Function Review

The function remains the most important concept through most of AP Calculus AB. I really expect you to have mastered all aspects related to functions before you enter the class. I review the major things you should be very proficient at below.

- Definition of a Function

> Definition: A **function** is a special relation, such that for any given input, there is, at most, one output.

In AP Calculus, you are expected to be able to solve problems analytically, graphically, and numerically. It is important, then, that you tell whether something is a function by looking at an equation, graph, or table of values.

Analytically: Solve the equation for the dependent variable (often y) and see if there are any possible inputs that will give you more than one output.

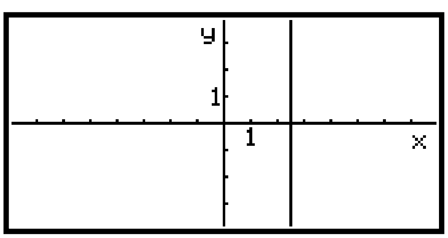
Graphically: Use the vertical line test.

Numerically: Examine the table of values to see if one input produces more than one output.

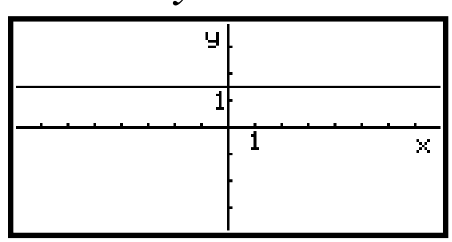
- Graphs of Basic Functions

Some functions' graphs show up over and over again, so I have them shown below. You need to have a basic understanding of all of these graphs, so that when you are given an equation of a function, you will already have a clear picture in your mind of what the graph looks like.

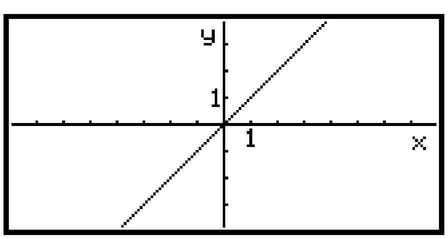
$x = k$



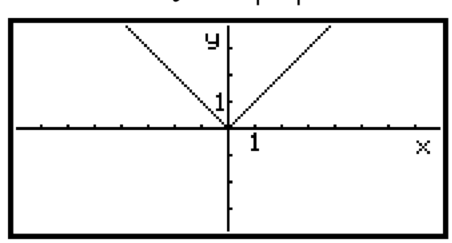
$y = k$



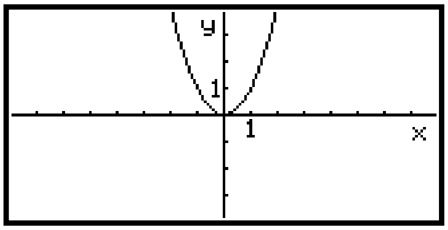
$y = x$



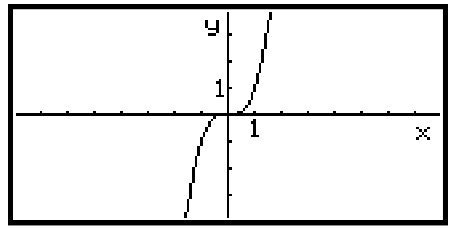
$y = |x|$



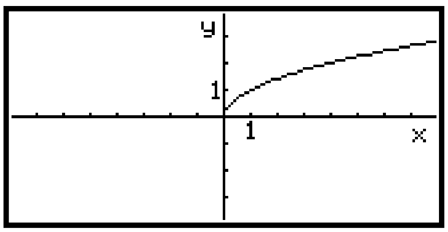
$y = x^2$



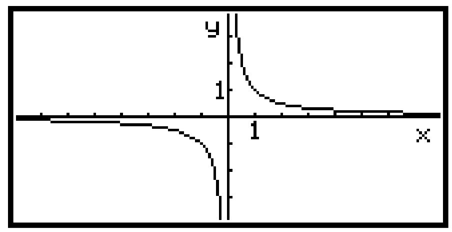
$y = x^3$



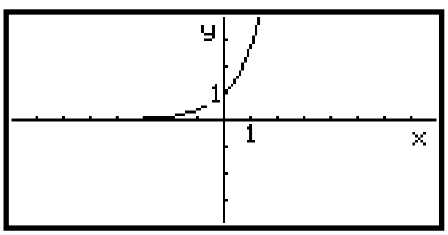
$y = \sqrt{x}$



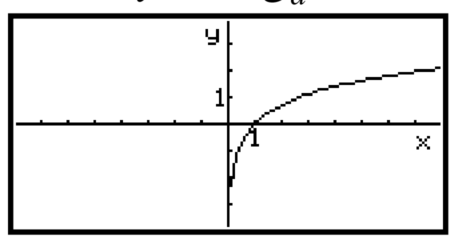
$y = \dfrac{1}{x}$



$y = a^x$



$y = \log_a x$



- Domain and Range

Definition: **Domain** is the set of all possible inputs to a function/relation.

Definition: **Range** is the set of all possible outputs of a function/relation.

There are several types of functions that have very predicable domains:

(1) Polynomial Functions

These functions have domains that consist of all real numbers.

(2) Rational Functions

The domain of this type of function is any number, so long as the denominator does not equal zero. So, to determine the domain of a rational function, set the denominator equal to zero, and that will tell you what the domain does **NOT** include.

(3) Radical Functions

These are functions that contain a radical. Some examples are $f(x)=\sqrt{x}$, $g(x)=\sqrt{x-2}$, and $h(x)=\sqrt{x+3}$. We know that the square root of a negative number does not exist. So, to determine the domain of a radical function, determine when the expression underneath the radical is greater than or equal to zero. This will tell you what the domain **DOES** include.

(4) Exponential Functions

These functions have domains that consist of all real numbers.

(5) Logarithmic Functions

These are functions that contain a logarithm. Some examples are $f(x)=\log x$, $g(x)=\log(x-2)$, and $h(x)=\log(x+3)$. We know that the logarithm of a non-positive number does not exist. So, to determine the domain of a logarithmic function, determine when the "result" portion of the logarithm is greater than zero.

(6) Trigonometric Functions

Function	**Domain**
$f(x)=\sin x$	$x\in(-\infty,\infty)$
$f(x)=\cos x$	$x\in(-\infty,\infty)$
$f(x)=\tan x$	$x\neq\frac{\pi}{2}+n\pi$, where n is any integer
$f(x)=\csc x$	$x\neq n\pi$, where n is any integer
$f(x)=\sec x$	$x\neq\frac{\pi}{2}+n\pi$, where n is any integer
$f(x)=\cot x$	$x\neq n\pi$, where n is any integer

As far as finding the range is concerned, there is no slick way to categorize how to answer those types of questions. In many cases, the best way to determine the range of a function is to examine its graph. From that standpoint, you should be able to determine the ranges of quadratic, exponential, logarithmic, and trigonometric functions without the assistance of a calculator.

- **Composition of Functions**

Definition: $(f\circ g)(x)$ means $f(g(x))$.

❍ Even and Odd Functions

Although you have been learning about this for some time without fully understanding what benefit this information provides, in calculus if you can recognize that a function is even or odd, you can save yourself a lot of work.

Definition: A function is **even** if $f(-x) = f(x)$. This means that the function's graph will have ***y*-axis symmetry**.

Definition: A function is **odd** if $f(-x) = -1 \cdot f(x)$. This means that the function's graph will have **origin symmetry**.

Remember, a function cannot ever have *x*-axis symmetry. Why not?

❍ Inverse Functions

There are several things about inverse functions that are important for you to know:

(1) What IS an inverse function?

An inverse function is a *function* that **<u>reverses</u>** what some other function does.

(2) How do you determine whether a function HAS an inverse?

The easiest way is to graph a function and see if the function passes the horizontal line test.

(3) How do you find the inverse of a function if it exists?

Given an equation for a function, you need to switch *x* and *y* and then solve for *y*.

(4) How do you determine whether two functions are inverses of each other?

Graphically, you can determine this if the two graphs reflect across the line $y = x$.

Analytically, *f* and *g* would be inverses of each other if $f(g(x)) = g(f(x)) = x$.

○ Polynomial Functions

The vast majority of your Precalculus studies deal with polynomial functions. In your Algebra classes, you tend to deal with polynomials of degree one (linear functions) and two (quadratic functions). You have learned recently about characteristics of all polynomial functions, and it is important to recall and remember several pieces of information related to polynomial functions.

(1) Equivalent Statements

If f is a polynomial function and a is a real number, the following statements are equivalent:

1. $x = a$ is a ***ZERO*** or ***ROOT*** of the function f.
2. $x = a$ is a ***SOLUTION*** of the equation $f(x) = 0$.
3. $(x - a)$ is a ***FACTOR*** of f.
4. $(a, 0)$ is an ***x-INTERCEPT*** of the graph of f.

(2) Leading Coefficient Test

This test tells us information about how the graph behaves as $x \to -\infty$ and as $x \to +\infty$. We concern ourselves with two pieces of information: whether the degree of the function is even or odd, and whether the leading coefficient is positive or negative.

		Leading Coefficient	
		POSITIVE	NEGATIVE
Degree	EVEN	Rises to the Right Rises to the Left	Falls to the Right Falls to the Left
	ODD	Rises to the Right Falls to the Left	Falls to the Right Rises to the Left

(3) Zeros and Turning Points

If a polynomial function f has degree n, the following statements are true:

1. The graph of f has, at most, $n - 1$ turning points.
2. The function f has, at most, n real zeros.
3. The function f has, exactly, n complex zeros.

(4) Multiplicity

There is only one solution to the equation $(x-2)^2 = 0$, but it occurs twice. This is known as a repeated zero, with multiplicity of 2. This has graphical implications.

1. If the multiplicity of a zero $x = a$ of a function is odd, the graph will **cross through** the *x*-axis at $(a,0)$.
2. If the multiplicity of a zero $x = a$ of a function is even, the graph will **touch, but not cross through** the *x*-axis at $(a,0)$.

(5) Intermediate Value Theorem

Although this theorem gave you fits when you first dealt with it, it is very straightforward. In order for it to apply, remember that the function must be continuous (thus, it always applies to polynomial functions). It states the following:

Let *a* and *b* be real numbers such that $a < b$. If $f(a) \neq f(b)$, then in the interval $[a,b]$, *f* takes on every value between $f(a)$ and $f(b)$.

In other words, if *f* is a continuous function, and $f(1) = -10$ and $f(2) = 3$, we know that somewhere between $x = 1$ and $x = 2$, *f* must equal zero, since $-10 < 0 < 3$. We are also assured that somewhere between $x = 1$ and $x = 2$, *f* must equal –5.6 as well, since $-10 < -5.6 < 3$. In actuality *any* number between –10 and 3 must be a function value for some value of *x* between $x = 1$ and $x = 2$. Usually, in calculus, however, we are interested in where a function equals zero, so we often use the Intermediate Value Theorem for this purpose.

(6) Rational Root Theorem

If our polynomial function is linear, we have a quick way to find the zero: set the function equal to zero, and isolate the variable. If the polynomial function is quadratic, we have a quick way to find the zero as well: use the quadratic formula. Although ways exist to find the zeroes of polynomial functions with degree greater than two, they are not very neat. If a polynomial function has *rational* zeros, there is a nifty way to determine the *possible* rational solutions. We still must check those possible rational solutions, so it is not nearly as nice as the quadratic case, but still doable. The Rational Root Theorem is used for this purpose:

If the polynomial function *f* has integer coefficients, every rational zero has the form of $\frac{p}{q}$, where *p* represents the set of possible factors of the constant term of *f* and *q* represents the set of possible factors of the leading coefficient of *f*.

(7) Complex Zeros

This is not really all that important to AP Calculus, but it is a good idea to remember that if $a+bi$ is a complex zero of a function, then it follows that its complex conjugate $a-bi$ must also be a zero of that function (i.e. complex zeros occur in conjugate pairs).

(8) Descartes' Rule of Signs

Also not so important to AP Calculus, but kind of nifty to remember, is the fact that we can obtain a good idea of the number of positive and negative real zeros of a polynomial function called *f*, by looking at the sign changes of *f*:

1. The number of *positive real zeros* of *f* is either equal to the number of sign changes of $f(x)$, or less than that number by an even integer.

2. The number of *negative real zeros* of *f* is either equal to the number of sign changes of $f(-x)$ or less than that number by an even integer.

For example, the function $f(x)=x^3-3x+2$ has two sign changes, so there are either two or zero positive real zeros. Since $f(-x)=-x^3+3x+2$ has one sign change, we can be certain that there will be one negative real zero for *f*.

○ Rational Functions and Asymptotes

Recall that rational functions are functions that can be expressed as the ratio of two polynomials. Rational functions' graphs possess the unique characteristic of asymptotes. There are three different types of asymptotes that you need to be familiar with.

(1) Vertical Asymptotes

These are caused by places where the function is undefined. This occurs when the denominator equals zero. For instance, $f(x)=\dfrac{1}{x-3}$ has a vertical asymptote at $x=3$. But, be careful. In Calculus, this concept will be extended a little bit. As Precalculus students, you were always exposed to the concept of vertical asymptotes on an elementary level. Notice that the function $g(x)=\dfrac{x-2}{x^2-4}$ has a vertical asymptote at $x=-2$, but not at $x=2$, as you would probably expect. Why is this so (Hint: Simplify the rational expression first)?

(2) Horizontal Asymptotes

This concept is very similar to vertical asymptotes, when we talk about graphs of rational functions, but what causes horizontal asymptotes is extremely different than what causes vertical asymptotes. Horizontal asymptotes refer to the behavior of functions as $x\to-\infty$ or as $x\to+\infty$. To determine horizontal asymptotes, you need to compare the degree of both numerator and denominator of the rational function:

1. If the degree of the numerator is **greater** than the degree of the denominator then no horizontal asymptote exists.

2. If the degree of the numerator is **less** than the degree of the denominator then the horizontal asymptote is $y=0$.

3. If the degree of the numerator is **equal** to the degree of the denominator then the horizontal asymptote is $y=\dfrac{\text{Leading Coefficient of Numerator}}{\text{Leading Coefficient of Denominator}}$.

(3) Slant Asymptotes

In the bigger scheme of things, this type of asymptote is not nearly as critical as the first two, but it is important to know. Slant asymptotes occur only if the degree of the numerator is one higher than the degree of the denominator. Through long division, the slant asymptote is determined. For example, the function $h(x)=\dfrac{3x^2-4}{x}$ has a slant asymptote at $y=3x$ because *x* is the quotient of $\dfrac{3x^2-4}{x}$ (Notice, the remainder is of no interest to us when determining the slant asymptote).

❍ Logarithmic Functions

It is important to recognize that logarithmic functions exist to provide inverses to exponential functions.

In essence, a logarithm is simply an exponent.

Recall that calculators only have two built in bases: $\log_{10}$, which is the **common logarithm** base, and is often just written as $\log$, and $\log_e$, which is the **natural logarithm** base, and is always written as $\ln$.

It is important to remember the change of base formula:

$$\log_a x = \frac{\log_b x}{\log_b a}$$

It is also important to remember the three properties of logarithms:

1. $\log_b (u \cdot v) = \log_b u + \log_b v$
2. $\log_b \left(\frac{u}{v} \right) = \log_b u - \log_b v$
3. $\log_b u^n = n \cdot \log_b u$

EXPLORATION

There is actually a companion to the change of base formula for exponential functions that is not really emphasized in your Precalculus studies, but will be important for us in calculus. We will do an exploration now to develop this relationship.

What is $10^{\log_{10} 6}$? What is $10^{\log 90}$? What is $e^{\ln 3}$? What is $e^{\ln 78}$?

Now, simplify the following expressions (which might not seem to have any relationship to the last set of questions asked of you).

$\left(x^3\right)^2 =$ $\left(x^m\right)^n =$

Given this reminder about properties of exponents, what does $x^{m \cdot n}$ equal?

Now, apply this recognition to $e^{(\ln 5)x}$. Rewrite this expression similar to the way you did $x^{m \cdot n}$ above.

Now, simplify the interior part of this expression.

Go through a similar process to simplify $e^{(\ln a)x}$, where a is any positive number.

This relationship is very important to remember:

Trigonometry Review

○ Trigonometric Chart Values

I know, I know, you dread the concept of a trig chart, and knowing all of the information on it. However, I know almost 100 former calculus students who will vouch for the fact that the better you know that information, the more you can focus on the calculus you will be learning.

It is also time for you to understand that the concept of a degree as a measure of angles is not really sufficient in the world of calculus. Again, the sooner you start to use radian measure as a way to understand angles, the better off you will be in AP Calculus.

Although your teachers have probably told you this before, I will reiterate that it is UNNECESSARY to memorize the whole trig chart. Knowing that the tangent of an angle is the ratio of sine to cosine makes it unnecessary to memorize the values of tangent for a given angle. It is also unnecessary to memorize the values of cosecant, secant, and cotangent, since these are just the multiplicative inverses of sine, cosine, and tangent.

Also, if you use the fact that the absolute values of trigonometric functions for angles with the same reference angle are equal, you can reduce the amount you have to memorize as well. Finally, remembering which trigonometric ratios are positive in which quadrants (**All Students Take Calculus**) will enable you to recall important trigonometric chart values without memorizing a great deal of information.

This abbreviated chart consists of the values you should know like the back of your hand (how well DO people know the back of their hands anyway? Do they know this body part better than the front of their hands?).

θ	$\sin\theta$	$\cos\theta$
0	0	1
$\pi/6$	1/2	$\sqrt{3}/2$
$\pi/4$	$\sqrt{2}/2$	$\sqrt{2}/2$
$\pi/3$	$\sqrt{3}/2$	1/2
$\pi/2$	1	0
π	0	-1
$3\pi/2$	-1	0
2π	0	1

❍ Trigonometric Identities

There are boatloads of trigonometric identities, but thankfully, there is only a handful that you should expect to see and use in AP Calculus. These are summarized for you below. Again, the better you know these identities, the more you can focus on the calculus you will be learning.

Even/Odd Identities:

$$\sin(-x) = -1 \cdot \sin x \qquad \cos(-x) = \cos x \qquad \tan(-x) = -1 \cdot \tan x$$

Pythagorean Identities:

$$\sin^2 x + \cos^2 x = 1 \qquad \tan^2 x + 1 = \sec^2 x \qquad 1 + \cot^2 x = \csc^2 x$$

Sum and Difference Identities:

$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y \qquad \cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$$

$$\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y}$$

Double Angle Identities:

$$\sin 2x = 2 \sin x \cos x \qquad \cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$$

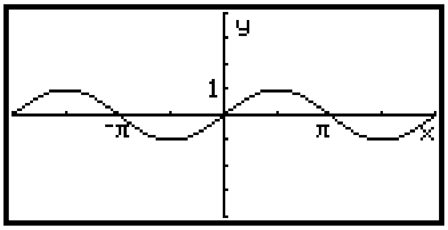
Power Reducing Identities:

$$\sin^2\left(\tfrac{x}{2}\right) = \frac{1 - \cos x}{2} \qquad \cos^2\left(\tfrac{x}{2}\right) = \frac{1 + \cos x}{2}$$

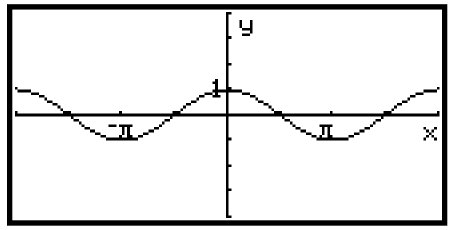
❍ Graphs of Trigonometric Functions

You should be familiar with the graphs of the six trigonometric functions:

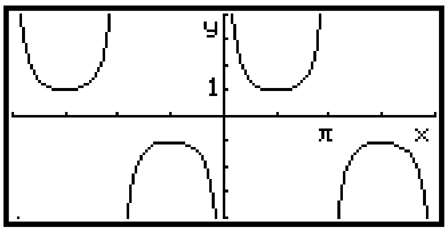
$y = \sin x$



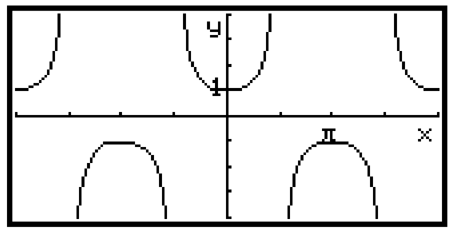
$y = \cos x$



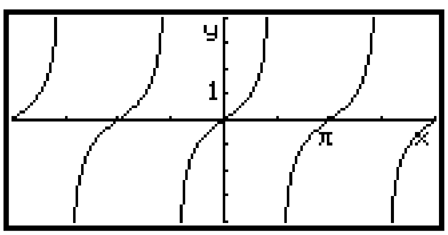
$y = \csc x$



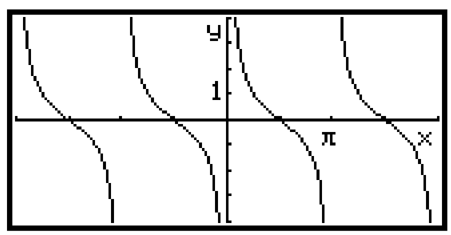
$y = \sec x$



$y = \tan x$



$y = \cot x$



Remember, the general formulas $y = A\sin\left(B\left(x - C\right)\right) + D$ and $y = A\cos\left(B\left(x - C\right)\right) + D$ provide us with very important information:

$$\text{Amplitude} = |A|$$

$$\text{Period} = \frac{2\pi}{|B|}$$

$$\text{Phase Shift} = C$$

$$\text{Vertical Shift} = D$$

For the general formula for tangent, recall that the $\text{Period} = \dfrac{\pi}{|B|}$.

○ Inverse Trigonometric Functions

You will also be required to remember information about inverse trigonometric functions, or the "arc" functions. Remember, trigonometric functions technically do not have inverses when we consider the entire domains of these functions. But if the following restrictions are placed on these functions, these functions will have inverses:

$$y = \sin x, \quad x \in \left[-\tfrac{\pi}{2}, \tfrac{\pi}{2}\right]$$
$$y = \cos x, \quad x \in [0, \pi]$$
$$y = \tan x, \quad x \in \left(-\tfrac{\pi}{2}, \tfrac{\pi}{2}\right)$$

Remember that the domain of a function becomes the range of the corresponding inverse function. So, the appropriate ranges for these inverse functions are as follows:

$$y = \sin^{-1} x = \arcsin x, \quad y \in \left[-\tfrac{\pi}{2}, \tfrac{\pi}{2}\right]$$
$$y = \cos^{-1} x = \arccos x, \quad y \in [0, \pi]$$
$$y = \tan^{-1} x = \arctan x, \quad y \in \left(-\tfrac{\pi}{2}, \tfrac{\pi}{2}\right)$$

Miscellaneous

The following topics are topics that past AP Calculus students have indicated they wished they had a better foundation on. They do not really follow any particular order.

- Completing the Square

Although $2(x-2)^2+5$ and $2x^2-8x+13$ are equivalent expressions, in calculus, we will want to be able to convert from one type of expression to the other. It is relatively easy to convert $2(x-2)^2+5$ to $2x^2-8x+13$ by expanding the expression and simplifying. However, we need to use the process of completing the square to go the other direction.

To complete the square, remember to do the following:

Step 1: Make sure the leading coefficient is one, by factoring the terms with *x* (Ignoring the constant term).

$$2x^2-8x+13=2(x^2-4x)+13$$

Step 2: Add the square of half of the coefficient of *x* to portion of the expression that you have factored.

$$2(x^2-4x)+13=2(x^2-4x+\mathbf{4})+13-\_\_\_$$

Step 3: To balance this new term, you must subtract the value of the term you introduce in Step 2 from the original constant term. Be careful, this is NOT always the same number you add in Step 2.

$$2(x^2-4x)+13=2(x^2-4x+4)+13-\mathbf{8}$$

Step 4: Factor the perfect square trinomial that you have just created and combine the constants.

$$2(x^2-4x+4)+13-\mathbf{8}=2(x-2)^2+5$$

❍ Dividing Polynomials

Often in calculus, you need to be able to divide to two polynomials, because the quotient and remainder are easier to work with than the initial rational function is. Remember, there are two ways to accomplish this task: long division and synthetic division. While synthetic division is certainly more convenient, that only works if the divisor is linear. Often when finding slant asymptotes, this is not the case, so you must recall the process of long division as well. Remember to write both the dividend and divisor in standard form, and remember to leave placeholders for missing terms. Then just concern yourself with the leading terms at each step along the way. For example, the division of $\left(2x^4+4x^3-5x^2+3x-2\right)\div\left(x^2+2x-3\right)$ would look like this:

$$\begin{array}{r|l} & \quad 2x^2 \qquad\quad +1 \\ \hline x^2+2x-3 & 2x^4+4x^3-5x^2+3x-2 \\ & -\left(2x^4+4x^3-6x^2\right) \\ \hline & \qquad\qquad\quad x^2+3x-2 \\ & \qquad\quad -\left(x^2+2x-3\right) \\ \hline & \qquad\qquad\qquad x+1 \end{array}$$

This tells us that the quotient of $\left(2x^4+4x^3-5x^2+3x-2\right)\div\left(x^2+2x-3\right)$ is $2x^2+1+\frac{x+1}{x^2+2x-3}$.

Synthetic division comes in handy for a couple of reasons. First of all, remember that the remainder from the process of synthetic division actually gives us the function value when the zero of the divisor is the input to the dividend. This fact (Remainder Theorem) is useful when using the Rational Root Theorem, since this is the fastest way to check and see if a possible rational root is actually a root of a function. Usually what students forget when using synthetic division is to include zeros for placeholders, and that they should ADD the numbers in given columns, not subtract, as we do in long division. The reason for this difference is that we use the zero of the divisor in the "elbow," so we take care of the subtraction at that point. Using $\left(x^4-10x^2-2x+4\right)\div(x+3)$ as an example:

	1	0	-10	-2	4
	↓	-3	9	3	-3
-3	1	-3	-1	1	1

This tells us that the quotient of $\left(x^4-10x^2-2x+4\right)\div(x+3)$ is $x^3-3x^2-x+1+\frac{1}{x+3}$. Again, remember this also tells us that if $f(x)=x^4-10x^2-2x+4$ then $f(-3)=1$.

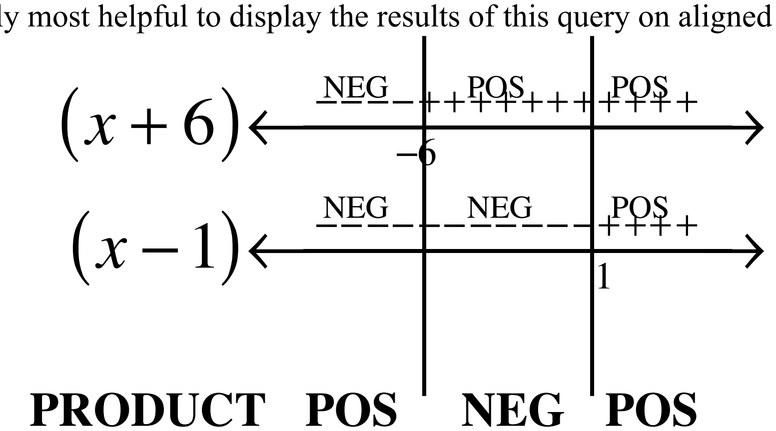
Solving Polynomial and Rational Inequalities

For some reason, students have difficulty solving things like $x^2 + 5x - 6 > 0$. They assume since $3x + 4 = 7$ is solved exactly like $3x + 4 < 7$, with the exception of the sign being used, $x^2 + 5x - 6 > 0$ must be solved exactly like $x^2 + 5x - 6 = 0$. Although the solutions to the quadratic equation $x^2 + 5x - 6 = 0$ are helpful in solving the related inequality, there is more work that needs to be done. When we solve $x^2 + 5x - 6 = 0$, we find the values of x that EQUAL zero. On either side of these "critical numbers" then, are sets of values that either satisfy the inequality or do not satisfy the inequality. The easiest way to determine which values are which is to look at a factored form of the expression. It is fast to tell whether each factor is positive or negative, and when multiplying numbers, if we know this about the factors, we can easily tell whether the final product is less than or greater than zero, without actually computing the products.

So when solving $x^2 + 5x - 6 > 0$, we want to do the following:

Step 1: Solve the related equation to obtain the "critical numbers." For $x^2 + 5x - 6 = 0$, $x = -6$ and $x = 1$.

Step 2: Find out where each factor (in this case $(x+6)$ and $(x-1)$) is positive and negative. It is usually most helpful to display the results of this query on aligned number lines.



Step 3: Determine which intervals satisfy the given inequality. Since we are asked to solve $x^2 + 5x - 6 > 0$, we are really interested in knowing when $x^2 + 5x - 6 = (x+6)(x-1)$ is *positive*. From the sign chart, we can see that this happens when $x < -6$ (negative times a negative produces a positive product), as well as when $x > 1$ (positive times a positive produces a positive product). Thus, the solution is $x \in (-\infty, -6) \cup (1, \infty)$.

The process for solving rational inequalities is very similar, since *quotients* will be positive and negative in the same way that *products* are. Of course all of this is based on the fact that the inequality is being compared to zero, so be sure to have one side of the inequality set equal to zero before beginning this process.

○ Absolute Value

Many students hold the simplistic idea that absolute value means you just chop off the negative sign if it is there. I would advise you to have a more sophisticated understanding of this concept by remembering two things:

Definition: $|x| = \begin{cases} -x, & x < 0 \\ x, & x \geq 0 \end{cases}$. This represents the **distance** of a number from zero.

There are three basic types of problems that you need to be able to solve with respect to absolute value.

Type 1: $|\text{BOB}| = 4$. What this means is to find all values for BOB that are **exactly** 4 units away from zero. Thus, you must find what makes $\text{BOB} = 4$ or $\text{BOB} = -4$ to come up with all of the solutions.

Type 2: $|\text{BOB}| < 4$. What this means is to find all values for BOB that are **less than** 4 units away from zero. Thus, you must find what makes $-4 < \text{BOB} < 4$.

Type 3: $|\text{BOB}| > 4$. What this means is to find all values for BOB that are **more than** 4 units away from zero. Thus, you must find what makes $\text{BOB} < -4$ or $\text{BOB} > 4$ to come up with all of the solutions.

❍ Summation and Sigma Notation

Wow, it may not seem like it from your Precalculus coverage, but sigma notation is very critical for calculus. See, in calculus, we are going to be adding lots of things together (not hard at all, huh?). It would get pretty tiring to write down the millions of things we are going to add together, so sigma notation is pretty important.

Definition: The sum of the first n terms of a sequence is represented by

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + a_4 + \square + a_n$$

where i is the **index of summation**, n is the **upper limit of summation**, and 1 is the **lower limit of summation**.

For instance, when you see $\sum_{i=1}^{3} 2i$, this means the sum of the first three terms of the sequence $a_i = 2i$, beginning with 1 as the first input. So,

$$\sum_{i=1}^{3} 2i = 2(1) + 2(2) + 2(3) = 2 + 4 + 6 = 12$$

❍ Solving Systems of Equations

In all of mathematics, not just calculus, it is often common to run into a system of equations with several unknowns. You are probably used to solving systems of two equations and two unknowns using either substitution or elimination (i.e. linear combinations). These methods are great, but start to become tedious for larger systems of equations. This is where matrices come in handy.

For instance, say you encountered the following system of equations with four unknowns:

$$\begin{aligned} a - \ 2b - \ c - \ 2d &= 13 \\ a - \ 5b - \ 2c - \ 3d &= -2 \\ 2a - \ 5b - \ 2c - \ 5d &= 0 \\ -a + \ 4b + \ 4c + \ 11d &= 0 \end{aligned}$$

We can rewrite this system with a coefficient matrix times a variable matrix, and set that equal to the result matrix:

$$\begin{bmatrix} 1 & -2 & -1 & -2 \\ 3 & -5 & -2 & -3 \\ 2 & -5 & -2 & -5 \\ -1 & 4 & 4 & 11 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 0 \\ 3 \end{bmatrix}$$

Of course, we want to get the variable matrix alone, and so we need to left multiply both sides of this equation by the inverse of the coefficient matrix. Now, it would be nice if you knew how to find the inverse of a 2x2 and 3x3 matrix by hand, but for larger matrices, the calculator is probably the best tool to accomplish this.

$$\begin{bmatrix} 1 & -2 & -1 & -2 \\ 3 & -5 & -2 & -3 \\ 2 & -5 & -2 & -5 \\ -1 & 4 & 4 & 11 \end{bmatrix}^{-1} \begin{bmatrix} 1 & -2 & -1 & -2 \\ 3 & -5 & -2 & -3 \\ 2 & -5 & -2 & -5 \\ -1 & 4 & 4 & 11 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 1 & -2 & -1 & -2 \\ 3 & -5 & -2 & -3 \\ 2 & -5 & -2 & -5 \\ -1 & 4 & 4 & 11 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ -2 \\ 0 \\ 3 \end{bmatrix}$$

This isolates the variable matrix and produces the following:

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} -32 \\ -13 \\ -37 \\ 15 \end{bmatrix}$$

Thus, $a = -32,\ b = -13,\ c = -37,$ and $d = 15$.

❍ Binomial Theorem

There will be occasions in calculus where you will want to have the expanded form of something like $(2x-1)^4$. Of course, you could expand this binomial out, one factor at a time, but this can be time consuming – time you will not have. So, it is important to remember the binomial theorem.

$$(x+y)^n = {}_nC_0 x^n y^0 + {}_nC_1 x^{n-1} y^1 + {}_nC_2 x^{n-2} y^2 + \square + {}_nC_{n-2} x^2 y^{n-2} + {}_nC_{n-1} x^1 y^{n-1} + {}_nC_n x^0 y^n$$

This may look confusing, but just remember to follow these three steps:

Step 1: Write the first term (which is 2*x*) in decreasing powers of *x*.

$$(2x)^4 \quad + (2x)^3 \quad + (2x)^2 \quad + (2x)^1 \quad + (2x)^0$$

Step 2: Write the second term (which is –1) in increasing powers of *x*.

$$(2x)^4(-1)^0 + (2x)^3(-1)^1 + (2x)^2(-1)^2 + (2x)^1(-1)^3 + (2x)^0(-1)^4$$

Notice how each individual term's powers add up to 4, which is the power we are raising the binomial to.

Step 3: Include the binomial coefficient from Pascal's Triangle.

$${}_4C_0(2x)^4(-1)^0 + {}_4C_1 \cdot (2x)^3(-1)^1 + {}_4C_2 \cdot (2x)^2(-1)^2 + {}_4C_3 \cdot (2x)^1(-1)^3 + {}_4C_4 \cdot (2x)^0(-1)^4$$

Now, you could evaluate all of these combinations, but recall that Pascal's Triangle makes it a tad bit faster to remember (as long as *n* is a relatively small number). It is a triangular shaped array of numbers that is obtained by adding the two numbers above an entry to obtain that entry, and that has 1s at the start and end of every row. I have listed the first seven rows of this triangle below:

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

1 7 21 35 35 21 7 1

The equation from Step 3 would look like this:

$$1 \cdot (2x)^4(-1)^0 + 4 \cdot (2x)^3(-1)^1 + 6 \cdot (2x)^2(-1)^2 + 4 \cdot (2x)^1(-1)^3 + 1 \cdot (2x)^0(-1)^4.$$

Thus,

$$(2x-1)^4 = 16x^4 - 32x^3 + 24x^2 - 8x + 1.$$

- **Vectors**

Remember that a vector is a quantity with both magnitude and direction, whereas a scalar is a quantity with only magnitude. We can express a vector by describing both its magnitude and direction (e.g. a force vector of 100 pounds with a direction angle of $60°$), or we can also describe a vector by providing its component form. For example, the vector above would have a component form of $\langle 50.000, 86.603 \rangle$ pounds or $50.000\mathbf{i} + 86.603\mathbf{j}$ pounds. The motivation for component form, you will recall, is because it is easier to add vectors in component form than if you were given two vectors' magnitudes and directions.

To convert a vector with magnitude *M* and direction angle θ, to component form:

$$\langle M\cos\theta, M\sin\theta \rangle$$

To convert a vector from component form to magnitude and direction, you need to utilize the following information:

$$M = \sqrt{x^2 + y^2}$$

where *x* is the horizontal component of the vector, and *y* the vertical component of the vector.

$$\theta^* = \left|\tan^{-1}\left(\frac{y}{x}\right)\right|$$

where θ^* is the reference angle of the direction angle.

For example, if the vector $\langle -\sqrt{3}, 1 \rangle$ were converted to magnitude and direction form, the magnitude would be 4, but the formula for the direction angle would produce an angle of $30°$. Looking at $\langle -\sqrt{3}, 1 \rangle$, we can tell that the direction angle must be in the second quadrant. Thus, the actual direction angle would not be $30°$, but a second quadrant angle with a reference angle of $30°$, which is $150°$.