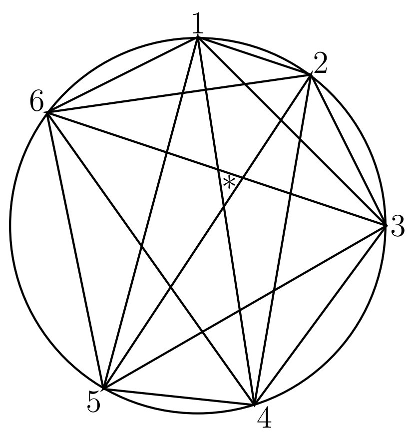
Math Challenge Problem - November 2004

Results and Solutions

Problem: n points are given on the circumference of a circle, and the chords determined by them are drawn. Suppose further that no 3 chords meet a common point inside the circle. Let $t(n)$ be the number of triangles determined by the chords which have all 3 vertices inside the circle. The diagram shows that $t(6) = 1$.

Find $t(2004)$.



Solution: Each triangle that we're trying to count is determined by 3 chords. Since all 3 vertices of the triangle lie inside the circle, these three chords must have distinct endpoints. That is, the three chords are determined by 6 different points on the circumference. So every triangle corresponds to a set of 6 points on the circumference.

On the other hand, every set of 6 points on the circumference determines exactly one triangle with all 3 vertices inside the circle. To see this, suppose that the points are $P_1, P_2, P_3, P_4, P_5, P_6$, in clockwise order. Then to form a triangle with all 3 vertices inside the circle, the 3 chords that we form must all intersect inside the circle. That can only happen in one way: we must select the chords P_1P_4, P_2P_5 and P_3P_6.

Therefore there is a one-to-one correspondence between the triangles we're counting and sets of 6 points on the circumference of the circle. Therefore $t(n) = \binom{n}{6}$ and, in particular, $T(2004) = \binom{2004}{6} = 89,289,444,610,966,600$ (if anyone cares).

Only one solution was submitted to this problem, but it was correct and well-written. The winner of this month's prize is **Daisydee Bautista**.

Follow-up Question: Let $s(n)$ be the number of points inside the circle at which chords intersect. For example, the diagram shows that $s(6) = 15$.

Find $s(2004)$.