a slave boy standing by. He makes certain perceptions of space awake in the mind of Meno's slave by directing his close attention to some simple facts of geometry.

By means of a few words and some simple forms we can repeat Plato's experiment on new ground.

Do we by directing our close attention on the facts of four dimensions awaken a latent faculty in ourselves? The old experiment of Plato's, it seems to me, has come down to us as novel as the day he incepted it, and its significance not better understood through all the discussion of which it has been the subjects.

Imagine a voiceless people living in a region where everything had a velvety surface, and who were thus deprived of all opportunity of experiencing what sound is. They could observe the slow pulsations of the air caused by their movements, and arguing from analogy, they would no doubt infer that more rapid vibrations were possible. From the theoretical side they could determine all about these more rapid vibrations. They merely differ, they would say, from slower ones, by the number that occur in a given time; there is a merely formal difference.

But suppose they were to take the trouble, go to the pains of producing these more rapid vibrations, then a totally new sensation would fall on their rudimentary ears. Probably at first they would only be dimly conscious of Sound, but even from the first they would become aware that a merely formal difference, a mere difference in point of number in this particular respect, made a great difference practically, as related to them. And to us the difference between three and four dimensions is merely formal, numerical. We can tell formally all about four dimensions, calculate the relations that would exist. But that the difference is merely formal does not prove that it is a futile and empty task, to present to ourselves as closely as we can the phenomena of four dimensions. In our formal knowledge of it, the whole question of its actual relation to us, as we are, is left in abeyance.

Possibly a new apprehension of nature may come to us through the practical, as distinguished from the mathematical and formal, study of four dimensions. As a child handles and examines the objects with which he comes in contact, so we can mentally handle and examine four-dimensional objects. The point to be determined is this. Do we find something cognate and natural to our faculties, or are we merely building up an artificial presentation of a scheme only formally possible, conceivable, but which has no real connection with any existing or possible experience?

This, it seems to me, is a question which can only be settled by actually trying. This practical attempt is the logical and direct continuation of the experiment devised in the "Meno."

Why do we think true? Why, by our process of thought, can we predict what will happen, and correctly conjecture the constitution of the things around us? This is a problem which every modern philosopher has considered, and of which Descartes, Leibnitz, Kant, to name a few, have given memorable solutions. Plato was the first to suggest it. And as he had the unique position of being the first devisor of the problem, so his solution is the most unique. Later philosophers have talked about consciousness and its laws, sensations, categories. But Plato never used such words. Consciousness apart from a conscious being meant nothing to him. His was always an objective search. He made man's intuitions the basis of a new kind of natural history.

In a few simple words Plato puts us in an attitude with regard to psychic phenomena—the mind—the ego—"what we are," which is analogous to the attitude scientific men of the present day have with regard to the phenomena of outward nature. Behind this first apprehension of ours of nature there is an infinite depth to be learned and known. Plato said that behind the phenomena of mind that Meno's slave boy exhibited, there was a vast, an infinite perspective. And his singularity, his originality, comes out most strongly marked in this, that the perspective, the complex phenomena beyond were, according to him, phenomena of personal experience. A footprint in the sand means a man to a being that has the conception of a man. But to a creature that has no such conception, it means a curious mark, somehow resulting from the concatenation of ordinary occurrences. Such a being would attempt merely to explain how causes known to him could so coincide as to produce such a result; he would not recognise its significance.

Plato introduced the conception which made a new kind of natural history possible. He said that Meno's slave boy thought true about things he had never learned, because his "soul" had experience. I know this will sound absurd to some people, and it flies straight in the face of the maxim, that explanation consists in showing how an effect depends on simple causes. But what a mistaken maxim that is! Can any single instance be shown of a simple cause? Take the behaviour of spheres, for instance; say those ivory spheres, billiard balls, for example. We can explain their behaviour by supposing they are homogenous elastic solids. We can give formulæ which will account for their behaviour in every variety. But are they homogenous elastic solids? No, certainly not. They are complex in physical and molecular structure, and atoms and ions beyond open an endless vista. Our simple explanation is false, false as it can be. The balls act as if they were homogenous elastic spheres. There is a statistical simplicity in the resultant of very complex conditions, which makes that artificial conception useful.

But its usefulness must not blind us to the fact that it is artificial. If we really look deep into nature, we find a much greater complexity than we at first suspect. And so behind this simple "I," this myself, is there not a parallel complexity? Plato's "soul" would be quite acceptable to a large class of thinkers, if by "soul" and the complexity he attributes to it, he meant the product of a long course of evolutionary change, whereby simple forms of living matter endowed with rudimentary sensation had gradually developed into fully conscious beings.

But Plato does not mean by "soul" a being of such a kind. His soul is a being whose faculties are clogged by its bodily environment, or at least hampered by the difficulty of directing its bodily frame—a being which is essentially higher than the account it gives of itself through its organs. At the same time Plato's soul is not incorporeal. It is a real being with a real experience. The question of whether Plato had the conception of non-spatial existence has been much discussed. The verdict is, I believe, that even his "ideas" were conceived by him as beings in space, or, as we should say, real. Plato's attitude is that of Science, inasmuch as he thinks of a world in Space. But, granting this, it cannot be denied that there is a fundamental divergence between Plato's conception and the evolutionary theory, and also an absolute divergence between his conception and the genetic account of the origin of the human faculties. The functions and capacities of Plato's "soul" are not derived by the interaction of the body and its environment.

Plato was engaged on a variety of problems, and his religious and ethical thoughts were so keen and fertile that the experimental investigation of his soul appears involved with many other motives. In one passage Plato will combine matter of thought of all kinds and from all sources, overlapping, interrunning. And in no case is he more involved and rich than in this question of the soul. In fact, I wish there were two words, one denoting that being, corporeal and real, but with higher faculties than we manifest in our bodily actions, which is to be taken as the subject of experimental investigation; and the other word denoting "soul" in the sense in which it is made the recipient and the promise of so much that men desire. It is the soul in the former sense that I wish to investigate, and in a limited sphere only. I wish to find out, in continuation of the experiment in the Meno, what the "soul" in us thinks about extension, experimenting on the grounds laid down by Plato. He made, to state this matter briefly, the hypothesis with regard to the thinking power of a being in us, a "soul." This soul is not accessible to observation by sight or touch, but it can be observed by its functions; it is the object of a new kind of natural history, the materials for constructing which lie in what it is natural for us to think. With Plato "thought" was a very wide-reaching term, but still I would claim in his general plan of procedure a place for the particular question of extension.

The problem comes to be, "What is it natural to us to think about matter *qua* extended?"

First of all, I find that the ordinary intuition of any simple object is extremely imperfect. Take a block of differently marked cubes, for instance, and become acquainted with them in their positions, You may think you know them quite well, but when you turn them round –rotate the block round a diagonal, for instance—you will find that you have lost track of the individuals in their new positions. You can mentally construct the block in its new position, by a rule, by taking the remembered sequences, but you don't know it intuitively. By observation of a block of cubes in various positions, and very expeditiously by a use of Space names applied to the cubes in their different presentations, it is possible to get an intuitive knowledge of the block of cubes, which is not disturbed by any displacement. Now, with regard to this intuition, we moderns would say that I have formed it by my tactual visual experiences (aided by hereditary predisposition). Plato would say that the soul had been stimulated to recognise an instance of shape which it knew. Plato would consider the operation of learning merely as a stimulus; we as completely accounting for the result. The latter is the more common-sense view. But, on the other hand, it presupposes the generation of experience from physical change. The world of sentient experience, according to the modern view, is closed and limited; only the physical world is ample and large and of ever-to-be-discovered complexity. Plato's world of soul, on the other hand, is at least as large and ample as the world of things.

Let us now try a crucial experiment. Can I form an intuition of a four-dimensional object? Such an object is not given in the physical range of my sense contacts. All I can do is present to myself the sequences of solids, which would mean the presentation to me under my conditions of a four-dimensional object. All I can do is to visualise and tactualise different series of solids which are alternative sets of sectional views of a four-dimensional shape.

If now, on presenting these sequences, I find a power in me of intuitively passing from one of these sets of sequences to another, of, being given one, intuitively constructing another, not using a rule, but directly apprehending it, then I have found a new fact about my soul, that it has a four-dimensional experience; I have observed it by a function it has.

I do not like to speak positively, for I might occasion a loss of time on the part of others, if, as may very well be, I am mistaken. But for my own part, I think there are indications of such an intuition; from the results of my experiments, I adopt the hypothesis that that which thinks in us has an ample experience, of which the intuitions we use in dealing with the world of real objects are a part; of which experience, the intuition of four-dimensional forms and motions is also a part. The process we are engaged in intellectually is the reading the obscure signals of our nerves into a world of reality, by means of intuitions derived from the inner experience.

The image I form is as follows. Imagine the captain of a modern battle-ship directing its course. He has his charts before him; he is in communication with his associates and subordinates; can convey his messages and commands to every part of the ship, and receive information from the conning-tower and the engine-room. Now suppose the captain immersed in the problem of the navigation of his ship over the ocean, to have so absorbed himself in the problem of the direction of his craft over the plane surface of the sea that he forgets himself. All that occupies his attention is the kind of movement that his ship makes. The operations by which that movement is produced have sunk below the threshold of his consciousness, his own actions, by which he pushes the buttons, gives the orders, are so familiar as to be automatic, his mind is on the motion of the ship as a whole. In such a case we can imagine that he identifies himself with his ship; all that enters his conscious thought is the direction of its movement over the plane surface of the ocean.

Such is the relation, as I imagine it, of the soul to the body. A relation which we can imagine as existing momentarily in the case of the captain is the normal one in the case of the soul with its craft. As the captain is capable of a kind of movement, an amplitude of motion, which does not enter into his thoughts with regard to the directing the ship over the plane surface of the ocean, so the soul is capable of a kind of movement, has an amplitude of motion, which is not used in its task of directing the body in the three-dimensional region in which the body's activity lies. If for any reason it became necessary for the captain to consider three-dimensional motions with regard to his ship. If for any reason it becomes necessary for the captain to consider three-dimensional motions with regard to his ship, it would not be difficult for him to gain the materials for thinking about such motions; all he has to do is to call his own intimate experiences into play. As far as the navigation of the ship, however, is concerned, he is not obliged to call on such experience. The ship as a whole simply moves on a surface. The problem of three-dimensional movement does not ordinarily concern its steering. And thus with regard to ourselves all the movements and activities which characterise our bodily organs are three-dimensional; we never need to consider the ampler movements. But we do more than use the movements of our body to effect our aims by direct means; we have now come to the pass when we act indirectly on nature, when we call processes into play which lie beyond the reach of any explanation we can give by the kind of thought which has been sufficient for the steering of our craft as a whole. When we come to the problem of what goes on in the minute, we find our habitual conceptions inadequate.

The captain in us must wake up to his own intimate nature, realise these functions of movement which are his own, and in virtue of his knowledge of them apprehend how to deal with the problem he has come to.

Think of the history of man. When has there been a time, in which his thoughts of form and movement were not exclusively of such varieties as were adapted for his bodily performance? We have never had a demand to conceive what our own most intimate powers are. But, just as little as by immersing himself in the steering of his ship over the plane surface of the ocean, a captain can loose the faculty of thinking about what he actually does, so little can the soul loose its own nature. It can be roused to an intuition that is not derived from the experience which the senses give. All that is necessary is to present some few of those appearances which, while inconsistent with three-dimensional matter, are yet consistent with our formal knowledge of four-dimensional matter, in order for the soul to wake up and not begin to learn, but of its own intimate feeling fill up the gaps in the presentiment, grasp the full orb of possibilities from the isolated points presented to it. In relation to this question of our perceptions, let me suggest another illustration, not taking it too seriously, only propounding it to exhibit the possibilities in a broad and general way.

In the heavens, amongst the multitude of stars, there are some which, when the telescope is directed on them, seem not to be single stars, but to be split up into two. Regarding these twin stars through a spectroscope, an astronomer sees in each a spectrum of bands of colour and black lines. Comparing these spectrums with one another, he finds that there is a slight relative shifting of the dark lines, and from that shifting he knows that the stars are rotating round one another, and can tell their relative velocity with regard to the earth. By means of his terrestrial physics he reads this signal of the skies. This shifting of lines, the mere slight variation of a black line in a spectrum, is very unlike that which the astronomer knows it means. But it is probably much more like what it means that the signals which the nerves deliver are like the phenomena of the outer world.

No picture of an object is conveyed through the nerves. No picture of motion, in the sense in which we postulate its existence, is conveyed through the nerves. The actual deliverances of which our consciousness takes account are probably identical for eye and ear, sight and touch.

If for a moment I take the whole earth together and regard it as a sentient being, I find that the problem of its apprehension is a very complex one, and involves a long series of personal and physical events. Similarly the problem of our apprehension is a very complex one. I only use this illustration to exhibit my meaning. It has this special merit, that, as the process of conscious apprehension takes place in our case in the minute, so with regard to this earth being, the corresponding process takes place in what is relatively to it very minute.

Now, Plato's view of a soul leads us to the hypothesis that that which we designate as an act of apprehension may be a very complex event, both physically and personally. He does not seek to explain what an intuition is; he makes it a basis from whence he sets out on a voyage of discovery. Knowledge means knowledge; he puts conscious being to account for conscious being. He makes an hypothesis of the kind that is so fertile in physical science—an hypothesis making no claim to finality, which marks out a vista of possible determination behind determination, like the hypothesis of space itself, the type of serviceable hypotheses.

And, above all, Plato's hypothesis is conducive to experiment. He gives the perspective in which real objects can be determined; and, in our present enquiry, we are making the simplest of all possible experiments—we are enquiring what it is natural to the soul to think of matter as extended.

Aristotle says we always use a "phantasm" in thinking, a phantasm of our corporeal senses a visualisation or a tactualisiation. But we can so modify that visualisation or tactualisation that it represents something not known by the senses. Do we by that representation wake up an intuition of the soul? Can we by the presentation of these hypothetical forms, that are the subject of our present discussion, wake ourselves up to higher intuitions? And can we explain the world around by a motion that we only know by our souls?

Apart from all speculation, however, it seems to me that the interest of these four-dimensional shapes and motions is sufficient reason for studying them, and that they are the way by which we can grow into a fuller apprehension of the world as a concrete whole.

SPACE NAMES.

If the words written on the squares drawn in fig. 1 are used as the names of the squares in the positions in which they are placed, is it evident that a combination of these names will denote a figure composed of the designated squares. It is found to be most convenient to take as the initial square that marked with an asterisk, so that the directions of progression are towards the observer and to the right. The directions of progression, however, are arbitrary, and can be chosen at will.

*

en	et	el
an	at	al
in	it	il

Fig. 1.

Thus *et*, *at*, *it an*, *al*, will denote a figure in the form of a cross composed of five squares.

Here, by means of the double sequence, *e*, *a*, *i*, and *n*, *t*, *l*, it is possible to name a limited collection of space elements.

The system can obviously be extended by using letter sequences of more numbers.

But, without introducing such a complexity, the principles of a space language can be exhibited, and a nomenclature obtained adequate to all the considerations of the preceding pages.

1. *Extension.*

Call the large squares in fig. 2 by the names written in them. It is evident that each can be divided as shown in fig. 1. Then the small square marked 1 will be "en" in "En," or "Enen." The square marked 2 will be "et" in "En" or "Enet," while the square marked 4 will be "em" in "Et" or "Eten." Thus the square 5 will be called "Ilil."

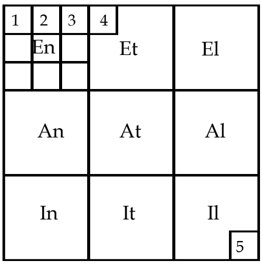


Fig. 2.

This principle of extension can be applied in any number of dimensions.

2. *Application to Three-Dimensional Space.*

To name a three-dimensional collocation of cubes take the upward direction first, secondly the direction towards the observer, thirdly the direction to his right hand.

These form a word in which the first latter gives the place of the cube upwards, the second letter its place towards the observer, the third letter its place to the right.

We have thus the following scheme, which represents the set of cubes of column 1, fig. 101, page 165.

We begin with the remote lowest cube at the left hand, where the asterisk is placed (this proves to be by far the most convenient origin to take for the normal system).

Thus "nen" is a "null" cube, "ten" a red cube on it, and "len" a "null" cube above "ten."

len	let	lel
lan	lat	lal
lin	lit	lil

ten	tet	tel
tan	tat	tal
tin	tit	til

*

nen	net	nel
nan	nat	nal
nin	nit	nil

By using a more extended sequence of consonants and vowels a larger set of cubes can be named.

To name a four-dimensional block of tesseracts it is simply necessary to prefix an "e," an "a," or an "i" to the cub names.

Thus the tesseract blocks schematically represented on page 165, fig. 101 are named as follows:—

elen	elet	elel	alen	alet	alel	ilen	ilet	ilel
elan	elat	elal	alan	alat	alal	ilan	ilat	ilal
elin	elit	elil	alin	alit	alil	ilin	ilit	ilil

eten	etet	etel	aten	atet	atel	iten	itet	itel
etan	etat	etal	atan	atat	atal	itan	itat	ital
etin	etit	etil	atin	atit	atil	itin	itit	itil

*

enen	enet	enel	anen	anet	anel	inen	inet	inel
enan	enat	enal	anan	anat	anal	inan	inat	inal
enin	enit	enil	anin	anit	anil	inin	init	inil
	1			2			3	

2. DERIVATION OF POINT, LINE, FACE, ETC. NAMES.

The principle of derivation can be shown as follows: Taking the square of squares

*

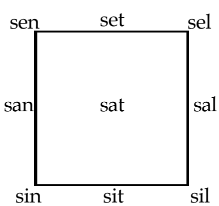
en	et	el
an	at	al
in	it	il

the number of squares in it can be enlarged and the whole kept the same size.

en	et	et	el
an	at	at	al
an	at	at	al
in	it	it	il

Compare fig. 79, p. 138, for instance, or the bottom layer of fig. 84.

Now use an initial "s" to denote the result of carrying this process on to a great extent, and we obtain the limit names, that is the point, line, area names for a square. "Sat" is the whole interior. The corners are "sen," "sel," "sin," "sil," while the lines are "san," "sal," "set," "sit."



I find that by the use of the initial "s" these names come to be practically entirely disconnected with the systematic names for the square from which they are derived. They are easy to learn, and when learned can be used readily with the axes running in any direction.

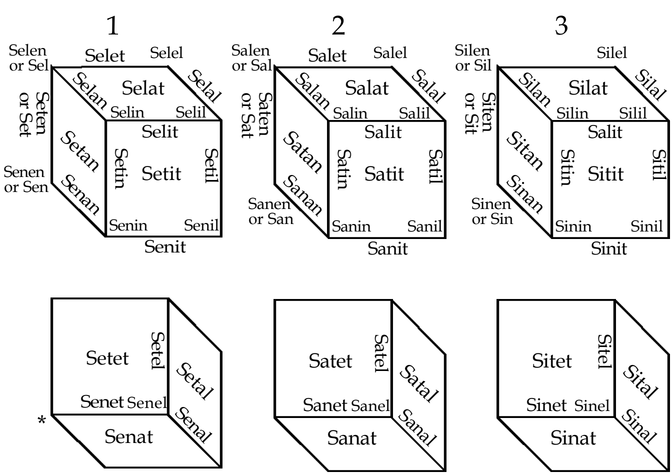
To derive the limit names for a four-dimensional rectangular figure, like the tesseract, is a simple extension of this process. These point, line, etc., names include those which apply to a cube, as will be evident on inspection of the first cube of the diagrams which follow.

All that is necessary is to place an "s" before each of the names given for a tesseract block. We then obtain apellatives which, like the colour names on page 174, fig. 103, apply to all the points, lines, faces, solids, and to the hypersolid of the tesseract. These names have the advantage over the colour marks that each point, line, etc., has its own individual name.

In the diagrams I give the names corresponding to the positions shown in the coloured plate or described on p. 174. By comparing cubes 1, 2, 3 with the first row of cubes in the coloured plate, the systematic names of each of the points, lines, faces, etc., can be determined. The asterisk shows the origin from which the names run.

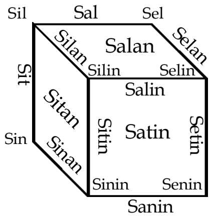
These point, line, face, etc. names should be used in connection with the corresponding colours. The names should call up coloured images of the parts named in their right connection.

It is found that a certain abbreviation adds vividness of distinction to these names. If the final "en" be dropped wherever it occurs the system is improved. Thus instead of "senen," "seten," "selen," it is preferable to abbreviate to "sen," "set," "sel," and also use "san," "sin" for "sanen," "sinen."



Interior Setat Interior Satat Interior Sitat

4



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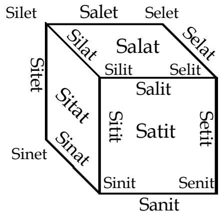

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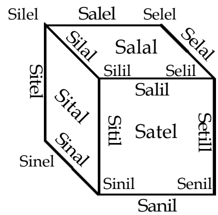
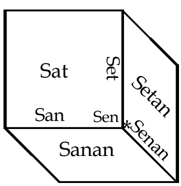
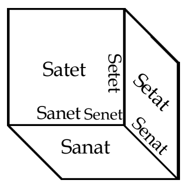
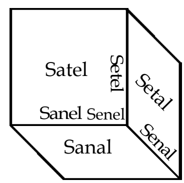




Interior Satan



Interior Satat



Interior Satal

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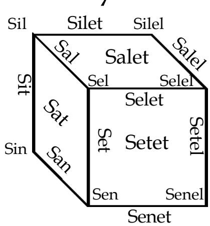

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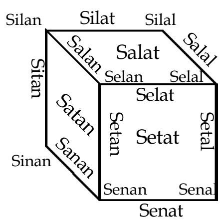

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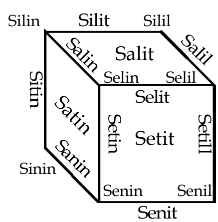
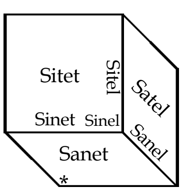
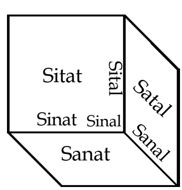
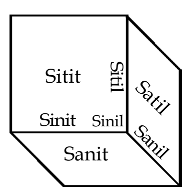




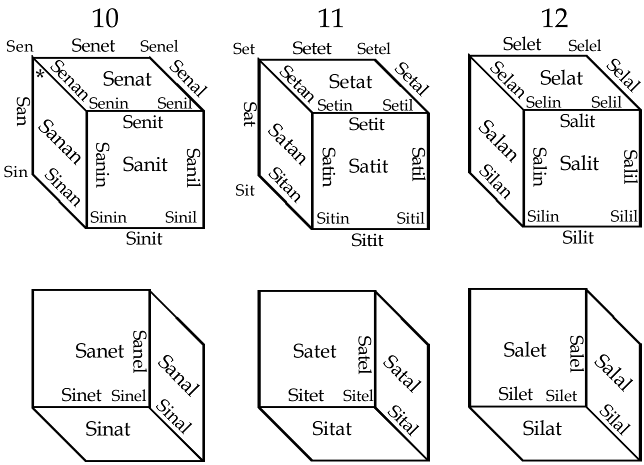
Interior Satet



Interior Satat



Interior Satit



Interior Sanat   Interior Satat   Interior Salat

We can now name any section. Take *e.g.* the line in the first cube from *senin* to *senel*, we should call the line running from *senin* to *senel*, *senin senat senel*, a line light yellow in colour with null points.

Here *senat* is the name for all of the line except its ends. Using "senat" in this way does not mean that the line is the whole of *senat*, but what there is of it is *senat*. It is a part of the *senat* region. Thus also the triangle, which has its three vertices in *senin*, *senel*, *selen*, is named thus:

Area: *setat*.
Sides: *setan*, *senat*, *setet*.
Vertices: *senin*, *senel*, *sel*.

The tetrahedron section of the tesseract can be thought of as a series of plane sections in the successive sections of the tesseract shown in fig. 114, p. 191. In b_0 the section is the one written above. In b_1 the section is made by a plane which cuts the three edges from *sanen* intermediate of their lengths and thus will be:

> Area: *satat*.
> Sides: *satan, sanat, satet*.
> Vertices: *sanan, sanet, sat*.

The sections in b_2, b_3 will be like the section in b_1 but smaller.

Finally in b_4 the section plane simply passes through the corner named *sin*.

Hence, putting these sections together in their right relation, from the face *setat*, surrounded by the lines and points mentioned above, there run:

> 3 faces: *satan*, *sanat*, *satet*
> 3 lines: *sanan*, *sanet*, *sat*

and these faces and lines run to the point *sin*. Thus the tetrahedron is completely named.

The octahedron section of the tesseract, which can be traced from fig. 72, p. 129 by extending the lines there drawn, is named:

Front triangle *selin*, *selat*, *selel*, *setal*, *senil*, *setit*, *selin* with area *setat*.

The sections between the front and rear triangle, of which one is shown in 1b, another in 2b, are thus named, points and lines, *salan*, *salat*, *salet*, *satet*, *satel*, *satal*, *sanal*, *sanat*, *sanit*, *satit*, *satin*, *satan*, *salan*.

The rear triangle found in 3b by producing lines is *sil*, *sitet*, *sinel*, *sinat*, *sinin*, *sitan*, *sil*.

The assemblage of sections constitute the solid body of the octahedron *satat* with triangular faces. The one from the line *selat* to the point *sil*, for instance, is named *selin*, *selat*, *selel*, *salet*, *salat*, *salan*, *sil*. The whole interior is *salat*.

Shapes can be easily cut out of cardboard which, when folded together, form not only the tetrahedron and the octahedron, but also samples of all the sections of the tesseract taken as it passes cornerwise through our space. To name and visualise with appropriate colours a series of these sections is an admirable exercise for obtaining familiarity with the subject.

Extension and Connection with Numbers

By extending the letter sequence it is of course possible to name a larger field. By using the limit names the corners of each square can be named.

Thus "en sen," "an sen," etc. will be the names of the points nearest the origin in "en" and in "an."

A field of points in which each one is indefinitely small is given by the names written below.

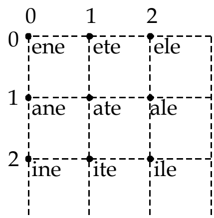
ensen	etsen	elsen
ansen	atsen	alsen
insen	itsen	ilsen

The squares are shown in dotted lines, the names denote the points. These points are not mathematical points, but really minute areas.

Instead of starting with a set of squares and naming them, we can start with a set of points.

By an easily remembered convention we can give names so such a region of points.

Let the space names with a final "e" added denote the mathematical points at the corner of each square nearest the origin. We have then



for the set of mathematical points indicated. This system is really completely independent of the area system and is connected with it merely for the purpose of facilitating the memory process. The word "ene" is pronounced like "eny," with just sufficient attention to the final vowel to distinguish it from the word "en."

Now, connecting the numbers 0, 1, 2 with the sequence *e*, *a*, *i*, and also with the sequence *n*, *t*, *l*, we have a set of points named as with numbers in a co-ordinate system. Thus "ene" is (0, 0); "ata" is (1, 1); "ite" is (2, 1). To pass to the area system the rule is that the name of the square is formed from the name of its point nearest to the origin by dropping the final *e*.

By using a notation analogous to the decimal system a larger field of points can be named. It remains to assign a letter sequence to the numbers from positive 0 to positive 9, and from negative 0 to negative 9, to obtain a system which can be used to denote both the usual co-ordinate system of mapping and a system of named squares. The names denoting the points all end with *e*. Those that denote squares end with a consonant.

There are many considerations which must be attended to in extending the sequences to be used, such as uniqueness in the meaning of the words formed, ease of pronunciation, avoidance of awkward combinations.

I drop "s" altogether from the consonant series and short "u" from the vowel series. It is convenient to have unsignificant letters at disposal. A double consonant like "st" for instance can be referred to without giving it a local significance by calling it "ust." I increase the number of vowels by considering a sound like "ra" to be a vowel, using, that is, the letter "r" as forming a compound vowel.

The series is as follows:—

CONSONANTS

	0	1	2	3	4	5	6	7	8	9
positive	n	t	l	p	f	sh	k	ch	nt	st
negative	s	d	th	h	v	m	g	j	nd	sp

VOWELS.

	0	1	2	3	4	5	6	7	8	9
positive	e	a	i	ee	æ	ai	ar	ra	ri	ree
negative	er	o	oo	io	œ	iu	or	ro	roo	rio

Pronunciation.—*e* as in men; *a* as in man; *i* as in in; *ee* as in between; *æ* as ay in may; ai as i in mine; *ar* as in art; *er* as ear in earth; *o* as in on; *oo* as oo in soon; *io* as in clarion; *œ* as oa in oat; *iu* pronounced like yew.

To name a point such as (23, 41) it is considered as (3, 1) on from (20, 40) and is called "ifeete." It is the initial point of the square *ifeet* of the area system.

The preceding amplification of a space language has been introduced merely for the sake of completeness. As has already been said nine words and their combinations, applied to a few simple models suffice for the purposes of our present enquiry.

TRANSCRIBER'S NOTE.

Do what thou wilt shall be the whole of the Law.

This e-text of *The Fourth Dimension* was initially key-entered from a poor quality facsimile of the first edition. Some sections in that printing (perpetrated by Kessinger Publications in Kila, Montana) were illegible and the colour plate was photocopied in black and white rendering it useless; these have been corrected based on other copies of the text; the colour plate has been reconstructed based on a colour photo-copy of the first edition and the descriptions in the text.

"A Language of Space" was not part of the first edition; it was issued as a pamphlet around the same time, and was bound in at the start of the Kessinger printing, separately paginated. In 1906 and subsequent printings of *The Fourth Dimension* it was included at the end as a second appendix.

All figures have been redrawn as vector art. In some of the views of the cube and tesseract in chapters XI-XIII, figures have been coloured instead of, or in addition to, placing the colour names or their initials on the figure. To make the nomenclature clear, wherever a particular colour is first used, the name is also printed. These colours are the same as those used in the colour plate; I have followed the original printing in rendering "null" as a light grey.

A few obvious typographical errors have been fixed, and some minor changes in punctuation made for the sake of clarity. Some errors in the mnemonics for the syllogism on p. 101 have been corrected.

In the first paragraph on p. 173 the second sentence originally ran "Then each of the eight lines of the cube …" and the calculation of the number of bounding squares of the tesseract was thus given as 12 + 8 = 20. This has been corrected as an obvious error (similarly in the last paragraph on p. 177).

Love is the law, love under will.

T.S.

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