Chapter 14 Section 3

(11) *Because* $\frac{\partial M}{\partial y} = 2x = \frac{\partial N}{\partial x} \rightarrow$ *vector field is conservative* $\therefore$ *independent of path*

$\vec{F}(x,y) = 2xy\mathbf{i} + x^2\mathbf{j} \quad \vec{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} \Leftrightarrow \vec{r}(t) = t\mathbf{i} + t^2\mathbf{j} \;\; 0 \le t \le 1$

$\therefore \vec{F}(x(t), y(t)) = 2t^3\mathbf{i} = t^2\mathbf{j} \qquad d\vec{r} = \mathbf{i} + 2t\mathbf{j}$

$\vec{F}\cdot d\vec{r} = 2t^3 + 2t^3 = 4t^3 \qquad \int_C \vec{F}\cdot d\vec{r} = \int_0^1 4t^3 = 1$

(15a) $\frac{\partial M}{\partial y} = 2y = \frac{\partial N}{\partial x} \therefore$ *independent of path*

Since $(0,0) \rightarrow (4,4)$ *choose* $\mathbf{PQ} = \langle x-0, y-0\rangle = t\langle 4,4\rangle = \langle 4t, 4t\rangle \Rightarrow x = y = 4t;\; 0 \le t \le 1$

$dx = dy = 4$

$\therefore \int_C y^2dx + 2xydy = \int_0^1 (64t^2 + 128t^2)dt = 64$

(b) $(-1,0) \rightarrow (1,0)$ *choose* $\mathbf{PQ} =< x-1, y >=< t,0 >\Rightarrow x = t+1, y = 0$

$\therefore \int_C y^2dx + 2xydy = 0$

$(c-d)$ *Since the paths are closed and the vector field is conservative the integral is zero*

(17) $\frac{\partial N}{\partial x} = 2x = \frac{\partial M}{\partial y} \;\therefore$ *conservative field*

Since $\vec{F}(x,y) = \nabla f$

$\int Mdx = \int Ndy \Leftrightarrow x^2y + h(x) = x^2y + (1/3)y^3 + g(y)$

$\therefore \;\; \nabla f = x^2y + (1/3)y^3 + K \quad (K \textit{ is any constant})$

$\int_C \vec{F}\cdot d\vec{r} = \left[x^2y + (1/3)y^3\right]_{(5,0)}^{(0,4)} = \frac{64}{3}$

(19) $\vec{F}(x,y,z) = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$

$\frac{\partial M}{\partial y} = z = \frac{\partial N}{\partial x},\; \frac{\partial N}{\partial z} = x = \frac{\partial P}{\partial y},\; \frac{\partial M}{\partial z} = y = \frac{\partial P}{\partial x} \qquad \therefore$ *conservative vector field*

$\nabla f = xyz + k$

(a) *Since* $\vec{r}(t) = t\mathbf{i} + 2\mathbf{j} + t\mathbf{k} \;\; 0 \le t \le 4 \qquad \vec{r}(0) = 2\mathbf{j} \Rightarrow$ *starting point* $= (0,2,0)$

$\vec{r}(4) = 4\mathbf{i} + 2\mathbf{j} + 4\mathbf{j} \Rightarrow$ *ending point* $= (4,2,4)$

$\int_C \vec{F}\cdot d\vec{r} = [xyz]_{(0,2,0)}^{(4,2,4)} = 32$

(21) $\vec{F}(x,y,z) = (2y+x)\mathbf{i} + (x^2 - z)\mathbf{j} + (2y-4z)\mathbf{k}$

$\frac{\partial M}{\partial y} = 2 \neq 2x = \frac{\partial N}{\partial x}$ $\therefore$ *field is path* **dependent**

(a) $\vec{r}(t) = t\mathbf{i} + t^2\mathbf{j} + \mathbf{k}\ \ 0 \le t \le 1$

$\vec{F}(x(t), y(t), z(t)) = (2t^2 + t)\mathbf{i} + (t^2 - 1)\mathbf{j} + (2t^2 - 4)\mathbf{k}$

$d\vec{r} = \mathbf{i} + 2t\mathbf{j}$ $\quad\quad\quad \vec{F} \cdot d\vec{r} = 2t^3 + 2t^2 - t$

$\int_0^1 (2t^3 + 2t^2 - t)dt = \frac{2}{3}$

(35) $\vec{F}(x,y) = 9x^2y^2\mathbf{i} + (6x^3y - 1)\mathbf{j}$

$\frac{\partial M}{\partial y} = 18x^2y = \frac{\partial N}{\partial y}$ *conservative field*

$\nabla f = 3x^3y^2 + h(x) = 3x^3y^2 - y + g(y) \Rightarrow \nabla f = 3x^3y^2 - y + K$

$W = \left[3x^3y^2 - y\right]_{(0,0)}^{(5,9)} = 30366$

(37) *choose the path traced out by,*

$\vec{r}(t) = 2\cos t\mathbf{i} + 2\sin t\mathbf{j};$ $\quad\quad \vec{r}\,'(t) = -2\sin t\mathbf{i} + 2\cos t\mathbf{j}$

$\left\|\vec{r}\,'(t)\right\| = \sqrt{4\sin^2 t + 4\cos^2 t} = 2$

$\therefore\ \vec{T}(t) = -\sin t\mathbf{i} + \cos t\mathbf{j};$ $\quad\quad \vec{T}\,'(t) = -\cos t\mathbf{i} - \sin t\mathbf{j};$ $\quad\quad \left\|\vec{T}\,'(t)\right\| = 1$

$\therefore\ \vec{N}(t) = -\cos t\mathbf{i} - \sin t\mathbf{j}$

The force is towards the center, that is the purpose of finding $\vec{N}(t)(=\vec{F}(x,y))$.

$\therefore \quad \vec{F} \cdot d\vec{r} = 2\sin t\cos t - 2\sin t\cos t = 0 \Rightarrow W = 0$

You can also use the fact that the field is conservative and closed to conclude that the work done is equal to zero.