Ramanujan And The Cubic Equation $3^3 + 4^3 + 5^3 = 6^3$

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"Politics is for the moment, but an equation is for eternity." – Albert Einstein[1]

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Contents:

I. Introduction

In this paper, we will tackle a well-known diophantine problem but focus on its solution in terms of *quadratic forms* including a very broad identity that covers several known ones. For example, how do we explain the fact that,

$$(3x^2+5xy-5y^2)^3 + (4x^2-4xy+6y^2)^3 + (5x^2-5xy-3y^2)^3 = (6x^2-4xy+4y^2)^3$$

which has as its first member the very intriguing equation $3^3 + 4^3 + 5^3 = 6^3$? Similarly, we have,

$$(x^2+9xy-y^2)^3 + (12x^2-4xy+2y^2)^3 = (9x^2-7xy-y^2)^3 + (10x^2+2y^2)^3$$

with $1^3 + 12^3 = 9^3 + 10^3$ as a first instance. The common sum of the second example, 1729, is associated with a famous anecdote[2] involving Ramanujan and Hardy and may be recognized by some as the *taxicab number* for $n = 2$, namely the smallest number representable in n ways as a sum of two positive cubes. These two identities, needless to say, were found by Ramanujan. No wonder he recognized 1729.

Why do these quadratic forms exist? It turns out they are *particular cases of a more general identity* and in this paper we will give this identity which seems to have been missed by Ramanujan. Before doing so, we can have the usual preliminaries. We will discuss parametric solutions to the diophantine equation,

$$a^3 + b^3 + c^3 = d^3 \qquad \text{(eq.1)}$$

or the equivalent forms,

$$a^3 + b^3 = c^3 + d^3 \qquad \text{(eq.2)}$$
$$a^3 + b^3 + c^3 + d^3 = 0 \qquad \text{(eq.3)}$$

This is a rather famous problem and understandably enough as it seems to be the logical next step after the Pythagorean equation $a^2 + b^2 = c^2$. Since integer solutions to the aforementioned equation are known as *Pythagorean triples,* by analogy perhaps it is permissible to call the general class of equations,

$$a_1^n + a_2^n + \ldots + a_{r-1}^n = a_r^n$$

as "*nth power r-tuples*", or "the sum of *r*-1 *n*th powers of *positive or negative* integers equal to an *n*th power", which itself is a particular instance of a still larger class of equations (equal sums of like powers) though we need not go into that here. The study of *n*th power *r*-tuples is notable in number theory as this was where Euler made a rare wrong conjecture. He conjectured that it took at least *n n*th powers to sum up to an *n*th power and we do now have examples for *n* up to 8, though skipping *n* = 6. While the conjecture was true for *n* = 3, it turned out to be false for *n* = 4, with eight counter-example *quartic quadruples* found so far by D. Bernstein, Noam Elkies, Roger Frye, and Allan MacLeod with the first (though not the smallest) by Elkies in 1988,

$$2682440^4 + 15365639^4 + 18796760^4 = 20615673^4$$

and the earlier one for *n* = 5 found by Lander and Parkin in 1966,

$$27^5 + 84^5 + 110^5 + 133^5 = 144^5$$

It's an open question whether there is a *sextic sextuple* (or *five* sixth powers equal to a sixth power) or even just a sextic 7-tuple, though hopefully the discovery of the latter may be just around the corner. In general, it is not known if there is an *n*th power *r*-tuple for *r* = *n* (or even *n*+1) for all *n* > 3. See Jean-Charles Meyrignac's "*Equal Sums Of Like Powers*" at http://euler.free.fr/ for more details.

For *odd* powers since we allow *positive or negative* solutions the form with the greatest symmetry is,

$$a_1^n + a_2^n + \ldots + a_r^n = 0$$

so what Lander and Parkin found was a *quintic quintuple*. There are only *three* such quintuples found, the second by Bob Scher and Ed Seidl in 1997,

$$5027^5 + 6237^5 + 14068^5 + (-220)^5 + (-14132)^5 = 0$$

and the third found by Jim Frye only last year, Sept 2004,

$$55^5 + 3183^5 + 28969^5 + 85282^5 + (-85359)^5 = 0$$

Thus what the author will refer to subsequently as *cubic quadruples* (a,b,c,d) will be solutions to the equation,

$$a_1^3 + a_2^3 + a_3^3 + a_4^3 = 0$$

unless indicated otherwise. By *Fermat's Last Theorem*, obviously there are no cubic triples in the integers.

II. Cubic Quadruples: $a^3 + b^3 + c^3 + d^3 = 0$

The smallest numerical solutions to this diophantine equation are (as eq.1),

$$3^3 + 4^3 + 5^3 = 6^3$$
$$1^3 + 6^3 + 8^3 = 9^3$$
$$7^3 + 14^3 + 17^3 = 20^3$$

and (as eq.2),

$$1^3 + 12^3 = 9^3 + 10^3$$
$$2^3 + 16^3 = 9^3 + 15^3$$
$$10^3 + 27^3 = 19^3 + 24^3$$

where, as was already pointed out, the sum $1^3 + 12^3 = 9^3 + 10^3 = 1729$ is taxicab(2). Taxicab(n) are known for n = 1-5, but n = 6 and above are unknown. Before we go to Ramanujan's quadratic identities, we will give various parametrizations. If we arbitrarily choose $a^3 + b^3 + c^3 = d^3$ (eq.1) as the form to solve, then Euler has *completely* solved this for positive or negative rational solutions and it is given by,

$$a = (1-(p-3q)(p^2+3q^2))r$$
$$b = ((p+3q)(p^2+3q^2)-1)r$$
$$c = ((p^2+3q^2)^2-(p+3q))r$$
$$d = ((p^2+3q^2)^2-(p-3q))r$$

where the variable r is a scaling factor reflecting the equation's homogeneity. (Appropriate sign changes will solve the other forms.) Note that the prevalence of the expression p^2+3q^2 is understandable as this solution takes advantage of the multiplicative properties of this algebraic form. This is also intimately connected to *Eisenstein integers*, numbers that involve the complex cube roots of unity.

It is still an open question however to find the complete characterization of eq.1 for all *positive integer* solutions, as pointed out by Dickson in the classic "*History of the Theory of Numbers, Vol 2*". For alternative solutions, Hardy and Wright in their "*An Introduction to the Theory of Numbers*" gave,

$$a^3(a^3 + b^3)^3 = a^3(a^3 – 2b^3)^3 + (a^3 + b^3)^3b^3 + (2a^3 – b^3)^3b^3$$
$$a^3(a^3 + 2b^3)^3 = a^3(a^3 – b^3)^3 + (a^3 – b^3)^3b^3 + (2a^3 + b^3)^3b^3$$

Solutions to eq.1 were also given by Francois Vieta as,

$$a = 2t^3-1,\ b = t(t^3-2),\ c = t^3+1,\ d = t(t^3+1)$$

and one found by this author,

$$n^3 + (3n^2+2n+1)^3 + (3n^3+3n^2+2n)^3 = (3n^3+3n^2+2n+1)^3$$

which automatically implies that *any* integer will appear as a non-trivial solution to cubic quadruples at least once. For eq.2, Ramanujan gave the very simple parametrization,

$$(a + x^2y)^3 + (bx + y)^3 = (ax + y)^3 + (b + x^2y)^3$$

where $a^2+ab+b^2 = 3xy^2$. For the more symmetric form (eq.3), we have one by Elkies,

$$a = -(s+r)t^2 +(s^2+2r^2)t-s^3+rs^2-2r^2s-r^3$$
$$b = t^3-(s+r)t^2 +(s^2+2r^2)t+rs^2-2r^2s+r^3$$
$$c = -t^3+(s+r)t^2 -(s^2+2r^2)t+2rs^2-r^2s+2r^3$$
$$d = (s-2r)t^2 +(r^2-s^2)t+s^3-rs^2+2r^2s-2r^3$$

As one can see, all of the parametrizations involve at least one cubic polynomial. What this paper will address will be the case of limiting it to quadratic expressions, in particular, the question asked by Ramanujan which he submitted to the Journal of the Indian Mathematical Society early last century. In Question 441 of the JIMS he asked,

"Show that $(3x^2+5xy-5y^2)^3 + (4x^2-4xy+6y^2)^3 + (5x^2-5xy-3y^2)^3 = (6x^2-4xy+4y^2)^3$ and *find other quadratic expressions satisfying similar relations*."

By expanding both sides, one can readily see that it is identically true. C. Hooley makes use of this identity to provide a lower bound for a certain estimate in his paper "*On the numbers that are representable as the sum of two cubes*" (1980). One proposed answer to Ramanujan's problem was given by S. Narayanan. Either he knew about or independently rediscovered Vieta's solution but by defining,

$$m = 2t^3-1,\ n = t(t^3-2),\ p = t^3+1,\ q = t(t^3+1)$$

(compare to Vieta's), then we get,

$$(mx^2-mxy-py^2)^3 + (nx^2-nxy+qy^2)^3 + (px^2+mxy-my^2)^3 = (qx^2-nxy+ny^2)^3$$

Interestingly enough, Narayanan's solution in turn was rediscovered by Marc Chamberland in the context of looking for solutions for eq.1 and not in order to answer Question 441. His parametrization was, given,

$$a^3 + b^3 + c^3 = d^3,\ \text{and}\ c(c^2 - a^2) = b(d^2 - b^2)$$

then,

$$(cx^2-cxy-ay^2)^3 + (bx^2-bxy+dy^2)^3 + (ax^2+cxy-cy^2)^3 = (dx^2-bxy+by^2)^3$$

(Though in his paper it was set as $y = 1$). By comparing Narayanan's and Chamberland's expressions, one can see that (m,n,p,q) = (c,b,a,d).

However, we will give a *much* more general answer to Ramanujan's question. What we would like to know is that given *any* known cubic quadruple (a,b,c,d), can we use this to find quadratic forms q_i such that,

$$q_1^3 + q_2^3 + q_3^3 + q_4^3 = 0?$$

The answer is *yes*. To prove this assertion, we give the beautiful identity,

Identity 1 (Id.1)

$(ax^2 + v_1xy + bwy^2)^3 + (bx^2 - v_1xy + awy^2)^3 + (cx^2 + v_2xy + dwy^2)^3 + (dx^2 - v_2xy + cwy^2)^3$

$= (a^3 + b^3 + c^3 + d^3)(x^2 + wy^2)^3$

where $v_1 = -(c^2-d^2)$, $v_2 = (a^2-b^2)$, $w = (a+b)(c+d)$.

If the right hand side of the equation is zero, then it automatically gives us a cubic quadruple in terms of quadratic forms. *Thus, we have reduced the problem to simply finding solutions to* $a^3 + b^3 + c^3 + d^3 = 0$*!* Since the complete positive and negative rational solution to this has already has been given by Euler, then for *every* cubic quadruple (a,b,c,d) there corresponds a quadratic form parametrization.

(For those who wish to verify this identity and the three others that will be given later, two of which rather complicated, there is a Microsoft Word file, www.geocities.com/titus_piezas/RamCube.doc so one can easily cut and paste them onto the computer algebra system of your choice, or the free one at www.quickmath.com.)

We can note three things. **First**, the *discriminant* f_i of a pair of quadratic forms, namely (q_1, q_2) and (q_3, q_4), are identical and given by,

$f_1 = v_1^2-4abw$, and $f_2 = v_2^2-4cdw$

Second, by permuting the values (a,b,c,d), there are 4!=24 possibilities such that $a^3 + b^3 + c^3 + d^3 = 0$. However, generically there are only three significant and distinct combinations: {(a,b),(c,d)}, {(a,c),(b,d)}, {(a,d),(b,c)}.

Example 1:

Let a = 3, b = 4, c = 5, d = -6, discriminants = 457, -791.

$$(3x^2+11xy-28y^2)^3 + (4x^2-11xy-21y^2)^3 + (5x^2-7xy+42y^2)^3 + (-6x^2+7xy-35y^2)^3 = 0$$

Example 2: (Ramanujan's)

Let a = 3, b = 5, c = 4, d = -6, discriminants = 85, -16*5.

$$(3x^2+5xy-5y^2)^3 + (5x^2-5xy-3y^2)^3 + (4x^2-4xy+6y^2)^3 + (-6x^2+4xy-4y^2)^3 = 0$$

Example 3:

Let a = 3, b = -6, c = 4, d = 5, discriminants = -9*23, 321.

$$(3x^2+3xy+18y^2)^3 + (-6x^2-3xy-9y^2)^3 + (4x^2-9xy-15y^2)^3 + (5x^2+9xy-12y^2)^3 = 0$$

(Note: I find it rather interesting that -23, 321 just so happen to have class number 3, as negative and positive square-free fundamental discriminants. See Mathworld's list of class numbers.)

So Ramanujan's identity was just one in a triplet with middle terms of opposite parity. **Finally**, there is no reason to stop with quadruples. If we let,

$$a^3 + b^3 + c^3 + d^3 = k$$

where k is either 0, 1, or more cubes (not necessarily positive), then we can go beyond Ramanujan's requirement and find quadratic expressions for r cubes equal to zero. For example, take the case of four cubes equal to a cube, the few smallest being,

$$1^3 + 1^3 + 5^3 + 6^3 = 7^3$$
$$3^3 + 3^3 + 7^3 + 11^3 = 12^3$$
$$1^3 + 5^3 + 7^3 + 12^3 = 13^3$$

However, there is the very nice instance $11^3 + 12^3 + 13^3 + 14^3 = 20^3$ so using our first identity (Id.1) we get,

$$(11x^2 + 27xy + 7452y^2)^3 + (12x^2 -27xy + 6831y^2)^3 + (13x^2 -23xy + 8694y^2)^3 + (14x^2 +23 xy + 8073y^2)^3 = 20^3(x^2 + 621y^2)^3$$

which is just one possible form, not to mention that we could just as well have used the smaller examples. One thing which we just pointed was that we took advantage of quadratic forms whose middle terms, given by v_1 and v_2, were of opposite parity. However, there *are* parametrizations where this is not the case. One such example is a version given by Bruce Reznick,

$$(3x^2-11xy+3y^2)^3 + (4x^2-4xy+6y^2)^3 + (5x^2-5xy-3y^2)^3 = (6x^2-8xy+6y^2)^3$$

So it seems that our identity, general though it is, is in fact still just a special case of something *even more* general and that is what we will find in the next section.

III. Derivation: The Hard Way

The identity (Id.4) we have given so far is a special case. In this section, we focus on the general case that does not depend on the opposite parity of middle terms. Let q_i be quadratic forms. Then what we are looking for is a solution to,

$$q_1^3 + q_2^3 + q_3^3 + q_4^3 = 0 \quad \text{(eq.4)}$$

or explicitly,

$$(ax^2 + v_1xy + v_2y^2)^3 + (bx^2 + v_3xy + v_4y^2)^3 + (cx^2 + v_5xy + v_6y^2)^3 + (dx^2 + v_7xy + v_8y^2)^3 = 0$$

where the v_i are unknowns. One way to solve this is to expand and collect terms in powers of x and y. We then have the system of equations, call it S_1,

$$p_0 = (a^3 + b^3 + c^3 + d^3)\ x^6$$
$$p_1 = (a^2v_1 + b^2v_3 + c^2v_5 + d^2v_7)\ x^5y$$
$$p_2 = (av_1^2 + a^2v_2 + bv_3^2 + b^2v_4 + cv_5^2 + c^2v_6 + dv_7^2 + d^2v_8)\ x^4y^2$$
$$p_3 = (v_1^3 + 6av_1v_2 + v_3^3 + 6bv_3v_4 + v_5^3 + 6cv_5v_6 + v_7^3 + 6dv_7v_8)\ x^3y^3$$
$$p_4 = (v_1^2v_2 + av_2^2 + v_3^2v_4 + bv_4^2 + v_5^2v_6 + cv_6^2 + v_7^2v_8 + dv_8^2)\ x^2y^4$$
$$p_5 = (v_1v_2^2 + v_1v_2^2 + v_1v_2^2 + v_1v_2^2)\ xy^5$$
$$p_6 = (v_2^3 + v_4^3 + v_6^3 + v_8^3)\ y^6$$

Note that this is a rather *natural* system and we wish to set all the $p_i = 0$. If $a^3 + b^3 + c^3 + d^3 = 0$, that takes care of p_0. Thus, we have six equations p_i and eight unknowns v_i. So it seems our system is under-determined. However, we can use the "*fait accompli*" principle (literally, "accomplished fact"): since Ramanujan's identity already exists, then this system must not be impossible to solve. What one can do is to use two known values from the identity, and resolve this system of six equations in six unknowns to its resolvent using a computer algebra system. To the surprise of the author, this equation of a humongous degree had *three* non-trivial linear factors (plus one repetition), one being Ramanujan's expected value. The next thing to ask would be what would happen if two *arbitrary* values were used. It again had three linear factors.

After more experimentation, the conclusion was that two of our v_i with odd subscript (in this paper, v_1 and v_3) must be *independent variables*, and we just have six equations in six unknowns. We can then resolve this system into a single equation. But was the fact that it had linear factors a peculiarity of the cubic quadruple (3,4,5,-6)? The incredible thing was that later it was realized that for *any* cubic quadruple (a,b,c,d), *it always had a linear factor*. Furthermore, it may have as many as three distinct linear factors, yielding quadratic forms with different discriminants.

To find these factors was no easy task. Since the resolvent was of a very high degree, it was a hard enough time dealing with numerical values, and much more if symbolic ones were used. What was needed was a shortcut. Perhaps one can use the *discriminant* of the quadratic forms (or in general, of the parametric polynomial) to generate additional information. Since the author observed that discriminants were always the same for a pair of q_i, then we can have two more equations,

$$v_1^2-4av_2 = v_3^2-4bv_4, \qquad v_5^2-4cv_6 = v_7^2-4dv_8$$

and we can use these to immediately solve for v_4 and v_6. Substituting into p_1, p_2, we can linearly solve for v_7 and v_8. Substituting all the known v_i into p_3, we get a cubic in v_5 (call this polynomial e_3) with v_2 and v_5 as the remaining unknowns. It was the experience of the author that it was at this point that when using numerical examples this cubic would factor, with one factor linear in v_1, v_3, v_5 (and the other unknown v_2 conveniently isolated in the quadratic factor). If indeed this will always happen for cubic quadruples, then we can eliminate one variable, say d, between $a^3 + b^3 + c^3 + d^3 = 0$ and e_3 and see if it factors.

That is easier say than done as factoring polynomials of relatively high degree and with so many symbolic variables (a,b,c,d,v_1,v_2,v_3,v_5) cannot be done quickly. So, we can use a trick that Euler used to find the complete parametrization of cubic quadruples. It is simply to do a linear substitution that reduces one variable to a lesser degree, namely,

$$a = p+s, \quad b = p-s, \quad c = q-r, \quad d = q+r$$

Substituting this into $a^3 + b^3 + c^3 + d^3 = 0$, we get,

$$p^3 + q^3 + 3qr^2 + 3ps^2 = 0 \qquad \text{(eq.5)}$$

with r and s only as second powers. This slight reduction is all we need and doing the same to the polynomial e_3, we eliminate the variable s between the two using resultants. The resulting polynomial does factor, one given as a quadratic in v_5. Solving this equation we find that its discriminant is given by,

$$-3p^5q^2r^2(p^3 + q^3 + 3qr^2)(v_1-v_3)^2$$

Using eq.5, we know that this is really,

$$-3p^5q^2r^2(-3ps^2)(v_1-v_3)^2$$

hence is a perfect square and is a linear factor in disguise. Simplifying, we get v_5 in terms of the independent variables v_1 and v_3 as,

$$v_5\ (6pqr) = (p^3+q^3+3q^2r+3p^2s)v_1 + (p^3+q^3+3q^2r-3p^2s)v_3$$

and we have our first linear factor! Using this value for v_5 (as well as the other substitutions) for any of the remaining polynomials p_4, p_5, or p_6, one can solve for the last unknown v_2 which turns out to be also linearly solvable, though only after some algebraic manipulation, and finally we have all our unknowns! We now give our second identity,

Identity 2 (Id.2)

If $a^3 + b^3 + c^3 + d^3 = 0$, then,

$$(ax^2 + v_1xy + v_2y^2)^3 + (bx^2 + v_3xy + v_4y^2)^3 + (cx^2 + v_5xy + v_6y^2)^3 + (dx^2 + v_7xy + v_8y^2)^3 = 0$$

with the solution (using the simplifying substitution a = p+s, b = p-s, c = q-r, d = q+r),

v_1, v_3 (free variables)

$$v_5\ (6pqr) = (p^3+q^3+3q^2r+3p^2s)v_1 + (p^3+q^3+3q^2r-3p^2s)v_3$$

$$v_7\ (6pqr) = -(p^3+q^3-3q^2r+3p^2s)v_1 - (p^3+q^3-3q^2r-3p^2s)v_3$$

and for even subscripts, first define the variables f_1 and f_2 as,

$$f_1 = -(4p^3+q^3)(pv_1-sv_1-pv_3-sv_3)^2/(12p^2qr^2)$$

$$f_2 = -(p^3+4q^3)(pv_1-sv_1-pv_3-sv_3)^2/(12pq^2r^2)$$

where we can express all the v_i in terms of these two as,

$$v_1^2-4av_2 = f_1, \quad v_3^2-4bv_4 = f_1, \quad v_5^2-4cv_6 = f_2, \quad v_7^2-4dv_8 = f_2$$

and one can easily solve for the v_i with even subscripts as they are only linear (and since the v_i with odd subscripts are already given). The new variables (p,q,r,s) can be expressed in terms of the originals as,

$$p = (a+b)/2, \quad s = (a-b)/2, \quad q = (c+d)/2 \quad r = (-c+d)/2$$

The expressions f_i in fact are the discriminants of the quadratic forms q_i. Note that if these are made *square-free* then they do not depend on v_1 and v_3 at all and in terms of the original variables, these are simply,

$f_1 = -(4(a+b)^3+(c+d)^3)/(3(c+d))$, $f_2 = -((a+b)^3+4(c+d)^3)/(3(a+b))$

(The explicit forms of the f_i were reverse-engineered from the derived expressions of the v_i. As the explicit forms of the v_i with even subscripts were unwieldy, it was more aesthetic to express them in terms of the f_i.)

If we substitute the formulas for the v_i into eq.4, what we get is the factorization,

$$q_1^3 + q_2^3 + q_3^3 + q_4^3 = (a^3 + b^3 + c^3 + d^3)\, P(x,y)$$

or equivalently, $q_1^3 + q_2^3 + q_3^3 + q_4^3 = (p^3 + q^3 + 3qr^2 + 3ps^2)\, P(x,y)$ depending on whether the original or substitute variables were used, though it would need a robust computer algebra system to do this symbolically. (It takes about 55 seconds on a Pentium 4.) P(x,y) is a complicated polynomial, not a perfect cube power, whose role is negligible if,

$$a^3 + b^3 + c^3 + d^3 = p^3 + q^3 + 3qr^2 + 3ps^2 = 0$$

since then the right hand side of eq.4 vanishes anyway. Later, we shall see that there may be a still *more* general identity where the right hand side need not vanish.

Example:

Let a = 3, b = 5, c = 4, d = -6, so we need to solve the equation,

$$(3x^2 + v_1xy + v_2y^2)^3 + (5x^2 + v_3xy + v_4y^2)^3 + (4x^2 + v_5xy + v_6y^2)^3 + (-6x^2 + v_7xy + v_8y^2)^3 = 0$$

From the substitution, we find that p = 4, s = -1, q = -1, r = -5. If we let $\mathbf{v_1 = 80m}$ and $\mathbf{v_3 = 80n}$, (to clear denominators) using the formulas we find all six unknown v_i as,

$v_5 = 64n$, $v_7 = 4(-5m-21n)$

and,

$v_2 = 5(-35m^2+170mn-51n^2)$, $v_4 = -425m^2+510mn+167n^2$,
$v_6 = 4(125m^2-150mn+109n^2)$, $v_8 = 2(-175m^2+130mn-207n^2)$

This is the "mother identity" that contains both Ramanujan's and Reznick's versions. By letting, $n = -m$, and after scaling down variables, we find this reduces to Ramanujan's,

$$(3x^2+5xy-5y^2)^3 + (5x^2-5xy-3y^2)^3 + (4x^2-4xy+6y^2)^3 = (6x^2-4xy+4y^2)^3$$

But if we let m = 11/80 and n = 1/16, we have Reznick's,

$$(3x^2-11xy+3y^2)^3 + (5x^2-5xy-3y^2)^3 + (4x^2-4xy+6y^2)^3 = (6x^2-8xy+6y^2)^3$$

so a judicious choice of small rational {m,n} can result in integer coefficients. If we set {a,b,c,d} = {3,4,5,-6} and {3,-6,4,5}, we can have two other "mother identities" with different discriminants. By making $v_3 = -v_1$, these will reduce, after scaling, to examples 1 and 3 given in the previous section.

IV. Derivation: The Easy Way

The alternative way to solve this is a much easier form which takes advantage of certain symmetries and the opposite parity of middle terms. Instead of eight v_i, we use only the three variables, v_1, v_2, w. Let,

$$(ax^2 + v_1xy + bwy^2)^3 + (bx^2 - v_1xy + awy^2)^3 + (cx^2 + v_2xy + dwy^2)^3 + (dx^2 - v_2xy + cwy^2)^3 = 0$$

$$p_0 = (a^3 + b^3 + c^3 + d^3)\ x^6$$
$$p_1 = (a^2v_1 - b^2v_1 + c^2v_2 - d^2v_2)\ x^5y$$
$$p_2 = (a^2bw + ab^2w + c^2dw + cd^2w + av_1^2 + bv_1^2 + cv_2^2 + dv_2^2)\ x^4y^2$$
$$p_2 = w(a^2bw + ab^2w + c^2dw + cd^2w + av_1^2 + bv_1^2 + cv_2^2 + dv_2^2)\ x^2y^4$$
$$p_1 = w^2(a^2v_1 - b^2v_1 + c^2v_2 - d^2v_2)\ xy^5$$
$$p_0 = w^3(a^3 + b^3 + c^3 + d^3)\ y^6$$

As one can see, the expressions are symmetric and if as before $a^3 + b^3 + c^3 + d^3 = 0$, then we really only have the two equations p_1 and p_2. Solving for the v_i and after simplification, we end up with,

$$v_1^2 = w(c-d)^2(c+d)/(a+b)$$

The function of the third variable *w* is then apparent as it enables us to make v_1 a rational square. By setting $w = (a+b)(c+d)$, then $v_1 = +/-(c^2-d^2)$. Using the negative case $v_1 = -(c^2-d^2)$, we find that $v_2 = (a^2-b^2)$, and so we get our identity (Id.1) given earlier,

$$(ax^2 + v_1xy + bwy^2)^3 + (bx^2 - v_1xy + awy^2)^3 + (cx^2 + v_2xy + dwy^2)^3 + (dx^2 - v_2xy + cwy^2)^3$$

$$= (a^3 + b^3 + c^3 + d^3)(x^2 + wy^2)^3$$

It seems odd Ramanujan didn't find this identity, considering his facility with algebraic manipulation, and that he was doing research in this area, as well as that it was relatively easy (in hindsight) to find. Since he usually submitted questions to JIMS the answers of which he already knew, it may be the case that he knew of this identity, or a version of it, and had it written somewhere. However, this is only a special case of the more general Id.2. (More accurately, a special case of *another* version of Id.2, where the v_i with even subscripts were in their derived form and more complicated than the ones given here. In that version, the polynomial P(x,y) when $v_3 = -v_1$ factors as a perfect cube and the identity, after scaling, reduces to Id.1.)

V. Cubic r-Tuples: $a_1^3 + a_2^3 + \ldots + a_r^3 = 0$

We now have the two identities Id.1 and Id.2, and we pointed out that while the former can be applied to cubic *r*-tuples, the latter is limited to quadruples. However, the former in turn is limited to forms with middle terms of opposite parity. We may want to know if we can find a general identity applicable to r-tuples $r > 4$ that does not take advantage of parity. In other words, we are looking for an *even* more general version of eq.4. To recall, eq.4 was,

$$q_1^3 + q_2^3 + q_3^3 + q_4^3 = 0$$

where the q_i are quadratic forms. But now we are after,

$$q_1^3 + q_2^3 + q_3^3 + q_4^3 = (a^3 + b^3 + c^3 + d^3)\, P(x, y)^3 \qquad \text{(eq.6)}$$

If we limit ourselves to quadruples, P(x,y) does not matter as the right hand side vanishes. However, as we mentioned earlier, if we do not wish for this side to vanish, if $a^3 + b^3 + c^3 + d^3 = k$ where k is one or more cubes (whether positive or negative), then we need to define P(x,y). In other words, we need to solve the general equation,

$$(ax^2 + v_1xy + v_2y^2)^3 + (bx^2 + v_3xy + v_4y^2)^3 + (cx^2 + v_5xy + v_6y^2)^3 + (dx^2 + v_7xy + v_8y^2)^3 = (a^3 + b^3 + c^3 + d^3)(x^2 + v_9xy + v_{10}y^2)^3 \qquad \text{(eq.7)}$$

If we expand this and collect powers similar to what we did earlier, then we now have *ten* unknowns but only six equations. However, if we use the same assumptions earlier, that two of the v_i are independent variables and pairs of q_i share the same discriminants (on the left hand side) hence giving two more equations, then we really have a system of eight equations in eight unknowns, call it S_2, and should be resolvable to a single equation. *Does the resolvent of S_2 generically have a non-trivial linear factor*? Heuristic evidence indicates it may be the case. It is hoped an interested reader can tackle this question as there may be more elegant methods to solve systems of equations.

For certain (a,b,c,d) not a cubic quadruple, we *did* manage to solve eq.7. These involve identities of form,

$$(t-p)^3 + (t+p)^3 + (t-q)^3 + (t+q)^3 = k$$

or equivalently, $2t(3p^2 + 3q^2 + 2t^2) = k$, which one can observe satisfies a+b = c+d. A numerical example was one given earlier, the nice equation $11^3 + 14^3 + 12^3 + 13^3 = 20^3$.

The author though had to use further conditions based on numerical experimentation, namely that *all four* q_i on the left hand side had the same discriminant and that the property of the (a,b,c,d) carries over to the q_i,

$$q_1+q_2 = q_3+q_4$$

and this gave more useful linear relations within the system. Then with a similar approach used earlier, we find our third identity,

Identity 3 (Id.3)

If a+b = c+d, (satisfied by a = t-p, b = t+p, c = t-q, d = t+q) then,

$$(ax^2 + v_1xy + v_2y^2)^3 + (bx^2 + v_3xy + v_4y^2)^3 + (cx^2 + v_5xy + v_6y^2)^3 + (dx^2 + v_7xy + v_8y^2)^3 = (a^3 + b^3 + c^3 + d^3)(x^2 + v_9xy + v_{10}y^2)^3$$

where, v_1, v_3 (free variables)

$$v_5\,(2qt) = -(p^2+q^2+pt-qt)v_1 - (p^2+q^2-pt-qt)v_3$$

$$v_7\,(2qt) = (p^2+q^2+pt+qt)v_1 + (p^2+q^2-pt+qt)v_3$$

$$v_9 = (v_1 + v_3)/(2t)$$

and for the others, first define the discriminants g_i as,

$$g_1 = (p^2+q^2-t^2)(pv_1+tv_1+pv_3-tv_3)^2/(4q^2t^2) \qquad \text{(LHS)}$$

$$g_2 = -(pv_1+tv_1+pv_3-tv_3)^2/(4q^2t^2) \qquad \text{(RHS)}$$

then,

$$v_1^2-4av_2 = g_1, \quad v_3^2-4bv_4 = g_1, \quad v_5^2-4cv_6 = g_1, \quad v_7^2-4dv_8 = g_1, \quad v_9^2-4v_{10} = g_2,$$

where the v_i with *even* subscript can easily be solved for. Finally,

$$p = (-a+b)/2, \ \ q = (-c+d)/2, \ \ t = (a+b)/2 = (c+d)/2$$

Note that the formulas are very suggestive of the ones found earlier, though cannot be derived by simply setting $p = q = t$ in the earlier formulas.

Example:

Let a = 11, b = 14, c = 12, d = 13,

$$(11x^2 + v_1xy + v_2y^2)^3 + (14x^2 + v_3xy + v_4y^2)^3 + (12x^2 + v_5xy + v_6y^2)^3 + (13x^2 + v_7xy + v_8y^2)^3 = 20^3(x^2 + v_9xy + v_{10}y^2)^3$$

so p = 3/2, q = 1/2, t = 25/2. Let $\mathbf{v_1 = 50m}$ and $\mathbf{v_3 = 50n}$ (to clear denominators) and we find the other v_i as,

$$v_5 = -30(2m-3n), \ \ v_7 = 10(11m-4n), \ \ v_9 = 2(m+n)$$

and,

$$v_2 = 5(2203m^2-3444mn+1353n^2) \qquad v_4 = 10(861m^2-1353mn+536n^2)$$
$$v_6 = 10(1012m^2-1601mn+637n^2) \qquad v_8 = 5(1901m^2-2948mn+1151n^2)$$
$$v_{10} = 5(157m^2-246mn+97n^2)$$

Finally, our fourth identity, by letting $v_3 = -v_1$ of Id.3 and after scaling, we get the simpler,

Identity 4 (Id. 4)

If a+b = c+d, then,

$$(ax^2 - v_1xy + by^2)^3 + (bx^2 + v_1xy + ay^2)^3 + (cx^2 + v_2xy + dy^2)^3 + (dx^2 - v_2xy + cy^2)^3 = (a^3 + b^3 + c^3 + d^3)(x^2 + y^2)^3$$

where $v_1 = -(c-d)$, $v_2 = -(a-b)$, though this again has middle terms with opposite parity.

There are many other particular equations with the same property as the example, a few of which are,

$$9^3 + 23^3 + 13^3 + 19^3 = 28^3$$

$13^3 + 23^3 + 15^3 + 21^3 = 30^3$

Equations whose terms are in arithmetic progression also belong to this class. One interesting example mentioned by Dave Rusin in a related context (connected to a certain elliptic curve) is given by,

$31^3 + 33^3 + 35^3 + 37^3 + 39^3 + 41^3 = 66^3$

which gives a plethora of possible relations such as 31+37=33+35, 31+41=33+39, etc.

VI. Conclusion: Beyond Cubics

Before we conclude our paper, there are some points that can be discussed. *First*, given the quadratic form identity $q_i^3 + q_2^3 + q_4^3 + q_4^3 = 0$, is it always the case that the discriminant is the same for at least a pair of q_i or can they all be different? It seems discriminants play a certain role in polynomial solutions of equal sums of like powers and it might be fruitful to take a second look at known parametrizations in terms of this point. *Second*, can the system of equations we called S_1 have more than *three* distinct linear solutions? Note that for (3,4,5,-6) there are actually four linear solutions, though with one repetition. *Third*, does the equation $q_1^3 + q_2^3 + q_3^3 + q_4^3 = (a^3 + b^3 + c^3 + d^3)\, q_5^3$ (eq.7) as S_2 *always* have a linear solution? So many questions and to think these came about because of Ramanujan's parametrization involving $3^3 + 4^3 + 5^3 = 6^3$. It seems there is more to this equation (and others of its kind) than meets the eye.

And as if these questions were not enough, there is still more room for investigation. The interesting thing is that Ramanujan had *another* magic trick up his sleeve in the form of a quadratic form identity for a different power, this time for *fourth* powers given by,

$$(2x^2-12xy-6y^2)^4 + (2x^2+12xy-6y^2)^4 + (3x^2+9y^2)^4 + (4x^2-12y^2)^4 + (4x^2+12y^2)^4 = (5x^2+15y^2)^4$$

with the first instance the very nice $2^4 + 2^4 + 3^4 + 4^4 + 4^4 = 5^4$.

It seems our cubic identity has a big brother, a general *quartic identity* and fortunately we have found its explicit form. But that is another story.

--End--

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Notes:

1. In addition to the "saying" $E = mc^2$, Einstein was noted for his many aphorisms, one of which was his sentiment regarding quantum mechanics, namely that "*God does not play dice with the universe.*". The significance of the saying cited at the start of this paper can be appreciated considering that in 1952, after the death of Israel's first president Chaim

Weizmann, the Israeli Cabinet offered the presidency to Einstein, which he politely refused. Perhaps the quotation would partly explain that refusal.

2. The anecdote regarding 1729 is well-known. Ramanujan was hospitalized in Putney and Hardy came in for a visit. In Hardy's words, "…I had ridden in taxi-cab no. 1729, and remarked that the number seemed to me rather a dull one, and that I hoped it was not an ill omen. 'No,' he replied, 'it is a very interesting number; it is the smallest number expressible as a sum of two cubes in two different ways.'". And thus this story immortalized this particular integer.

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