Math Challenge Problem - January 2005

Results and Solutions

Problem: What's the units digit of the real number

$$(7 + 5\sqrt{2})^{2005}\,?$$

(Note that the units digit is the first digit to the left of the decimal place.)

Solution: Let $a = (7 + 5\sqrt{2})^{2005}$ and $b = (7 - 5\sqrt{2})^{2005}$. The reason we introduce b is because $a + b$ is an integer and b is a very small number: since $5\sqrt{2} = \sqrt{50}$ and $7 < \sqrt{50} < 8$, it follows that $-1 < 7 - 5\sqrt{2} < 0$ which implies that $-1 < b < 0$. This means that if we can find the last digit of $a + b$, we have found the units digit of a because $a = (a + b) + (-b)$ and $-b$ is a very small positive number.

Now expanding using the Binomial Theorem, we get

$$a + b = \sum_{i=0}^{2005} \binom{2005}{i} 7^{2005-i}(5\sqrt{2})^i + \sum_{i=0}^{2005} \binom{2005}{i} 7^{2005-i}(-5\sqrt{2})^i.$$

Looking at the terms in these two series, we see that they are the same when i is even, but have opposite sign when i is odd (and hence sum to zero). So we can express this as twice the sum over the even values of i, say $i = 2j$. Therefore

$$a + b = 2\sum_{j=0}^{1002} \binom{2005}{2j} 7^{2005-2j}(5\sqrt{2})^{2j} = \sum_{j=0}^{1002} 2\binom{2005}{2j} 7^{2005-2j} 50^j.$$

Now all of the terms in this series are divisible by 10 except for the first term ($j = 0$). So the last digit of $a+b$ is equal to the last digit of $2 \cdot 7^{2005}$. Since $7^4 = 2401$, the last digit of 7^{4k} is 1, for any positive integer k. So finally, $2 \cdot 7^{2005} = 2 \cdot 7 \cdot 7^{2004} = 14 \cdot 7^{4 \cdot 501}$ has 4 as its last digit. Therefore the last digit of $a + b$ is 4 and that is the units digit of a.

Two solutions were submitted to this problem. One had the correct answer and one did not. The prize-winner for this month is **Erik Youngson**.