(25) $\iint_R f(x,y)dA = \int_0^{\pi/2}\int_0^1 r^3 \sin\theta\cos\theta dr d\theta = \frac{1}{4}\int_{\theta=0}^{\theta=\pi/2} \sin\theta\cos\theta d\theta$

let $u = \sin\theta \Rightarrow du = \cos\theta d\theta$

$\therefore u(\pi/2) = 1, u(0) = 0$

$\therefore \frac{1}{4}\int_{\theta=0}^{\pi/2} \sin\theta\cos\theta d\theta = \frac{1}{4}\int_{u=0}^{u=1} u du = \frac{1}{8}$

(27) $V = \int_0^{2\pi}\int_0^5 r^2 dr d\theta = \frac{125}{3}\int_0^{2\pi} d\theta = \frac{125\pi}{6}$

(29) $x^2 - 4x + y^2 = 0 \Leftrightarrow r^2\cos^2\theta - 4r\cos\theta + r^2\sin^2\theta = 0$

$= r^2(\cos^2\theta + \sin^2\theta) - 4r\cos\theta = 0$

$\Leftrightarrow r^2 = 4r\cos\theta \Leftrightarrow r = 4\cos\theta$

$\therefore V = 2\int_0^{\pi/2}\int_0^{4\cos\theta} r\sqrt{16 - r^2}\, dr d\theta = -\int_0^{\pi/2}\int_{u=16}^{16\sin^2\theta} \sqrt{u}\, du d\theta = \frac{-2}{3}\int_0^{\pi/2} \left[u^{3/2}\right]_{16}^{16\sin^2\theta} d\theta$

let $u = 16 - r^2 \Rightarrow du = -2r dr \quad \therefore u(4\cos\theta) = 16(1 - \cos^2\theta) = 16\sin^2\theta$

u(0)=16

$= \frac{128}{3}\int_0^{\pi/2} (1 - \sin^3\theta) d\theta = \frac{128}{3}[\theta]_0^{\pi/2} - \frac{128}{3}\int_0^{\pi/2} \sin^3\theta d\theta$

$= \frac{64\pi}{3} - \frac{128}{3}\left(\frac{2}{3}\right) = \frac{64}{9}(3\pi - 4)$

Note: The sine integral was computed using Wallis's Formula (see web page for more info).

(31) Give $z = \sqrt{16 - x^2 - y^2} = \sqrt{16 - r^2}$

$V_{hemisphere} = \int_0^{2\pi}\int_0^4 r\sqrt{16 - r^2}\, dr d\theta = \frac{-1}{2}\int_0^{2\pi}\int_{u=16}^{u=0} \sqrt{u}\, du d\theta = \frac{64}{3}\int_0^{2\pi} d\theta = \frac{128\pi}{3}$

let $u = 16 - r^2 \Rightarrow du = -2r dr \quad \therefore u(4) = 0, u(0) = 16$

$V_{cylinder} = \int_0^{2\pi}\int_0^a r\sqrt{16 - r^2}\, dr d\theta = \frac{-1}{2}\int_0^{2\pi}\int_{u=16}^{u=16-r^2} \sqrt{u}\, du d\theta = \frac{1}{3}\int_0^{2\pi} \left[16^3 - (16 - a^2)^{3/2}\right] d\theta$

$= \frac{128\pi}{3} - \frac{2\pi(16 - a^2)^{3/2}}{3}$

Note that $\frac{1}{2}V_{hemisphere} = \frac{1}{2}(\frac{128\pi}{3}) = \frac{64\pi}{3}$

$V_{hemisphere} - V_{cylinder} = \frac{128\pi}{3} + \frac{2\pi(16 - a^2)^{3/2}}{3} - \frac{128\pi}{3} = \frac{2\pi(16 - a^2)^{3/2}}{3}$

thus, $V_{(1/2)hemisphere} = V_{hemisphere} - V_{cylinder} \Leftrightarrow \frac{64\pi}{3} = \frac{2\pi(16 - a^2)^{3/2}}{3} \Rightarrow a = \sqrt{16 - (32)^{3/2}}$