RECURSIVE MULTISCALE ESTIMATION OF SPACE-TIME RANDOM FIELDS

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ABSTRACT

We recently developed a multiscale recursive estimation procedure for the estimation of large-scale dynamic systems. The procedure propagates multiscale models for the estimation errors more efficiently than the Kalman filter's propagation of the error covariances, with a resulting computational complexity of $\mathcal{O}(N)$ and $\mathcal{O}\left(N^{3/2}\right)$, where N is the number of variables estimated, for 1-D and 2-D dynamic systems, respectively. To further reduce the computational cost, we introduce in this paper a new class of reduced-order spatially-interpolated multiscale models, and demonstrate their use in remote sensing.

1. INTRODUCTION

The statistical estimation of two-dimensional dynamic systems is of great interest in many applications, such as ocean-height[2] or ocean-temperature (Fig. 1) mapping, where methods for assimilating data with dynamics have hitherto been limited. Even systems governed by simple dynamics such as diffusion present a challenge, due to the large state-dimensions (compared to one-dimensional problems) and to the time-varying nonstationary statistics (compared to most static problems).

In our past work [3, 4] we adapted a multiscale stochastic modeling and estimation methodology to the estimation of 1-D and small 2-D dynamic systems. Instead of propagating error covariances over time, like the Kalman filter, the multiscale recursive estimation algorithm propagates *models* for the estimation error, not covariances, and does so efficiently.

This paper addresses the challenges encountered in larger problems than studied earlier. Specifically, we consider a class of reduced-order models and interpolation methods for estimation of 2-D processes that further reduce the computational cost, and present preliminary dynamic ocean-temperature estimation results.

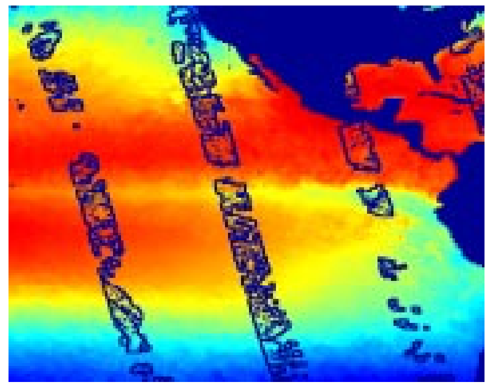

Fig. 1. Mean ocean temperature and superimposed measurements.

2. DYNAMIC ESTIMATION

Consider a dynamic system,

$$\boldsymbol{z}(t+1) = \boldsymbol{A}_d\, \boldsymbol{z}(t) + \boldsymbol{w}_d(t), \tag{1}$$

The form of least-squares time-recursive estimators (which includes the optimal Kalman filter) consists of two stages. One, the update stage,

$$\hat{\boldsymbol{z}}(t|t) = \hat{\boldsymbol{z}}(t|t-1) + \hat{\boldsymbol{\chi}}(t|t-1), \tag{2}$$

takes the measurements into account, where $\hat{\boldsymbol{\chi}}(t|t-1)$ is the estimate of the prediction error $\boldsymbol{\chi}(t|t-1)$. The measurement update step is essentially a static estimation problem, for which efficient methods are well-studied. Much more troubling is the prediction stage:

$$\hat{\boldsymbol{z}}(t+1|t) = \boldsymbol{A}_d\, \hat{\boldsymbol{z}}(t|t), \tag{3}$$

which accounts for time. The spatial mixing, introduced by all but the most trivial dynamics, destroys the particular statistical structure which one might wish to assume (via sparse matrices, preconditioning, or multiscale methods) to gain efficiency in the update stage.

This paper investigates ways of propagating multiscale models through the prediction step for fast estimation for large problems.

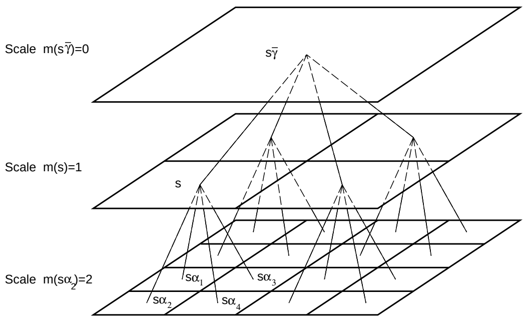

Fig. 2. An example multiscale quad-tree for modeling two-dimensional fields.

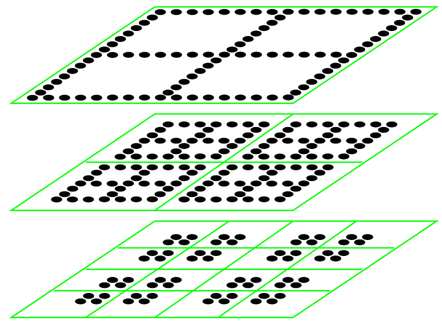

Fig. 3. The non-redundant linear functionals for modeling a 17×17 2-D MRF on a three-level tree.

3. MULTISCALE ESTIMATION

The multiscale recursive estimator in [4] achieves its computational efficiency by propagating a *model* for the estimation errors χ through time without explicitly calculating or storing the second-order statistics $\boldsymbol{P}_\chi(t|t)$. The error fields are modeled on tree structures, such as that of Fig. 2 on which each node has $q = 4$ children:

$$\boldsymbol{x}(s;\cdot|\cdot) = \boldsymbol{A}(s;\cdot|\cdot)\boldsymbol{x}(s\bar{\gamma};\cdot|\cdot) + \boldsymbol{B}(s;\cdot|\cdot)\boldsymbol{w}(s;\cdot|\cdot). \quad (4)$$

$\boldsymbol{x}(s;\cdot|\cdot)$ is the state at s, which indexes the nodes of the tree, and where $\boldsymbol{w}(s;\cdot|\cdot)$ is a white noise process uncorrelated with $\boldsymbol{x}(0;\cdot|\cdot)$. The $q+1$ subtrees connected to node s are conditionally decorrelated by the state $\boldsymbol{x}(s;\cdot|\cdot)$, which makes possible an efficient scale-recursive estimation algorithm on the tree.

We are interested in defining the multiscale state at s as a linear function of the process $\boldsymbol{\chi}$ of interest:

$$\boldsymbol{x}(s;\cdot|\cdot) = \boldsymbol{L}(s;\cdot|\cdot)\boldsymbol{\chi}(\cdot|\cdot). \quad (5)$$

Given (5) and $\boldsymbol{P}_\chi(\cdot|\cdot)$, the model parameters $\boldsymbol{A}(s;\cdot|\cdot)$ and $\boldsymbol{B}(s;\cdot|\cdot)$ are determined from the joint statistics between $\boldsymbol{x}(s;\cdot|\cdot)$ and $\boldsymbol{x}(s\bar{\gamma};\cdot|\cdot)$. The challenge arises in obtaining a model for the predicted error: to find $\boldsymbol{A}(s;t{+}1|t)$ and $\boldsymbol{B}(s;t{+}1|t))$ we need the joint statistics between

$$\begin{aligned} \boldsymbol{x}(s;t+1|t) &= \boldsymbol{L}(s;t+1|t)\boldsymbol{A}_d\boldsymbol{\chi}(t|t) \quad (6)\\ \boldsymbol{x}(s\bar{\gamma};t+1|t) &= \boldsymbol{L}(s\bar{\gamma};t+1|t)\boldsymbol{A}_d\boldsymbol{\chi}(t|t) \quad (7)\end{aligned}$$

which is much more involved than the joint statistics between $\boldsymbol{x}(s;t|t)$, $\boldsymbol{x}(s\bar{\gamma};t|t)$ due to the mixing effect of temporal dynamics $\boldsymbol{A}_d$. The specific additional joint statistics which are required depend on the choice of $\boldsymbol{L}(s)$ and on the dynamics. For nearest-neighbour dynamics, the non-redundant state assignment of 3 is effective, since the pixels adjacent to node s lie on the parent or child of s, greatly limiting the additional joint statistics involved.

4. REDUCED-ORDER 2-D MODELS

Given the degree of correlation between neighboring elements of typical 2-D random fields, a reduced-order state that models only a subsampled set of the boundary points, may adequately capture the correlation structure of the field and leader to faster estimation. However for non-redundant models (as in Fig. 3), where each field pixel appears in only one tree state, a subsampled state means that some elements of the field of interest appear nowhere on the tree. Although it is possible to ignore these elements for the purposes of static estimation, in performing the prediction step, estimates and error statistics will be required.

Let us denote the boundary points not represented in $\boldsymbol{x}(s)$ by $\boldsymbol{\zeta}(s)$. The simplest solution is to linearly predict the missing points from each state:

$$\boldsymbol{\zeta}(s) = \boldsymbol{M}(s)\boldsymbol{x}(s) + \boldsymbol{w}_\zeta(s\alpha_\zeta), \quad (8)$$

where $\boldsymbol{M}(s) = \boldsymbol{P}_{\zeta,x}(s)\boldsymbol{P}^{-1}(s)$. This approach is rather limited, in that it ignores the 2-D structure of the field, and neglects to use nearby pixels from other state elements in the interpolation process.

A more sophisticated method is to interpolate $\boldsymbol{\zeta}(s)$ based on all nearby state elements; that is,

$$\boldsymbol{\zeta}(s) = \boldsymbol{M}(s)\begin{bmatrix} \boldsymbol{x}(s) \\ \boldsymbol{x}(s\bar{\gamma}) \\ \boldsymbol{x}(s\alpha_1) \\ \vdots \\ \boldsymbol{x}(s\alpha_q) \end{bmatrix} + \boldsymbol{w}_\zeta(s\alpha_\zeta). \quad (9)$$

The advantage of such a 2-D interpolation method is that it leads to better estimates and realized error variances; on the other hand, one pays a computational penalty, especially during the prediction step, since a greater number of joint statistics need to be computed in order to compute the model parameters.

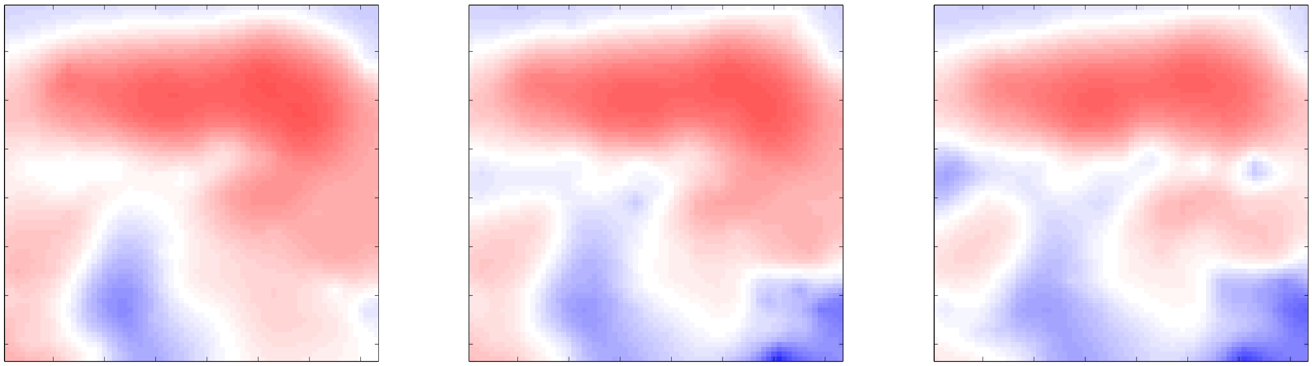

Fig. 4. Dynamic estimation of Pacific surface-temperature, three days apart.

5. RESULTS

We will model our dynamics as a diffusion process,

$$\frac{\partial z}{\partial t} = \nabla^2 z - \beta \cdot z + \gamma \cdot \omega, \tag{10}$$

where z is the temperature distribution; ω is white Gaussian noise, and β and γ are constants. Fig. 5 shows a 17×17 pinned cooling sheet example with four point measurements. We compare the realized statistics and fractional variance reduction

$$\mathrm{FVR} = \frac{\mathrm{Var(process)} - \mathrm{Var(updated\ error)}}{\mathrm{Var(process)}} \tag{11}$$

of two multiscale estimators — one using dense boundary points and the other subsampled boundaries with 2-D interpolation. Fig. 4 shows a small part of a more ambitious exercise — the dynamic estimation of the ocean surface temperature over six months.

The non-redundant linear functionals work well for modeling the estimation errors under a variety of more general conditions, e.g., when and the number and locations of measurements are time-varying. Since the problem is time-varying and does not attain steady-state, we compare the multiscale estimates with the optimal ones.

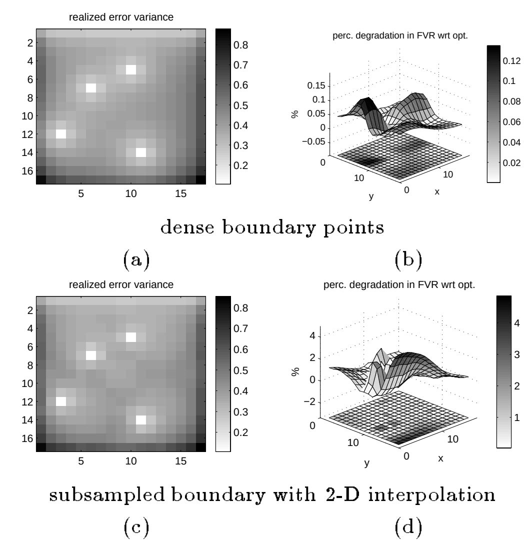

Fig. 5. Performance comparison of suboptimal multiscale steady-state estimators; (a), (c) Realized updated error variances. (b), (d) Degradation in multiscale FVR, within about 2% of the optimum.

6. REFERENCES

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