Multiscale Realization and Estimation for Space and Space-Time Problems

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Abstract — **Although there have been significant advances in realization techniques for a class of multiscale stochastic models introduced by Willsky, Benveniste, Chou, et al. [1], there remain a number of important open problems. We address two of them in this paper. The first is a new static realization method that is two orders of magnitude more efficient than methods previously available [2], but no less general. The second is the extension of this multiscale methodology to modeling dynamic systems.**

I. Introduction to Multiscale Models

In the multiscale framework of [1], zero-mean random fields are modeled on multiscale tree structures whose scale-to-scale relationship is expressed as:

$$x(s) = A(s)x(s\bar{\gamma}) + B(s)w(s), \tag{1}$$

where s is a node on the tree, $s\bar{\gamma}$ is its parent (see Fig. 1), and $w(s)$ is a white noise process uncorrelated with the root-node state $x(0)$, which is zero-mean with covariance $P_{x(0)}$. Conditioned on $x(s)$ the subprocesses of $x(\cdot)$ indexed on the disjoint subtrees extending from node s are mutually conditionally uncorrelated. This Markov property leads to an efficient scale-recursive smoothing algorithm which is a generalization of the Rauch-Tung-Striebel smoother [1]. Estimation error statistics and a multiscale model for the error process are produced with no additional computations.

II. Efficient Multiscale Realization

The multiscale realization problem is to find the multiscale model parameters $\{A(s), B(s)\}$ so that the finest scale process has statistics which exactly or approximately match some desired statistics. Once this problem is solved, the multiscale model provides an implicit specification of a conventional second-order prior model for a signal or an image mapped to the finest scale. Fast estimation, likelihood calculation, or sample path generation can then be accomplished.

The class of so-called internal models provides a convenient parameterization for the realization problem. An internal model is one for which each state $x(s)$ is a linear function of the size-N finest scale process, χ:

$$x(s) = L^T(s)\chi\,. \tag{2}$$

The mapping from the set of linear functionals $\{L(s)\}$ and the desired fine scale statistics $P_\chi = E\left[\chi\chi^T\right]$ to the parameters $\{A(s), B(s)\}$ is easy. The main problem, therefore, is finding the linear functionals.

Finding the linear functionals, $\{L(s)\}$ is challenging because often a reduced-order approximate realization of P_χ is desired. Therefore, one would like not only to find the linear functionals, but also to prioritize them so it is clear what to discard when a reduced order model is required. Several solutions to this problem, of finding the linear functionals in prioritized order, have been proposed. Our approach [2] has the advantage of being extremely fast—$\mathcal{O}(N^2)$—without compromising accuracy.

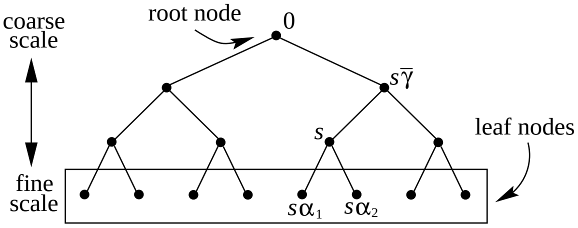

Figure 1: A dyadic tree.

III. Multiscale Estimation of Dynamic Systems

Our proposed multiscale dynamic estimation algorithm is best explained by drawing a parallel with the discrete-time Kalman filter. The strategy of our algorithm is to propagate the multiscale *models* for the updated and predicted estimation errors through time much like the Kalman filter propagates the error covariances, but in a computationally efficient manner and without ever computing or storing the full error covariance matrix. We assume a fixed set of linear functionals for modeling the estimation errors, but allow the model parameters to be modified over time. The update step incorporating new measurements at time t can be interpreted as *static* estimation of the predicted error. The challenge lies in the prediction step which requires untangling the spatial mixing due to the temporal dynamics.

Take a discretized 1-D diffusion process as an example [3]. We found the steady-state estimation errors to be close to a 1-D Markov random field, which can be modeled by keeping the end points of $\chi(s)$ as linear functionals at node s, where $\chi(s)$ is the portion of χ that descends from s. The resulting approximate multiscale estimator achieves $\mathcal{O}(N)$ computational complexity, verses Kalman filter's $\mathcal{O}(N^3)$, with minimal performance degradation.

References

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