

Chapter 3: Motion in Two Dimensions

Objectives: At the conclusion of this chapter, we'll be able to deal with 2-dimensional ballistics calculations, such as the flight of a projectile through an arc to a distant target. In addition, we'll learn about relative motion in two dimensions, and be able to solve simple (and not so simple) problems.

3.1 Two-Dimensional Motion

In the previous chapter, vectors were one-dimensional. In that case, it's safe to more or less ignore vectors altogether, and treat all the equations as scalar equations. Now, however, there's an additional direction, and the geometric power of vector algebra and calculus has to be brought on line.

The position of an object is described by two numbers, an x-coordinate and y-coordinate. When a baseball arcs through the sky after a powerful stroke, we could measure its position at each instant, then graph the result. We'd find that for each time, t , there would be both an x and y *coordinate* of the position of the ball in space. These may be cast as a position vector can be written as a vector, in terms of an x-component and a y-component. Here, however, we'll just note that since two coordinates are needed to describe the position, we'll need twice as many equations to describe the ballistics. (In the calculus-based physics course, more emphasis will be placed on the mathematical formalism.)

In addition, we will consider only those situations where the acceleration is constant. Calculus is required when this isn't the case, and it's not hard, but will not be discussed here.

For constant acceleration, the velocity and position of an object can be written as

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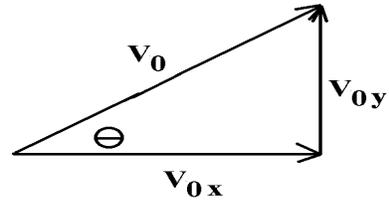
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These are almost exactly the same equations as in the case of one dimension, but there are twice as many.

Definitions

$a_x = \text{constant acceleration in the } x\text{-direction}$

$a_y =$ constant acceleration in the y-direction
 $v_x =$ velocity in the x-direction, at any time
 $v_y =$ velocity in the y-direction, at any time
 $v_{0x} =$ initial velocity in the x-direction
 $v_{0y} =$ initial velocity in the y-direction
 $x_0 =$ initial position, x
 $y_0 =$ initial position, y



The main difference in these equations, compared to the one-dimensional equations, lies in the velocity. The two components are related trigonometrically as two sides of a right triangle, with the hypotenuse equal to v_0 , the initial velocity:

$$\sqrt{\quad}$$

Figure 1: Velocity Vector

Of course, for an angle θ with respect to the x-direction, the usual trigonometric relations hold:

These equations are good for constant acceleration problems, and while they appear complicated, they are solved by the same methods, basically, as one-dimensional problems. There's just more algebra.

Notice the equations are the same as before, except with subscripts on the velocities to indicate velocity in the x and y directions, respectively. These velocities are obtained off a right triangle diagram.

Example 1. A shell is fired from a cannon at an angle of 60 degrees and initial speed of 200 meters per second. What are the initial speeds in the x-direction and y-direction?

Solution: Here, $\theta = 60^\circ$ and $v_0 = 200$ m/s. Plugging these numbers into the above equations results in

$$\frac{\sqrt{\quad}}{\quad} \quad \sqrt{\quad}$$

Example 2. A pitcher throws a ball horizontally at 40 m/s. His hand is 1.5 meters high when he releases the ball. How far does it travel before it hits the ground?

Solution: First, assemble all the constants. We have

Next, plug these values into the four ballistics equations:

All the information we need is inside these four equations.

(A) We want to know where it hits the ground. For this, we only need the last two equations. Setting $y=0$ will tell us how long it takes to go the vertical distance from $y=1.5$ meters down to $y=0$. Putting this into the x -equation will tell us how far horizontally it has gone.

$$\frac{\sqrt{\quad}}{\quad} \quad \sqrt{\quad}$$

Looks a little low maybe in the dirt by the time it crosses the plate.

GENERAL PROCEDURE

Step 1. Write down the ballistics equations.

Step 2. Plug in all initial conditions.

Step 3. Set each of the four equations equal to a known quantity, or a quantity desired to be known.

Step 4. Solve simultaneously for the unknowns.

3.2 Relative Velocity in Two Dimensions

Next we come to relative velocity in two dimensions. This is just the same equation as for one dimension:

It's a little more complicated for 2-dee, of course, but the idea is the same. The equation reads: "The velocity of A relative C is equal to the velocity of A relative B plus the velocity of B relative C. Notice that B, on the right hand side, appears "upstairs" and "downstairs". This is a major tip-off. An easy way to remember this is to isolate the subscripts, and ask whether

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makes sense as a division problem, which, of course, it does. If this is the case, you've named and arranged the terms correctly. The rest of the problem is simply a matter of plugging in the known quantities.

PROCEDURE

Step 1. Write down all three velocities with appropriate subscripts, e.g. A relative B means "the motion of A relative B."

Step 2. Assemble them so that the two terms with a letter appearing both upstairs and downstairs are on the same side.

Step 3. Solve for the unknown velocity.

Example 3. A man drives a car north at 80 mph while a policeman on motorcycle drives east at 60 mph. What is the velocity of the car relative the motorcycle? The speed of the car relative the motorcycle?

Solution: Let the velocity of the car relative the Earth be \mathbf{v}_{CE} , and let the velocity of the motorcycle relative the Earth be \mathbf{v}_{ME} . We are seeking \mathbf{v}_{CM} , the velocity of the car relative the motorcycle. Since \mathbf{M} appears upstairs and downstairs in the index, those terms shall be together on one side, and the remaining term on the other, just like the model. So

We have

Plugging them in, get

That s the relative velocity. The magnitude is the speed:

3.3 More Examples

3.3.1. Ballistics in Two Dimensions

Example 4. Boating. A boat travels 30 degrees South of West at 20 m/s. Find the components of its velocity.

Solution: As usual, North is positive y, East is positive x. Then:

Example 5. Cannon shot. A cannon fires a projectile at 300 m/s at an angle of thirty degrees above the horizontal from an initial height of zero. Find the maximum height.

Solution: Only the vertical component is important here, since nothing is asked about the horizontal motion. The initial velocity in the vertical direction is

The equation for the y-coordinate is given by

The equation for the velocity in the y-direction at any time is set equal to zero, since the y-velocity at maximum height is zero (though the horizontal velocity, of course, is not zero). This gives the time it takes to reach the maximum height.

The maximum height is then

Example 6. Jet Flight. Suppose a jet flying at constant velocity rises 500 meters while traveling one kilometer horizontally, all in 10 seconds. (A) Find the velocity. (B) Find the speed. (C) Find the angle of attack.

Solution: This is a straightforward application of the definitions.

(A)

(B)

(C)

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Example 8. Bombs Away ! A jet in horizontal flight at 800 kilometers per hour, one kilometer off the ground, drops a bomb. (A) How far from its original drop point does it hit the ground? (B) What's the velocity as it hits the ground?

Solution: This problem is relatively easy because the jet is flying horizontally, so there is hardly any trigonometry. First, let's put everything into MKS.

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Neglecting air resistance, there are no forces in the x-direction. So the ballistics equations read:

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Find out how long it takes for the bomb to hit the ground by setting $y=0$:

$$\sqrt{\quad}$$

Now we stick this time into the x-equation to find out how far from $x=0$ the bomb goes before hitting ground:

For part B, use the velocity equations.

The speed is given by the theorem of Pythagoras:

$$\sqrt{\quad} \quad \sqrt{\quad}$$

Example 9. Archery. An archer, in a bunker just below ground level, fires an arrow at 40 meters per second at a thirty degree angle with respect to the horizontal. The arrow is released at ground level. (A) What maximum height above the ground does it reach? (B) How far does it go before it hits the ground? (C) What's the velocity vector at that time? Neglect air friction.

Solution: First, let's do the trigonometry.

$$\frac{\sqrt{\quad}}{\quad} \quad \sqrt{\quad}$$

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Call the point of release $(x,y)=(0,0)$. The ballistics equations for constant acceleration read

$$\begin{aligned}
 & - \\
 & - \\
 & \sqrt{}
 \end{aligned}$$

(A) At the maximum height, .

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Plugging this into the y-equation gives:

(B) In this case, we can use symmetry to determine how far it went. Time of flight is obviously double the time it takes to get to the maximum height, so

$$\sqrt{} \quad \sqrt{}$$

(C) The velocity vector at the final instant before it hits the ground is found by plugging into the velocity equations:

$$\sqrt{}$$

This last result, at impact, is actually -20 m/s by symmetry, but round-off error caused the slight difference.

Example 11. Daredevil and the Canyon. A daredevil decides to jump the Grand Canyon on a motorcycle. The far side is 200 meters lower than the side he's on. (A) Neglecting air friction, at what speed must he make the jump, if the ramp will launch him at a 30 degree angle? (B) How high will he go?

Solution: Assume the canyon is 1000 meters across at this point. Choose the opposite side to have height position $y=0$. Neglecting air friction, the acceleration will be vertical only, so

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The y-component reads

—

(A) We'd like the guy to make it, which means

$$\frac{v \sin 30^\circ}{\sqrt{2}}$$

Substitute this time into the y-equation, and set the equation equal to zero, getting:

$$\left(\frac{\quad}{\sqrt{\quad}} \right) \quad \left(- \right) \frac{\quad}{\sqrt{\quad}}$$

(B) at the maximum height, so:

The answer is relative the lower cliff, which corresponds to $y=0$.

3.3.2. Relative Velocity in Two Dimensions

Example 12. Train relative Helicopter. A train is going north at 30 m/s while a helicopter is going southeast at 40 m/s. (A) What is the velocity vector of the train relative the helicopter? (B) What is the speed of the helicopter relative the train?

Solution: We have three quantities: $\mathbf{v}_{T/E}$, the velocity of the train relative the Earth, $\mathbf{v}_{H/E}$, the velocity of the helicopter relative the Earth, and $\mathbf{v}_{H/T}$, the velocity of the helicopter relative the train. Looking for a letter appearing both in the top and bottom of the subscript results in

Two of these are known:

Plug these in:

Solve this for .

$$\sqrt{\quad} \quad \sqrt{\quad} \quad \quad \quad \sqrt{\quad} \quad \sqrt{\quad}$$

The speed can be found with the theorem of Pythagoras:

$$\sqrt{\quad}$$

Example 13. Indian in a Canoe. A river runs east at 2 m/s. An indian in a canoe paddles north across the river at 3 m/s relative the river. (A) What is the velocity of the canoe relative the shore? (B) If the river is 200 meters across, how far to the east does the canoe go before reaching the opposite shore?

Solution: The three objects here are the canoe (C), the shore (S), and the river (R). The shore and the Earth are the same thing. Let east be positive x, north be positive y. First, we identify the three velocities:

Next, arrange the relative velocity equation so that the two terms with a letter upstairs in one and downstairs in the other are on the same side, and plug in the vectors already found.

So $(2,3)$ is the velocity of the canoe relative the shore. For part B, use the fact that $D=Vt$ for constant velocity: