ACCELERATION WAVES IN THE VON KARMAN PLATE THEORY[1]

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1   Introduction

The von Kármán plate theory is governed by two coupled nonlinear fourth-order partial differential equations in three independent variables (Cartesian coordinates on the plate middle-plane x^1, x^2 and the time x^3) and two dependent variables (the transversal displacement function w and Airy's stress function Φ), namely

$$
\begin{aligned}
&D\Delta^2 w - \varepsilon^{\alpha\mu}\varepsilon^{\beta\nu} w_{,\alpha\beta}\Phi_{,\mu\nu} + \rho w_{,33} = 0, \\
&(1/Eh)\,\Delta^2\Phi + (1/2)\,\varepsilon^{\alpha\mu}\varepsilon^{\beta\nu} w_{,\alpha\beta}w_{,\mu\nu} = 0,
\end{aligned}
\tag{1}
$$

where Δ is the Laplace operator with respect to x^1 and x^2, $D = Eh^3/12(1-\nu^2)$ is the bending rigidity, E is Young's modulus, ν is Poisson's ratio, h is the thickness of the plate, ρ is the mass per unit area of the plate middle-plane, $\delta^{\alpha\beta}$ is the Kronecker delta symbol and $\varepsilon^{\alpha\beta}$ is the alternating symbol. Here and throughout the work: Greek (Latin) indices range over 1, 2 (1, 2, 3), unless explicitly stated otherwise; the usual summation convention over a repeated index is used and subscripts after a comma at a certain function f denote its partial derivatives, that is $f_{,i} = \partial f/\partial x^i$, $f_{,ij} = \partial f/\partial x^i\partial x^j$, etc.

The von Kármán equations (1) describe entirely the motion of a plate, the membrane stress tensor $N^{\alpha\beta}$, moment tensor $M^{\alpha\beta}$, shear-force vector Q^α, strain tensor $E^{\alpha\beta}$ and bending tensor $K_{\alpha\beta}$ being given in terms of w and Φ through the following expressions:

$$
N^{\alpha\beta} = \varepsilon^{\alpha\mu}\varepsilon^{\beta\nu}\Phi_{,\mu\nu}, \ \ M^{\alpha\beta} = -D\left\{(1-\nu)\delta^{\alpha\mu}\delta^{\beta\nu} + \nu\delta^{\alpha\beta}\delta^{\mu\nu}\right\}w_{,\mu\nu},
$$
$$
Q^\alpha = M^{\alpha\mu}_{,\mu} + N^{\alpha\mu}w_{,\mu}, \ \ E^{\alpha\beta} = (1/Eh)\left\{(1+\nu)\varepsilon^{\alpha\mu}\varepsilon^{\beta\nu} - v\delta^{\alpha\beta}\delta^{\mu\nu}\right\}\Phi_{,\mu\nu}, \ \ K_{\alpha\beta} = w_{,\alpha\beta}.
$$

The theory under consideration allows an exact variational formulation, the von Kármán equations being the Euler-Lagrange equations [1] associated with the action functional

$$
I[w, \Phi] = \int\int\int L\,dx^1 dx^2 dx^3, \ \ L = T - \Pi
\tag{2}
$$

where

$$
\begin{aligned}
\Pi \ &= \ (D/2)\left\{(\Delta w)^2 - (1-\nu)\,\varepsilon^{\alpha\mu}\varepsilon^{\beta\nu} w_{,\alpha\beta}w_{,\mu\nu}\right\} \\
&- \ (1/2Eh)\left\{(\Delta\Phi)^2 - (1+\nu)\,\varepsilon^{\alpha\mu}\varepsilon^{\beta\nu}\Phi_{,\alpha\beta}\Phi_{,\mu\nu}\right\} + (1/2)\varepsilon^{\alpha\mu}\varepsilon^{\beta\nu}\Phi_{,\alpha\beta}w_{,\mu}w_{,\nu},
\end{aligned}
$$

is the strain energy per unit area of the plate middle-plane and

$$
T = (\rho/2)\,(w_{,3})^2,
$$

is the kinetic energy per unit area of the plate middle-plane.

2 Conservation laws

In the recent paper [2], all Lie point symmetries of system (1) are shown to be variational symmetries of the functional (2), and all corresponding (via Noether's theorem) conservation laws admitted by the smooth solutions of the von Kármán equations are established. Each such conservation law is a linear combination of the basic linearly independent conservation laws

$$\frac{\partial \Psi_{(j)}}{\partial x^3} + \frac{\partial P^{\mu}_{(j)}}{\partial x^{\mu}} = 0 \ \ (j = 1, 2, \ldots, 14),$$

whose densities $\Psi_{(j)}$ and fluxes $P^{\mu}_{(j)}$ are presented (together with the generators of the respective symmetries) on the Table 1 below in terms of Q^{α}, $M^{\alpha\beta}$, $G^{\alpha\beta}$ and F^{α},

$$G^{\alpha\beta} = (1/Eh)\left\{(1+\nu)\delta^{\alpha\mu}\delta^{\beta\nu} - \nu\delta^{\alpha\beta}\delta^{\mu\nu}\right\} \Phi_{,\mu\nu} - (1/2)\,\varepsilon^{\alpha\mu}\varepsilon^{\beta\nu}w_{,\mu}w_{,\nu}, \quad F^{\alpha} = G^{\alpha\nu}_{,\nu}.$$

Table 1 Conservation laws

w - translations	**transversal linear momentum** (first von Kármán eqn)
$X_1 = \frac{\partial}{\partial w}$	$P^{\alpha}_{(1)} = -Q^{\alpha}, \ \Psi_{(1)} = \rho\,w_{,3}$
Φ - translations	**compatibility condition** (second von Kármán eqn)
$X_{14} = \frac{\partial}{\partial \Phi}$	$P^{\alpha}_{(14)} = F^{\alpha}, \ \Psi_{(14)} = 0$
time - translations	**energy**
$X_4 = \frac{\partial}{\partial x^3}$	$P^{\alpha}_{(4)} = -w_{,3}Q^{\alpha} - \Phi_{,3}F^{\alpha} + w_{,3\beta}M^{\alpha\beta} + \Phi_{,3\beta}G^{\alpha\beta}$ $\Psi_{(4)} = T + \Pi$
x^1 & x^2- translations	**wave momentum**
$X_2 = \frac{\partial}{\partial x^1}$	$P^{\alpha}_{(2)} = \delta^{\alpha 1}L + w_{,1}Q^{\alpha} + \Phi_{,1}F^{\alpha} - w_{,1\beta}M^{\alpha\beta} - \Phi_{,1\beta}G^{\alpha\beta}$ $\Psi_{(2)} = -\rho\,w_{,1}w_{,3}$
$X_3 = \frac{\partial}{\partial x^2}$	$P^{\alpha}_{(3)} = \delta^{\alpha 2}L + w_{,2}Q^{\alpha} + \Phi_{,2}F^{\alpha} - w_{,2\beta}M^{\alpha\beta} - \Phi_{,2\beta}G^{\alpha\beta}$ $\Psi_{(3)} = -\rho\,w_{,2}w_{,3}$
rotations	**moment of the wave momentum**
$X_6 = x^2\frac{\partial}{\partial x^1} - x^1\frac{\partial}{\partial x^2}$	$P^{\alpha}_{(6)} = x^2 P^{\alpha}_{(2)} - x^1 P^{\alpha}_{(3)} + \varepsilon^{\ \mu}_{\nu}w_{,\mu}M^{\alpha\nu} + \varepsilon^{\ \mu}_{\nu}\Phi_{,\mu}G^{\alpha\nu}$ $\Psi_{(6)} = x^2\,\Psi_{(2)} - x^1\,\Psi_{(3)}$
rigid body rotations	**angular momentum**
$X_7 = x^1\frac{\partial}{\partial w}$	$P^{\alpha}_{(7)} = M^{\alpha 1} - x^1 Q^{\alpha} + w\varepsilon^{\alpha\nu}\Phi_{,\nu 2}, \ \Psi_{(7)} = \rho x^1 w_{,3}$
$X_8 = x^2\frac{\partial}{\partial w}$	$P^{\alpha}_{(8)} = M^{\alpha 2} - x^2 Q^{\alpha} + w\varepsilon^{\nu\alpha}\Phi_{,\nu 1}, \ \Psi_{(8)} = \rho x^2 w_{,3}$
scaling	
$X_5 = x^{\mu}\frac{\partial}{\partial x^{\mu}} + 2x^3\frac{\partial}{\partial x^3}$	$P^{\alpha}_{(5)} = x^1 P^{\alpha}_{(2)} + x^2 P^{\alpha}_{(3)} - 2x^3 P^{\alpha}_{(4)} - w_{,\beta}M^{\alpha\beta} - \Phi_{,\beta}G^{\alpha\beta}$ $\Psi_{(5)} = x^1\,\Psi_{(2)} + x^2\,\Psi_{(3)} - 2x^3\,\Psi_{(4)}$
Galilean boost	**center-of-mass theorem**
$X_9 = x^3\frac{\partial}{\partial w}$	$P^{\alpha}_{(9)} = -x^3 Q^{\alpha}, \ \Psi_{(9)} = \rho\left(x^3 w_{,3} - w\right)$
$X_{10} = x^1 x^3\frac{\partial}{\partial w}$	$P^{\alpha}_{(10)} = x^3 P^{\alpha}_{(7)}, \ \Psi_{(10)} = x^1\,\Psi_{(9)}$
$X_{11} = x^2 x^3\frac{\partial}{\partial w}$	$P^{\alpha}_{(11)} = x^3 P^{\alpha}_{(8)}, \ \Psi_{(11)} = x^2\,\Psi_{(9)}$
$X_{12} = x^1\frac{\partial}{\partial \Phi}$	$P^{\alpha}_{(12)} = x^1 F^{\alpha} - G^{\alpha 1}, \ \Psi_{(12)} = 0$
$X_{13} = x^2\frac{\partial}{\partial \Phi}$	$P^{\alpha}_{(13)} = x^2 F^{\alpha} - G^{\alpha 2}, \ \Psi_{(13)} = 0$

3   Balance laws

Given a region Ω in the plate middle-plane with sufficiently smooth boundary Σ of outward unit normal n_α, a balance law

$$\frac{d}{dt}\int_\Omega \Psi_{(j)}dx^1dx^2 + \int_\Sigma P^\alpha_{(j)}n_\alpha d\Sigma = 0, \tag{3}$$

corresponds to each of the conservation laws listed in Table 1. It holds, just as the respective conservation law, for every smooth solution of the von Kármán equations.

The balance laws are applicable even if Ω is intersected by a discontinuity (singular) manifold (on which the corresponding densities $\Psi_{(j)}$ and fluxes $P^\alpha_{(j)}$ may suffer jump discontinuities) provided that the integrals exist. We are ready now to extend the "continuous" von Kármán plate theory so as to cover situations when some physical quantities suffer jump discontinuities at a certain curve.

4   Acceleration waves

Definition 1 *A discontinuity solution of the von Kármán equations is a couple of functions (w, Φ), defined in a certain region Ω, such that the two balance laws corresponding to the von Kármán equations themselves, namely*

$$\frac{d}{dt}\int_{\tilde\Omega} \rho w_{,3}dx^1dx^2 - \int_{\tilde\Sigma} Q^\alpha \tilde n_\alpha d\Sigma = 0, \quad \int_{\tilde\Sigma} F^\alpha \tilde n_\alpha d\Sigma = 0, \tag{4}$$

hold $\forall \tilde\Omega \subset \Omega$ with boundary $\tilde\Sigma$ of outward unit normal $\tilde n_\alpha$, and (w, Φ) is a solution of the (local) von Kármán equations (1) almost everywhere in Ω except for a moving curve Γ at which some of the derivatives of w or Φ have jumps.

Definition 2 *A discontinuity solution of the von Kármán equations is an acceleration wave if at the wave front – a smoothly propagating connected singular curve Γ –*

$$\Gamma : \gamma(x^1, x^2, x^3) = 0, \ \ (x^1, x^2) \in \Omega \subset \mathbf{R}^2, \ \ x^3 \in \mathbf{R}^+, \ \ \gamma \in C^1(\Omega \times \mathbf{R}^+),$$

we have

$$[w] = [\Phi] = [w_{,i}] = [\Phi_{,i}] = 0, \ \ [w_{,33}] \neq 0. \tag{5}$$

(Here and in what follows, the square brackets are used to denote the jump of any field f across the curve Γ, i.e., $[f] = f_2 - f_1$, where f_2 and f_1 are the limit values of f behind Γ and ahead of Γ.)

The moving curve Γ divides the region Ω into two parts Ω^+ and Ω^- and forms the common border between them. It is assumed that ahead of the wave front (in the region Ω^+) we have the known unperturbed fields $w^+(x^1, x^2, x^3)$, $\Phi^+(x^1, x^2, x^3)$ and behind it (in the region Ω^-) – the unknown perturbed fields $w^-(x^1, x^2, x^3)$, $\Phi^-(x^1, x^2, x^3)$. At the wave front Γ, we have the jump conditions (5).

The jumps of the derivatives of w and Φ across Γ are permissible if they obey the compatibility conditions following by Hadamard's lemma [3]. Thus

Proposition 1 *If $[w_{,33}] \neq 0$, then*

$$[w_{,\alpha\beta}] = \lambda n_\alpha n_\beta, \quad [w_{,\alpha 3}] = -\lambda C n_\alpha, \quad [w_{,33}] = \lambda C^2,$$

where λ is an arbitrary factor, C and n_α,

$$C = -\left|\nabla\gamma\right|^{-1}\partial\gamma/\partial x^3, \quad n_\alpha = \left|\nabla\gamma\right|^{-1}\partial\gamma/\partial x^\alpha, \quad \left|\nabla\gamma\right| = \sqrt{(\partial\gamma/\partial x^1)^2 + (\partial\gamma/\partial x^2)^2},$$

are the speed of displacement and the direction of propagation of the wave front Γ.

Proposition 2 *If at least one of the third derivatives of w suffers a jump at Γ, then the compatibility conditions for the jumps of the third derivatives of the displacement field across Γ are:*

$$\begin{aligned}
[w_{,\alpha\beta\gamma}] &= \lambda^* n_\alpha n_\beta n_\gamma + \partial\lambda/\partial s\,(n_\alpha n_\beta t_\gamma + n_\alpha t_\beta n_\gamma + t_\alpha n_\beta n_\gamma) \\
&+ \lambda a\,(t_\alpha t_\beta n_\gamma + t_\alpha n_\beta t_\gamma + n_\alpha t_\beta t_\gamma),
\end{aligned}$$

where t_α is the unit tangent vector to Γ, λ^* is an arbitrary factor, while $\lambda = [w_{,\alpha\beta}]\,n^\alpha n^\beta$ and $a = t_\alpha\partial n^\alpha/\partial s$, s being the natural parameter (arc-length) of the curve Γ.

Proposition 3 *If at least one of the second derivatives of Φ suffers a jump at Γ, then*

$$[\Phi_{,\alpha\beta}] = \mu n_\alpha n_\beta, \quad [\Phi_{,\alpha 3}] = -\mu C n_\alpha, \quad [\Phi_{,33}] = \mu C^2,$$

where μ is an arbitrary factor, are the compatibility conditions for the jumps of the second derivatives of the stress field across Γ.

Proposition 4 *If at least one of the third derivatives of Φ suffer a jump at Γ, then the compatibility conditions for the jumps of the third derivatives of the stress field across Γ are:*

$$\begin{aligned}
[\Phi_{,\alpha\beta\gamma}] &= \mu^* n_\alpha n_\beta n_\gamma + \partial\mu/\partial s\,(n_\alpha n_\beta t_\gamma + n_\alpha t_\beta n_\gamma + t_\alpha n_\beta n_\gamma) \\
&+ \mu a\,(t_\alpha t_\beta n_\gamma + t_\alpha n_\beta t_\gamma + n_\alpha t_\beta t_\gamma),
\end{aligned}$$

where $\mu = [\Phi_{,\alpha\beta}]n^\alpha n^\beta$ and μ^* is an arbitrary factor.

According to the divergence theorem (see e.g. [4]), a couple of functions $(w,\ \Phi)$ suffering jump discontinuities at a singular curve Γ is a discontinuity solution of the von Kármán equations in the sense of Definition 1 iff the following jump conditions

$$C\,[\rho w_{,3}] + [Q^\alpha]\,n_\alpha = 0, \quad [F^\alpha]\,n_\alpha = 0, \tag{6}$$

hold at Γ, and a balance law of form (3) holds on this solution iff at Γ:

$$C\left[\Psi_{(j)}\right] - \left[P^\alpha_{(j)}\right]n_\alpha = 0. \tag{7}$$

Definition 2, Propositions 1, 2, 3, 4 and jump conditions (6) imply that:

Proposition 5 *If an acceleration wave in the von Kármán plate theory is such that $[w_{,\alpha\beta\gamma}] \neq 0$ ($[\Phi_{,\alpha\beta\gamma}] \neq 0$) at the curve of discontinuity Γ, then $\lambda^* = -\lambda a$ ($\mu^* = -\mu a$).*

Given a discontinuity solution of the von Kármán equations, the two corresponding balance laws (4) being satisfied, the other balance laws do not necessarily hold for this solution. The jump conditions associated with the most important conservation laws from Table 1 are derived using (7) and presented on the Table 2 below, where $X_j^+(w_{,\beta})$ and $X_j^+(\Phi_{,\beta})$ denote the limit values of $X_j(w_{,\beta})$ and $X_j(\Phi_{,\beta})$ ahead of Γ:

Table 2 Jump conditions

time - translations	**energy**
$X_4 = \frac{\partial}{\partial x^3}$	$\frac{C}{2}\left(D\lambda^2 - \frac{\mu^2}{Eh}\right) = \left(D\lambda X_4^+(w_{,\beta}) - \frac{\mu}{Eh}X_4^+(\Phi_{,\beta})\right)n^\beta$
x^1 & x^2- translations	**wave momentum**
$X_2 = \frac{\partial}{\partial x^1}$	$\frac{1}{2}\left(D\lambda^2 - \frac{\mu^2}{Eh}\right)n^1 = -\left(D\lambda X_2^+(w_{,\beta}) - \frac{\mu}{Eh}X_2^+(\Phi_{,\beta})\right)n^\beta$
$X_3 = \frac{\partial}{\partial x^2}$	$\frac{1}{2}\left(D\lambda^2 - \frac{\mu^2}{Eh}\right)n^2 = -\left(D\lambda X_3^+(w_{,\beta}) - \frac{\mu}{Eh}X_3^+(\Phi_{,\beta})\right)n^\beta$
rotations	**moment of the wave momentum**
$X_6 = x^2\frac{\partial}{\partial x^1} - x^1\frac{\partial}{\partial x^2}$	$\frac{1}{2}\varepsilon^\alpha_\beta n_\alpha x^\beta\left(D\lambda^2 - \frac{\mu^2}{Eh}\right) =$ $-\left\{D\lambda\left(\varepsilon^\alpha_\beta w_{,\alpha} + X_6^+(w_{,\beta})\right) - \frac{\mu}{Eh}\left(\varepsilon^\alpha_\beta \Phi_{,\alpha} + X_6^+(\Phi_{,\beta})\right)\right\}n^\beta$
scaling	
$X_5 = x^\mu\frac{\partial}{\partial x^\mu} + 2x^3\frac{\partial}{\partial x^3}$	$\frac{1}{2}\left(x^\alpha n_\alpha - 2Cx^3\right)\left(D\lambda^2 - \frac{\mu^2}{Eh}\right) =$ $-\left\{D\lambda\left(w_{,\beta} + X_5^+(w_{,\beta})\right) - \frac{\mu}{Eh}\left(\Phi_{,\beta} + X_5^+(\Phi_{,\beta})\right)\right\}n^\beta$

The center-of-mass theorem holds for any discontinuity solutions of the von Kármán equations. The balance laws, associated with the infinitesimal symmetries X_7, X_8, X_{10} and X_{11} hold iff $\lambda = 0$, while those associated with X_{12} and X_{13} – iff $\mu = 0$. For this reason, there do not exist acceleration waves in the von Kármán plate theory satisfying all balance laws.

Obviously, when dealing with discontinuity solutions, from physical point of view it seems reasonably that at least the balance of energy should hold in addition to the balance laws corresponding to the fundamental equations considered. Observing Table 2 it is evident that for acceleration waves propagating into an undisturbed plate the balance of energy implies also the balances of wave momentum, moment of the wave momentum as well as the balance related to the scaling symmetry.

5   Examples

As an example, we consider acceleration waves such that behind and ahead of the wave front the plate motion is described by solutions of the von Kármán equations invariant under the group generated by X_3 and $X_2 + (1/c)X_4$, where c is an arbitrary constant. The most general form of such group-invariant solutions is

$$w = u(\xi) = u_0 + u_1\xi + u_2\sin\omega\xi + u_3\cos\omega\xi,$$
$$\Phi = \varphi(\xi) = \varphi_0 + \varphi_1\xi + \varphi_2\xi^2 + \varphi_3\xi^3,$$

where $\xi = x^1 - cx^3$, u_j, φ_j are arbitrary constants, and $\omega = c\sqrt{\rho/D}$. Let (u^+, φ^+) – an arbitrary solution of that kind – describes the plate motion ahead of the wave front. Then, Definition 2 and Propositions 1 to 4 imply that each acceleration wave of the type considered reads

$$
u = \begin{cases} u^+ + c_1(1 - \cos\omega\xi), & \xi < 0, \\ u^+, & \xi > 0, \end{cases} \qquad \varphi = \begin{cases} \varphi^+ + c_2\xi^2, & \xi < 0, \\ \varphi^+, & \xi > 0, \end{cases} \tag{8}
$$

where c_1 and c_2 are arbitrary constants, but $c_1 \neq 0$; the wave front in this case is the moving straight line $\Gamma : \xi = 0$. In general, however, an acceleration wave of form (8) does not satisfy the balance laws other than (4). Indeed, after a little manipulation, the jump conditions from Table 2 simplify to

$$
DEh\omega^4 c_1(c_1 - 2u_3^+) = 4c_2(c_2 + 2\varphi_2^+), \tag{9}
$$
$$
DEh\omega^2 c_1(u_1^+ + \omega u_2^+) = 2c_2\varphi_1^+, \tag{10}
$$

where u_j^+, φ_j^+ are the constants in u^+ and φ^+, respectively. The jump condition (9) is necessary and sufficient for the balances of energy, wave momentum and moment of wave momentum to hold, while the balance related to X_5 requires both (9) and (10). The second relation is treated in a different manner according to the acceleration wave under consideration. If the wave is such that $c_2 = 0$, then (10) holds for this wave only if $u_1^+ = -\omega u_2^+$. On the other hand, if we consider waves with $c_2 \neq 0$, then choosing the coefficient φ_1^+ in a suitable manner, we could satisfy (10) identically (note that adding a linear function of the independent variables to Airy's stress function does not change the membrane stress tensor $N^{\alpha\beta}$). Hence, the balance associated with the scaling symmetry (X_5) holds only for acceleration waves of form (8) satisfying (9) and which are such that either $c_2 \neq 0$ or $u_1^+ = -\omega u_2^+$.

Another example, discussed in details in [2], is an axisymmetrically expanding acceleration wave composed by solutions of the von Kármán equations that are joined invariants of the rotation (X_6) and scaling (X_5) symmetries.

References

[1] P. J. Olver, *Applications of Lie Groups to Differential Equations*, 2nd edition, Graduate Texts in Mathematics, Vol. 107, Springer-Verlag, New York, 1993.

[2] P. Djondjorov and V. Vassilev, Conservation laws and group-invariant solutions of the von Kármán equations, *Int. J. Non-Linear Mechanics*, **31** (1996), 73–87.

[3] C. Truesdell, *A First Course in Rational Continuum Mechanics*, The Johns Hopkins University, Baltimore, Maryland, 1972.

[4] D. Edelin, *Nonlocal variations and invariance of fields*, American Elsevier Publishing Company, Inc., New York, 1969