$$\ln_a x = \frac{\ln x}{\ln a}$$

Proof

$$Since \quad a^{\ln_a x} = x$$

$$\ln(a^{\ln_a x}) = \ln x$$

$$Since \quad \ln b^x = x \ln b \Rightarrow \ln(a^{\ln_a x}) = \ln x \Leftrightarrow (\ln_a x)(\ln a) = \ln x$$

$$\therefore \qquad \ln_a x = \frac{\ln x}{\ln a} \qquad Q.E.D.$$

$$\ln b^x = x \ln b$$

Proof

$$Note: \; \frac{d[\ln b^x]}{dx} = \frac{b^x(\ln b)}{b^x} = \ln b$$

$$\Rightarrow \qquad \int d[\ln b^x] = \int (\ln b)dx \Leftrightarrow \ln b^x = (\ln b)x + C$$

$$if \qquad x = 0:$$

$$\ln b^x = \ln b^0 = \ln 1 = 0 \qquad and \qquad (\ln b)x + C = 0 + C$$

$$\therefore \qquad 0 = 0 + C \qquad \Rightarrow \qquad C = 0$$

$$\therefore \qquad \ln b^x = (\ln b)x. \qquad Q.E.D.$$

$$\frac{d[a^x]}{dx} = a^x(\ln a)$$

Proof

$$Since \quad e^{\ln a} = a \qquad \Rightarrow \qquad \ln a^x = \ln(e^{(\ln a)x})$$

$$\therefore \qquad \frac{d[a^x]}{dx} = \frac{d[e^{(\ln a)x}]}{dx}$$

$$let \qquad u = (\ln a)x \Rightarrow du = (\ln a)dx \Leftrightarrow \frac{du}{dx} = \ln a$$

$$By\,the\,Chain\,Rule \quad \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

$$\frac{d[e^{(\ln a)x}]}{dx} = \frac{d[e^u]}{du}\frac{du}{dx} = e^u \ln a = a^x(\ln a) \qquad Q.E.D.$$