**SESSION II**

# 2. Anharmonic Free and Forced Oscillations

## 2.1 Formulation of the Problem

An exactly harmonic potential seldom occurs in nature; a small anharmonicity is almost always present. In analytic calculations such perturbation terms present considerable difficulties. In numerical calculations on the computer, on the other hand, it makes scarcely any difference whether the potential is harmonic or anharmonic. In what follows we shall again consider the one-dimensional motion of a point mass with mass $M = l\ kg$. Friction will be ignored. The potential of the restoring force has the form:

$$V(x) = A\frac{|x|^{B+1}}{(B+1)} \qquad (2.1)$$

The restoring force is then:

$$K(x) = -A|x|^{B}\frac{x}{|x|}. \qquad (2.2)$$

If we also take into account a harmonic driving force the equation of motion becomes:

$$-M\frac{d^2x}{dt^2} - A|x|^{B}\frac{x}{|x|} + C\cos\omega t = 0\ . \qquad (2.3)$$

Figure 2.1 shows a few potential forms. For $B \rightarrow \cdot$, *we* get an inclined plane in each of the positive and negative x-directions. When the driving force vanishes this case can be treated quite simply even by analytic means. One would have the solution of the free-falling body both for positive and negative *x*-values and could match the solutions at the origin. The oscillation period would then increase with the square root of the amplitude. When $B = 1$ we again have the harmonic case, and in the absence of the driving force the oscillation period is independent of the amplitude. For large values of $B$ we approach the case of rigid reflecting walls, and the oscillation period becomes shorter with increasing amplitude.

It is interesting to study the effect of the driving force. It pushes the mass to and fro

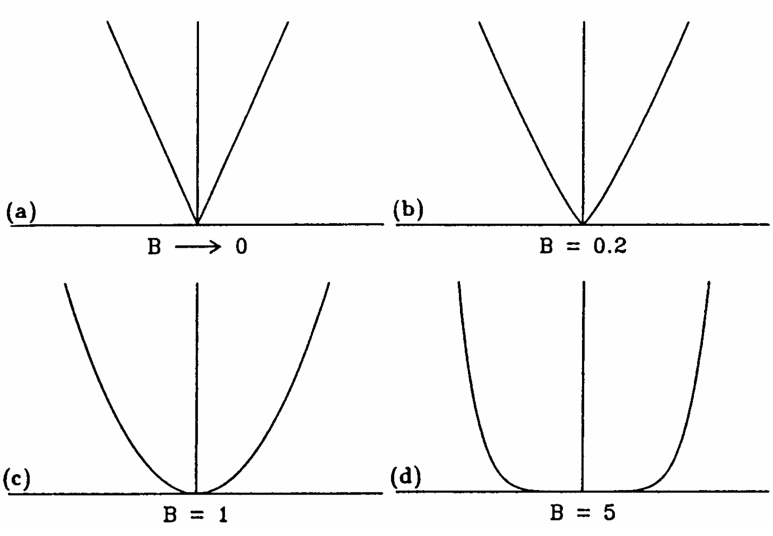


*Fig. 2.1a-d The potential V(x) for values of B.*

with a fixed predetermined frequency ω. What will happen if we start in the rest position and let the force operate on the mass? The driving force can operate effectively, i.e. impart energy, only if its frequency corresponds well with the oscillation frequency. For an anharmonic oscillation this will occur only when a certain amplitude is reached. Let us leave it to the computer to show us what will happen!

## 2.2 Numerical Treatment

### 2.2.1 Improvement of the Euler Method

The exercises in ***Session I*** have shown that the Euler method requires long computation times, in order to achieve accurate results. If the motion is complicated, e.g. by strong variations in the curvature of the solution, then the Euler method is no longer useful. We must therefore look for a better method.
The Runge-Kutta method, which we shall use here, is somewhat more difficult to understand than the Euler method. As a bridge to the Runge-Kutta method, we shall therefore discuss first a possible improvement of the Euler method. We again consider a first order differential equation of the type:

$$\frac{dy}{dt} = f(y,t) \ , \tag{2.4}$$

In order to be able to apply the technique of the Taylor expansion, we shall assume in what follows that $f(y, t)$ is differentiable a sufficient number of times with respect to $y$ and $t$. Then, writing down the first few terms of the Taylor expansion of $y(t)$ and applying (2.4), we obtain:

$$\begin{aligned} y(t+h) &= y(t) + h\frac{dy(t)}{dt} + \frac{h^2}{2}\frac{d^2y(t)}{dt^2} + O(h^3) \\ &= y(t) + hf(y(t),t) + \frac{h^2}{2}\frac{df(y(t),t)}{dt} + O(h^3) \ . \end{aligned} \tag{2.5}$$

To differentiate $f(y, t)$ with respect to $t$ we apply (1.2)

$$\frac{df(y(t),t)}{dt} = \frac{f(y(t+h),t) - f(y(t),t)}{h} + O(h) \ . \tag{2.6}$$

and replace the quantity $y(t+h)$ in (2.6) using (1.9). Substitution in (2.5) leads after some simple manipulation to

$$y(t+h) = y(t) + h\frac{f(y(t) + hf(y(t),t),t+h) + f(y(t),t)}{2} + O(h^3) \ . \tag{2.7}$$

We thus obtain an improved recursion formula, whose error term is now only of order $h^3$. One sees immediately where the improvement lies, compared with (1.9): instead of the gradient of the solution curve at the mesh point $t$, we now use the mean value of the gradients at the mesh points $t$ and $t + h$. In order to specify the gradient at the mesh point $t + h$, one actually needs to know already the solution curve at this point. Since this is not the case, it is substituted by the approximation for $y(t + h)$ obtained from the usual Euler formula (1.9). The error arising from this is of order $h^3$ and is accordingly of the same order as the error already incurred in (2.5).

For practical computation one often introduces the following abbreviations:

$$k^{(1)} = f\left(y(t),t\right),$$
$$k^{(2)} = f\left(y(t) + hk^{(1)},t + h\right) \tag{2.8}$$

In this abbreviated notation the formula for the improved the Euler method is as follows:

$$y(t+h) = y(t) + \frac{h}{2}\left(k^{(1)} + k^{(2)}\right) + O\left(h^3\right), \tag{2.9}$$

The generalisation to several coupled equations follows immediately, since we have nowhere explicitly used the fact that y(t) is a scalar function. We may regard (2.4) as a system of equations for vector functions:

$$\frac{dy_i(t)}{dt} = f_i\left(y_1(t), y_2(t), \ldots y_n(t); t\right), i = 1, \ldots n \tag{2.10}$$

From (2.8) we then write:

$$k_i^{(1)} = f_i\left(y_1(t), y_2(t), \ldots . y_n(t); t\right),$$
$$k_i^{(2)} = f_i\left(y_1(t) + hk^{(1)}, y_2(t) + hk_2^{(1)}, \ldots y_n(t) + k_n^{(1)}; t + h\right) \tag{2.11}$$

and hence from (2.9)

$$y_i(t+h) = y_i(t) + \frac{h}{2}\left(k_i^{(1)} + k_i^{(2)}\right) + O\left(h^3\right), i = 1, \ldots n, \tag{2.12}$$

### 2.2.2 The Runge-Kutta Method

The improved Euler method is better than the ordinary Euler method by one order in h. One can carry the improvement further, raising the order in h by the addition of further terms of the Taylor series (2.5). The best known of the methods obtained in this way is the Runge-Kutta method. Besides the gradients at the beginning and the end of the interval, it also uses the gradient at the middle of the interval, the values of the solution functions $y_i(t)$ at the middle and end of the interval being suitably extrapolated from the values at the beginning of the interval. Both in theoretical derivation and in practical application the Runge-Kutta method is similar to the improved Euler method. The error term in the recursion formula, however, is of order $h^5$, i.e. the Runge-Kutta method gives an improvement of a further two orders in *h* compared with the improved Euler method. What this means in practice, we shall see in the exercises.

The Runge-Kutta method uses four gradients, which one denotes by $k^{(l)}$ to $k^{(4)}$**.** The quantities $k^{(2)}$ and $k^{(3)}$ are gradients at the middle of the interval calculated in different ways. Their average value is used. Using the notation of (2.11), one calculates the gradients $k^{(1)}$ to $k^{(4)}$ as follows:

$$k_i^{(1)} = f_i\left(y_1(t), \ldots . y_n(t); t\right), \tag{2.13a}$$

$$k_i^{(2)} = f_i\left( y_1(t) + \frac{h}{2}k_1^{(1)}, \ldots y_n(t) + \frac{h}{2}k_n^{(1)}; t + \frac{h}{2}\right) \tag{2.13b}$$

$$k_i^{(3)} = f_i\left( y_1(t) + \frac{h}{2}k_1^{(2)}, \ldots y_n(t) + -k_n^{(2)}; t + \frac{h}{2}\right) \tag{2.13c}$$

$$k_i^{(4)} = f_i\left(y_1(t) + hk_1^{(3)}, \ldots y_n(t) + hk_n^{(3)}; t + h\right) \tag{2.13c}$$

The recursion formula of the Runge-Kutta method then becomes:

$$y_i(t+h) = y_i(t) + h\left(\frac{k_i^{(1)}}{6} + \frac{k_i^{(2)}}{3} + \frac{k_i^{(3)}}{3} + \frac{k_i^{(4)}}{6}\right) + O(h^5), i = 1,...n, \qquad (2.14)$$

For the derivation of the method refer to **[E. Issacson,H.B. Keller: Analysis of Numerical Functions (John Wiley and Sons, Inc., New York, 1966].**

The Runge-Kutta method was for a long time the most frequently used and best method for the numerical solution of ordinary differential equations, and even today it is often used. In recent year, however, a number of other methods, especially so-called predictor-corrector methods, have achieved prominence. Since the Runge-Kutta method is easy to understand and simple to program, we shall use it here.

## 2.3 Programming

Let us once again consider the system of differential equations (1.12) presented in SESSION I. By a similar procedure we obtain from (2.3) the system of equations.

$$\frac{dy_1(t)}{dt} = y_2(t) \qquad (2.15a)$$

$$\frac{dy_2(t)}{dt} = -\frac{A}{M}|y_1(t)|^B \frac{y_1(t)}{|y_1(t)|} + \frac{C}{M}\cos \omega t. \qquad (2.15b)$$

The right-hand sides of the system of differential equations (2.10) are accordingly:

$$f_1(y_1(t), y_2(t); t) = y_2(t) \qquad (2.16a)$$

$$f_2(y_1(t), y_2(t); t) = -\frac{A}{M}|y_1(t)|^B \frac{y_1(t)}{|y_1(t)|} + \frac{C}{M}\cos \omega t. \qquad (2.16b)$$

Proceed then in a similar manner to Session I.

## 2.4 Exercises

**2.4.1** Test the accuracy of the Runge-Kutta method in the case of a free harmonic oscillation $B = 1$ and in the case of a strongly anharmonic oscillation ($B = 5$).
*Hint:* If the mesh width h is too big, very strong forces can lead to exponent overflow. In this case abort the program and restart it with new parameters!

**2.4.2** Compare the solution curves of the anharmonic oscillations calculated with various values for *B*.

**2.4.3** Switch on the driving force, and study the resonance effects in harmonic and anharmonic oscillations.

## 2.5 Solutions to the Exercises

**2.5.1** It is surprising that the Runge-Kutta method with 5 mesh points per oscillation period already gives a profile similar to the true solution curve for the harmonic oscillation. The error in this case is essentially to he found in the damping: whereas the true solution without friction and without driving force is un-damped, the approximate solution shows damping. With 10 mesh points per oscillation period, scarcely any visible error remains in the graphical output.

For anharmonic oscillations with B >2 the rapid variations in the curvature of the solution curve impose high demands on the solution method. With 10 mesh points per oscillation period the inaccuracy for B = 5 is still easily detected. Increasing the number of mesh points by a factor of 5 or 10, however, leads even here to a satisfactory accuracy for the graphical output.

Figure 2.2 shows a harmonic oscillation calculated by the Runge-Kutta method with only 5 mesh points per oscillation period.

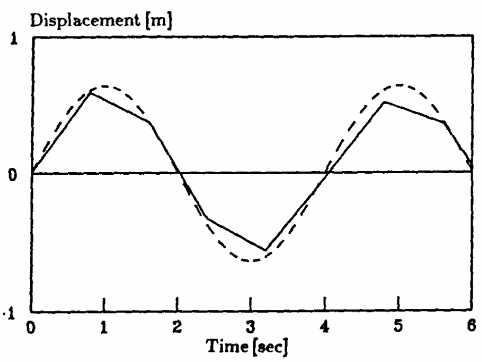


*Fig. 2.2. Harmonic oscillation calculated by the Runge-Kutta method using only 5 mesh points per oscillation period; the broken line shows the true solution for comparison*

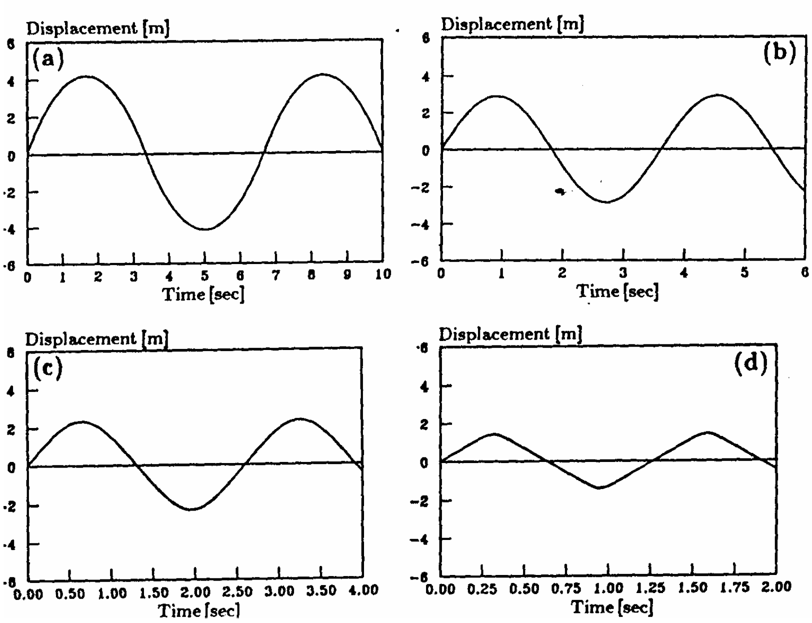


*Fig.2.3a-d Anharmonic and harmonic oscillations:*
*a) Anharmonic oscillation with B = 0.00001,*
*b) Harmonic oscillation,*
*c) Anharmonic oscillation with B = 2,*
*d) Strongly anharmonic oscillation with B = 10*

**2.5.2** Figure 2.3 shows oscillation curves for *B = 0.00001, B = 1, B = 2* and *B = 10.* When *B = 0.00001* the restoring force is, apart from a change of sign at *x = 0*, independent of the displacement (the value *B = 0.00001* stands for the value B = 0, which is not allowed by the program). The oscillation curve accordingly consists of a sequence of parabolas. For comparison the harmonic oscillation *(B = 1)* is shown. When *B = 2* the point mass moves in a cubic potential. At the turning points the restoring force is stronger than in the harmonic oscillation. The reversal of motion is accordingly sharper. This effect is shown still more strongly in the case $B = 10$. Here the potential climbs so steeply that the reversal of motion assumes the character of an elastic reflection. In the remaining part of the motion the forces are then comparatively small and have only a slight effect on the velocity of the point mass.

**2.5.3** The frequency of the free harmonic oscillation is ω = *SQRT*(*A/-M*). If one uses this value in the driving force one has the appropriate starting condition for an oscillation with increasing amplitude (resonance). As friction is ignored in our calculation the amplitude of the oscillation will increase without limit.

When the frequency of the driving force differs slightly from the resonance frequency, e.g. by about 10%, then a beat occurs: depending on the phase difference between the oscillation and the driving force, energy is fed into, or taken from, the oscillation; see for example Fig. 2.4.

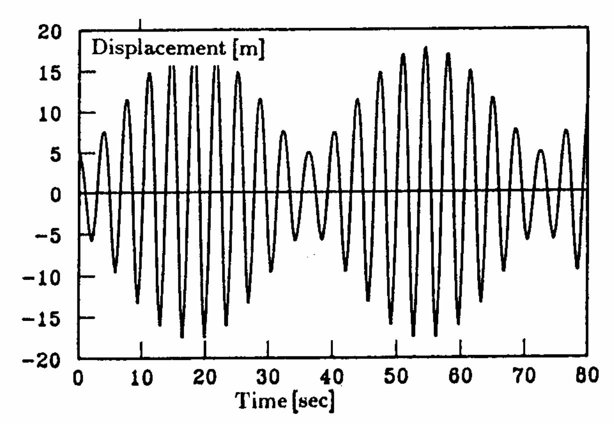


*Fig.2.4. Forced harmonic oscillations. The frequency of the driving force is about 10% greater than the frequency of the free oscillation.*

With a potential, which is only slightly anharmonic (e.g. $B = 1.1$) one can also obtain beats. In this case the amplitude of the oscillation cannot increase without limit, as the anharmonicity ensures that the driving force and the velocity are from time to time in opposition. With stronger anharmonicity a pronounced beat no longer occurs, because the driving force and the oscillation fall so quickly out of phase. An extreme case is shown in Fig. 2.5. A strong driving force pushes the point mass to and fro. In its motion it occasionally runs into the potential wall ($B = 5$) and gets reflected.

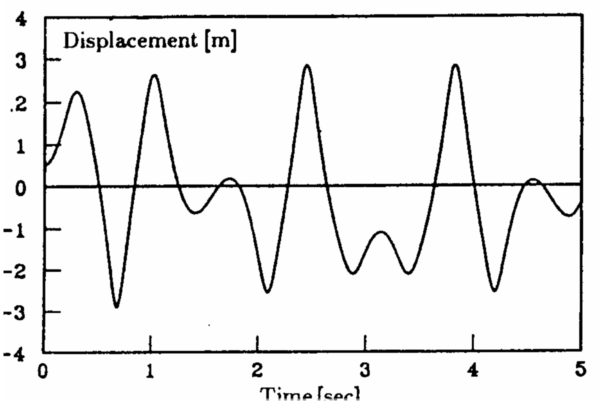


*Fig.2.5. Point mass under the influence of a strong driving force and a strongly anharmonic (B=5) restoring force.*