

(77) Average Displacement : A damping force affects the vibration of a spring so that the displacement of the spring is $y = e^{-4t}(\cos 2t + 5\sin 2t)$. Find the average value of y on the interval from $t = 0$ to $t = \pi$.

Recall that the average value of a function $f(x)$ is defined to be $\frac{1}{b-a} \int_a^b f(x) dx$

Therefore, the average value of $y = e^{-4t}(\cos 2t + 5\sin 2t)$ from $t = 0$ to $t = \pi$ is the integral

$$\frac{1}{\pi} \int_0^{\pi} e^{-4t} (\cos 2t + 5\sin 2t) dt = \frac{1}{\pi} \left[\int_0^{\pi} e^{-4t} \cos 2t dt + 5 \int_0^{\pi} e^{-4t} \sin 2t dt \right]$$

Using the formulas from exercise 65 and 66 the integrals are as follows,

$$\frac{1}{\pi} \int_0^{\pi} e^{-4t} \cos 2t dt = \frac{1}{\pi} \left[\frac{e^{-4t} (-4 \cos 2t + 2 \sin 2t)}{20} \right]_0^{\pi} = \frac{1}{20\pi} [e^{-4\pi} (-4 \cos 2\pi) - (-4 \cos 0)] = \frac{1}{5\pi} (-e^{-4\pi} + 1)$$

$$\frac{5}{\pi} \int_0^{\pi} e^{-4t} \sin 2t dt = \frac{5}{\pi} \left[\frac{e^{-4t} (-4 \sin 2t - 2 \cos 2t)}{20} \right]_0^{\pi} = \frac{1}{4\pi} [e^{-4\pi} (-2 \cos 2\pi) - (-2 \cos 0)] = \frac{1}{2\pi} (-e^{-4\pi} + 1)$$

$$\therefore \frac{1}{\pi} \left[\int_0^{\pi} e^{-4t} \cos 2t dt + 5 \int_0^{\pi} e^{-4t} \sin 2t dt \right] = \frac{1}{5\pi} (-e^{-4\pi} + 1) + \frac{1}{2\pi} (-e^{-4\pi} + 1) = \frac{7}{10\pi} (1 - e^{-4\pi})$$