SESSION III

3. Coupled Harmonic Oscillations

3.1 Formulation of the Problem

Coupled oscillations occur in many regions of physics. The Raman and infrared spectra, for example, have their origin in the coupled oscillations of atoms within the molecule. The analysis of these oscillations gives information not only on the structure of the molecule but also on the binding forces. Coupled oscillations occur in technology when machine components run roughly. In the design of anchorages, resonance frequencies as well as damping play an important role.

In this chapter we consider as a simple example the coupled harmonic oscillation of two point masses, each weighing *1 kg*. They are linked to their rest positions by harmonic restoring forces. The displacements from the rest positions are denoted by x_l and x_2 If the two displacements axe not equal, an additional harmonic force appears, which couples the two bodies to one another. The equation of motion is the following coupled system of differential equations:

$$-M\frac{d^2x_1}{dt^2}-C_1x_1+C(x_2-x_1)=0 \ ,$$
$$-M\frac{d^2x_2}{dt^2}-C_2x_2-C(x_2-x_1)=0 \, . \qquad (3.1)$$

In particular, these equations describe the motion of sympathetic pendulums (see Fig. 3.1). When the displacements are small one has approximately harmonic restoring forces. The force constants C_l and C_2 are, determined by the lengths of the pendulums l_i and the force of gravity Mg. The coupling spring has the force constant C. If the pendulum lengths are not equal, then C_l # C_2, i.e. the sympathetic pendulums are "out of tune".

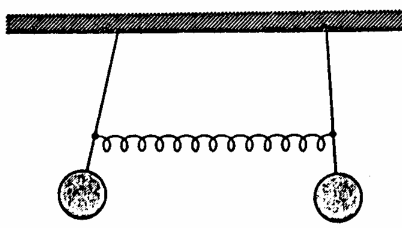

Fig. 3.1. Sympathetic pendulums

With the abbreviations:

$$A_{11}=-(C_1+C)/M \ , \quad A_{12}=C/M$$
$$A_{21}=C/M \, , \qquad A_{22}=-(C_2+C))/M \qquad (3.2)$$

Equation (3.1) can be brought into the general form

$$\frac{d^2x_1}{dt^2}=A_{11}x_1+A_{12}x_2$$
$$\frac{d^2x_2}{dt^2}=A_{21}x_1+A_{22}x_2 \, . \qquad (3.3)$$

Generalisation to m bodies gives

$$\frac{d^2x_i}{dt^2}=\sum_{j=1}^{m}A_{ij}x_j \ , i=1,...m \qquad (3.4)$$

If necessary frictional and driving forces can be added on the right hand sides of (3.4). We have at the outset mentioned technical applications. We recommend to our readers an interesting example for private study. It is the "earthquake-proof skyscraper". In earthquake areas such as, e.g. Tokyo, it has been discovered that skyscrapers should be built not too rigid, if they are to withstand severe earthquakes. It is better to build structures capable of oscillation, and to provide for adequate damping. By the methods, which we have now acquired, one can simulate on the computer the behaviour of a multi-storeyed building during an earthquake. One makes the approximate assumption that the whole mass of the building is concentrated in the ceilings (see Fig. 3.2). The steel framework of the building furnishes the restoring forces, when ceilings of successive storeys undergo different lateral displacements. The earthquake is simulated, by causing the ground floor to move to and fro in a stipulated manner. The program developed in this chapter can be extended with little effort to calculate the coupled motions of the ceilings and show the solutions $x_i(t)$ on the screen. It will be observed that the building can undergo dangerous characteristic "eigen-oscillations" if no damping is present. With appropriate oscillation damping built in, however, it can be shown on the screen that a skyscraper can well withstand moderately strong earthquakes.

3.2 Numerical Method

As already mentioned in ***Exercise I*** (Sect.1.2.1), a coupled system of second order differential equations can be reduced to a coupled system of first order differential equations. Using the notation:

$$y_i(t) = x_i(t), i = 1,...n'$$
$$y_{i+n'}(t) = \frac{dx_i(t)}{dt}, i = 1,...n' \qquad (3.5)$$

one obtains from (3.4) the system of equations (n =2n')

$$\frac{dy_i(t)}{dt} = f_i(y_i(t),.....y_n(t);t), i = 1,...n \qquad (3.6)$$

with

$$f_i = y_{n'+i}(t), \qquad f_{n'+i} = \sum_{j=1}^{n'} A_{ij} y_j(t), i = 1,...n' \qquad (3.7)$$

In order to solve the system of differential equations (3.6), we shall employ the Runge-Kutta method presented in, ***Exercise II*** (Sect. 2.2.2).

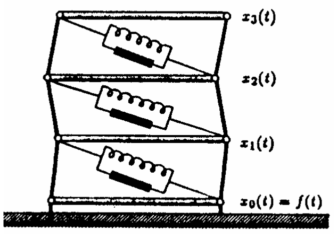

Fig. 3.2. Simulation of the oscillation of a building during an earthquake: the mass is concentrated in the ceilings, restoring and frictional forces act diagonally across the storeys, the ground floor in oscillated to and fro by the earthquake according to the function f(t)

3.3 Programming

The functions f_i on the right-hand sides of our system of equations (3.6) are, according to (3.7) and (3.2),

$$f_1 = y_3(t), \tag{3.8a}$$

$$f_2 = y_4(t), \tag{3.8b}$$

$$f_3 = -\left(\frac{C_1}{M} + \frac{C}{M}\right) y_1(t) + \frac{C}{M} y_2(t), \tag{3.8c}$$

$$f_4 = \frac{C}{M} y_1(t) - \left(\frac{C_2}{M} + \frac{C}{M}\right) y_2(t). \tag{3.8d}$$

Proceed in a similar manner as in Sessions I and II.

3.4 Exercises

3.4.1 Study the nature of the oscillations of sympathetic pendulums ($C_l = C_2$) with weak coupling ($C = C_1 /10$) and with strong coupling ($C = 10C_1$).

3.4.2 Choose a moderately strong coupling parameter C and unequal force constants C_1 , C_2. Study the energy transfer between the two point masses for various initial conditions. Are there initial conditions for which no transfer of energy takes place?

3.5 Solutions to the Exercises

NOTE: Output may vary according to input parameters.

3.5.1 With weak coupling as a rule one pendulum transfers energy to the other pendulum until it no longer has any left. Then the process is reversed, see Fig. 3.7. One should notice the relative phases of the oscillation curves. The pendulum which, is delivering the energy has a phase lead of up to 90". During the course of the energy transfer the phase lead is decreased and eventually becomes negative, when the driving pendulum becomes the driven. No transfer of energy takes place if the two pendulums are oscillating with equal amplitude either in phase or 180' out of phase.

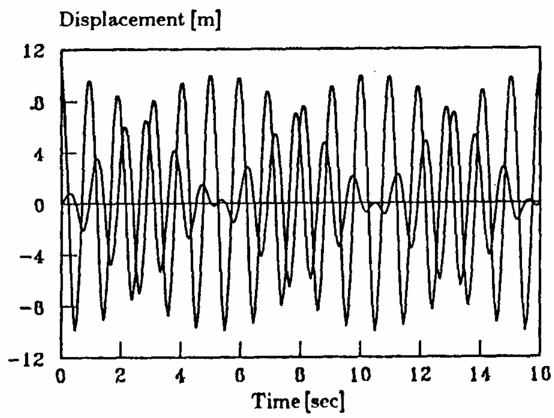

Fig.3.7.Oscillations of sympathetic pendulums with weak coupling ($C_l = C_2 = 39.5\ Nm^{-1}$, $C = 3.95\ Nm^{-1}$).

With strong coupling the two pendulums can oscillate relative to one another with a short oscillation period, whilst their combined centre of mass executes a pendulum motion with longer oscillation period, see Fig. 3.8.

3.5.2 With unequal force constants C_1 and C_2 (unequal pendulum lengths) the energy is now completely transferred only in one direction (see Fig. 3.9). In the other direction, the transfer of energy in only partial

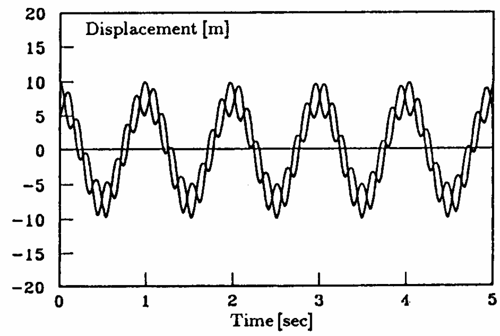

Fig.3.8. Oscillations of sympathetic pendulum with strong coupling (C_1 = C_2 = 39.5 Nm^{-1}, C = 395 Nm^{-1})

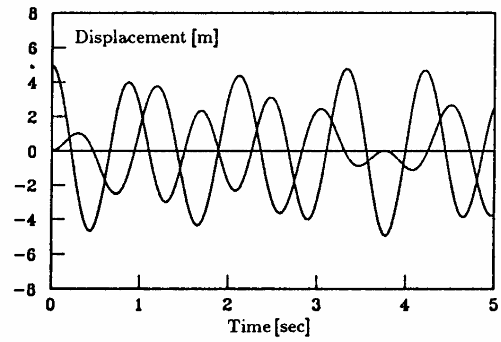

Fig.3.9. Coupled oscillation with C_1 = 40, C_2=30 and C = 10 Nm^{-1}

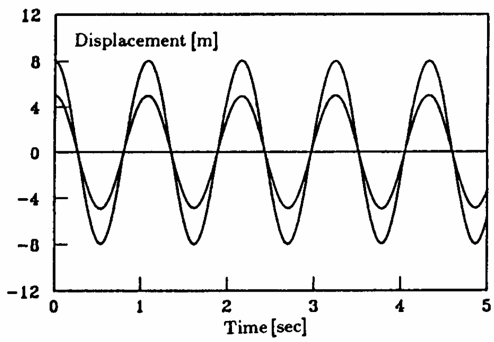

Fig.3.10. One of the two eigen-oscillations of the system of two point masses; the force constants are the same as in Fig. 6.9

There are initial conditions for which no energy transfer takes place. Fig.3.10 shows an oscillation with the same force constants as in Fig.3.9, but with the initial conditions $y_1(0) = 5m$, $y_2(0) = 8m$, $y_3(0) = y_4(0) = 0$. Such an oscillation is called an eigen-oscillation of the system. In our example there are two eigen-oscillations. Their oscillation periods are about 1.08 and 0.84s. The search for the initial conditions leading to eigen-oscillations is equivalent to the search for so-called normal coordinates. The latter are obtained by an orthogonal transformation from the coordinates used by us. The transformation to normal coordinates causes the non-diagonal elements of the matrix (A_{ij}) of the equations (3.4) to vanish.