Math Challenge Problem - September 2004

Results and Solutions

Problem: A triangle like the one shown is constructed with the odd numbers from 1 to 999 in the first row. Each number in the triangle, except those in the first row, is the sum of the two numbers above it. What number occupies the lowest vertex of the triangle?

$$\begin{array}{ccccccccc} 1 & & 3 & & 5 & & 7 & & 9 \\ & 4 & & 8 & & 12 & & 16 & \\ & & 12 & & 20 & & 28 & & \\ & & & 32 & & 48 & & & \\ & & & & 80 & & & & \end{array}$$

Solution: Let $a_{i,j}$ be the jth entry (from the left) in the ith row (from the top). We're given that $a_{1,j} = 2j-1$ for $j = 1, 2, \ldots, 500$ and that $a_{i+1,j} = a_{i,j} + a_{i,j+1}$ for all $i \geq 1$. Using this notation, the problem asks us to find $a_{500,1}$. After looking at smaller cases and searching for patterns, we may observe that the leftmost entry in the ith row is $2^{i-1}i$ and that consecutive entries in this row differ by 2^i. This leads us to make the following guess, which we'll prove by induction.

Claim: $a_{i,j} = 2^{i-1}i + 2^i(j-1) = 2^{i-1}(i+2j-2)$.

Proof: We'll prove it by induction on i. For $i = 1$, we have $a_{1,j} = 2^0(1+2j-2) = 2j-1$, which is true. Assume that the claim is true for $i = k$, that is,

$$a_{k,j} = 2^{k-1}(k+2j-2).$$

But then

$$\begin{aligned} a_{k+1,j} &= a_{k,j} + a_{k,j+1} = 2^{k-1}(k+2j-2) + 2^{k-1}(k+2(j+1)-2) \\ &= 2^{k-1}(2k+4j-2) = 2^k(k+2j-1) = 2^k(k+1+2j-2). \end{aligned}$$

So the result holds for $i = k+1$ whenever it holds for $i = k$. Therefore the claim is true for all i by mathematical induction. □

Finally, the answer to the problem is given by $a_{500,1}$ which equals $500 \cdot 2^{499}$.

Four solutions were submitted to this problem. Three were correct and the fourth contained a minor arithemetic error. Correct solutions were submitted by **Joel Peters-Fransen**, **Justyna Swistak** and **Erik Youngsen**. The prize-winner, Joel, was selected by random draw.

Other Approaches: There are many other ways to prove the result; most use induction in some way.

Erik noticed that if you take the average (mean) of the values in each row, then the mean of the values in row $i+1$ is twice the mean of the ith row. This can be proved by induction and then used to get the result.

Joel worked out a general formula for the bottom number which works no matter what numbers are in the first row. To see how his method works, let's look at a small example with numbers a, b, c, d, e in the first row:

$$\begin{array}{ccccccccc} a & & b & & c & & d & & e \\ & a+b & & b+c & & c+d & & d+e & \\ & & a+2b+c & & b+2c+d & & c+2d+e & & \\ & & & a+3b+3c+d & & b+3c+3d+e & & & \\ & & & & a+4b+6c+4d+e & & & & \end{array}$$

If one notices that the coefficients of a, b, c, d, e in the last row are the binomial coefficients $\binom{4}{0}$, $\binom{4}{1}$, $\ldots$, $\binom{4}{4}$, then it's possible to prove that, in general, if we start with a row of n numbers, we'll have a sum involving binomial coefficients $\binom{n-1}{i}$. Joel proved this by induction. (It's also possible to give a direct counting argument for this fact.) To finish off the given question, we now have to simplify the sum

$$S = \binom{499}{0}1 + \binom{499}{1}3 + \cdots + \binom{499}{498}997 + \binom{499}{499}999.$$

Joel's approach was to write the terms in the opposite order and use the fact that $\binom{n}{r} = \binom{n}{n-r}$ to obtain

$$S = \binom{499}{0}999 + \binom{499}{1}997 + \cdots + \binom{499}{498}3 + \binom{499}{499}1.$$

Adding the two expressions for S, we have

$$2S = 1000\left[\binom{499}{0} + \binom{499}{1} + \cdots + \binom{499}{499}\right] = 1000(2^{499}).$$

Therefore $S = 500 \cdot 2^{499}$.