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Symmetry Groups, Conservation Laws and Group-Invariant Solutions of the Marguerre-von Kármán Equations[1]

1 Introduction

Marguerre's theory for large deflection of thin isotropic elastic shells [1] leads to the following system of two coupled nonlinear fourth-order partial differential equations

$$\begin{aligned} &D\Delta^2 w - \varepsilon^{\alpha\mu}\varepsilon^{\beta\nu} w_{;\alpha\beta}\Phi_{;\mu\nu} - \varepsilon^{\alpha\mu}\varepsilon^{\beta\nu} b_{\alpha\beta}\Phi_{;\mu\nu} = p, \\ &(1/Eh)\Delta^2\Phi + (1/2)\varepsilon^{\alpha\mu}\varepsilon^{\beta\nu} w_{;\alpha\beta} w_{;\mu\nu} + \varepsilon^{\alpha\mu}\varepsilon^{\beta\nu} b_{\alpha\beta} w_{;\mu\nu} = q, \end{aligned} \tag{1}$$

in two independent variables – the coordinates on the shell middle-surface F, and two dependent variables – the transversal displacement function w, and Airy's stress function Φ, with right-hand sides appearing when the shell is subjected to an external transversal load and nonuniform heating. Here and throughout: $\varepsilon^{\alpha\beta}$ is the alternating tensor of F; $b_{\alpha\beta}$ is the curvature tensor of F; D, E and h are the bending rigidity, Young's modulus and thickness of the shell, respectively, which are supposed to be given constants; a semicolon is used for covariant differentiation with respect to the metric tensor $a_{\alpha\beta}$ of the surface F; Δ is the Laplace-Beltrami operator on F; Greek (Latin) indices range over 1, 2 (1, 2, 3), unless explicitly stated otherwise; the usual summation convention over a repeated index (one subscript and one superscript) is used.

This theory assumes that the intrinsic geometry of the shell middle-surface F should be Euclidean or approximately Euclidean in the following sense. Let (x^1, x^2, z) be a fixed right-handed rectangular Cartesian coordinate system in the 3-dimensional Euclidean space in which the middle-surface F of a shell is embedded, and let this surface be given by the equation

$$F : z = f(x^1, x^2),\ (x^1, x^2) \in \Theta \subset \mathbf{R}^2,$$

where $f : \mathbf{R}^2 \to \mathbf{R}$ is a single-valued smooth function possessing as many derivatives as may be required on a certain domain of interest Θ. Let us take x^1, x^2 to serve as coordinates on the surface F. Then, relative to this coordinate system, the components of the fundamental tensors and the alternating tensor of F are given by the expressions:

$$a_{\alpha\beta} = \delta_{\alpha\beta} + f_{,\alpha}f_{,\beta}, \quad b_{\alpha\beta} = a^{-1/2} f_{,\alpha\beta}, \quad \varepsilon^{\alpha\beta} = a^{-1/2} e^{\alpha\beta}, \tag{2}$$

where $a = \det(a_{\alpha\beta}) = 1 + (f_{,1})^2 + (f_{,2})^2$; $\delta_{\alpha\beta} = \delta^{\alpha\beta}$ is the Kronecker delta symbol; $e^{\alpha\beta}$ is the alternating symbol; subscripts after a comma at a certain function f denote its partial derivatives with respect to the coordinates on F. If the inequalities

$$|f_{,\alpha}|\,|f_{,\beta}| \le \varepsilon^2 \ll 1, \quad \varepsilon = const,$$

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hold for every point $(x^1, x^2) \in \Theta$ (such a shell is said to be shallow on the domain Θ), then the quadratic terms in the right-hand sides of expressions (2) are small compared to unity, they may be neglected, and thus allowing for a relative error of order $O(\varepsilon^2)$ one may regard the intrinsic geometry of the shell middle-surface F as Euclidean and (x^1, x^2) may be thought of as an Euclidean coordinate system on F, in which:

$$a_{\alpha\beta} = \delta_{\alpha\beta},\ b_{\alpha\beta} = f_{,\alpha\beta},\ \varepsilon^{\alpha\beta} = e^{\alpha\beta},$$

and the mean curvature H of the surface F and its Gaussian curvature K read

$$H = (1/2)a^{\alpha\beta}b_{\alpha\beta} = (1/2)\delta^{\alpha\beta}f_{,\alpha\beta},\ K = (1/2)\varepsilon^{\alpha\mu}\varepsilon^{\beta\nu}b_{\alpha\beta}b_{\mu\nu} = (1/2)e^{\alpha\mu}e^{\beta\nu}f_{,\alpha\beta}f_{,\mu\nu}$$

(note that the latter is not necessarily equal to zero within the allowed relative error).

Equations (1) are often referred to as Marguerre–von Kármán (MvK) equations to reflect the fact that they are an extension of the von Kármán equations for large bending of plates [2] (including the latter as a special case corresponding to $b_{\alpha\beta} = 0$) to the shallow shells. Actually (1) describe the state of equilibrium of the shell, but introducing, according to d'Alembert principle, the inertia force $-\rho w_{,33}$ in the right-hand side of the first MvK equation, $w_{,33}$ being the second derivative of the displacement field with respect to the time $t \equiv x^3$ and ρ – the mass per unit area of the shell middle-surface, one can extend (1) to describe the dynamic behaviour of shallow shells.

Applying the equivalence transformation $(x^1, x^2, w, \Phi) \mapsto (x^1, x^2, W, \Phi)$, $W = w + f$, to the time-independent MvK equations and $(x^1, x^2, x^3, w, \Phi) \mapsto (x^1, x^2, x^3, W, \Phi)$ – to the time-dependent ones one can map, see [3], the MvK equations to the von Kármán equations

$$\begin{aligned} &D\Delta^2 W - \varepsilon^{\alpha\mu}\varepsilon^{\beta\nu}W_{;\alpha\beta}\Phi_{;\mu\nu} = P, \\ &(1/Eh)\,\Delta^2\Phi + (1/2)\,\varepsilon^{\alpha\mu}\varepsilon^{\beta\nu}W_{;\alpha\beta}W_{;\mu\nu} = Q, \end{aligned} \tag{3}$$

and

$$\begin{aligned} &D\Delta^2 W - \varepsilon^{\alpha\mu}\varepsilon^{\beta\nu}W_{;\alpha\beta}\Phi_{;\mu\nu} + \rho W_{,33} = P, \\ &(1/Eh)\,\Delta^2\Phi + (1/2)\,\varepsilon^{\alpha\mu}\varepsilon^{\beta\nu}W_{;\alpha\beta}W_{;\mu\nu} = Q, \end{aligned} \tag{4}$$

respectively, where

$$P = 2Da^{\mu\nu}H_{;\mu\nu} + p, \quad Q = K + q.$$

Hereafter (3) and (4) will be referred to as the time-independent and time-dependent MvK equations respectively. In both cases, the moment tensor $M^{\alpha\beta}$, membrane stress tensor $N^{\alpha\beta}$, and shear-force vector Q^α are given in terms of W and Φ by the expressions

$$\begin{aligned} M^{\alpha\beta} &= D\left\{(1-\nu)a^{\alpha\mu}a^{\beta\nu} + \nu a^{\alpha\beta}a^{\mu\nu}\right\}\{W_{;\mu\nu} - f_{;\mu\nu}\}, \\ N^{\alpha\beta} &= \varepsilon^{\alpha\mu}\varepsilon^{\beta\nu}\Phi_{;\mu\nu},\ Q^\alpha = M^{\alpha\mu}_{;\mu} + N^{\alpha\mu}\{W_{;\mu} - f_{;\mu}\}, \end{aligned}$$

and the in-plane displacements v^α can be found solving the overdetermined system

$$v_{\alpha;\beta} + v_{\beta;\alpha} = (2/Eh)\left\{(1+\nu)\varepsilon_\alpha^{\ \mu}\varepsilon_\beta^{\ \nu} - \nu a_{\alpha\beta}a^{\mu\nu}\right\}\Phi_{;\mu\nu} - \{W_{;\alpha} - f_{;\alpha}\}\{W_{;\beta} - f_{;\beta}\},$$

the second one of the MvK equations being its compatibility condition.

2 Symmetry groups[1]

The following is known [5] for the symmetry groups of the homogeneous MvK equations.
Proposition 1 *The homogeneous time-independent MvK equations (3) admit the group* $G_{(S)}$ *generated by the basic vector field (operators):*

$$Y_1 = \frac{\partial}{\partial W},\ Y_2 = \frac{\partial}{\partial x^1},\ Y_3 = \frac{\partial}{\partial x^2},\ Y_4 = x^2\frac{\partial}{\partial x^1} - x^1\frac{\partial}{\partial x^2},\ Y_5 = x^1\frac{\partial}{\partial \Phi},$$
$$Y_6 = x^2\frac{\partial}{\partial \Phi},\ Y_7 = \frac{\partial}{\partial \Phi},\ Y_8 = x^1\frac{\partial}{\partial W},\ Y_9 = x^2\frac{\partial}{\partial W},\ Y_{10} = x^1\frac{\partial}{\partial x^1} + x^2\frac{\partial}{\partial x^2}.$$

Proposition 2 *The homogeneous time-dependent MvK equations (4) admit the group* $G_{(D)}$ *generated by the basic vector field:*

$$X_1 = \frac{\partial}{\partial W},\ X_2 = \frac{\partial}{\partial x^1},\ X_3 = \frac{\partial}{\partial x^2},\ X_4 = \frac{\partial}{\partial x^3},\ X_5 = x^1\frac{\partial}{\partial x^1} + x^2\frac{\partial}{\partial x^2} + 2x^3\frac{\partial}{\partial x^3},$$
$$X_6 = x^2\frac{\partial}{\partial x^1} - x^1\frac{\partial}{\partial x^2},\ X_7 = x^1\frac{\partial}{\partial W},\ X_8 = x^2\frac{\partial}{\partial W},\ X_9 = x^3\frac{\partial}{\partial W},\ X_{10} = x^1x^3\frac{\partial}{\partial W},$$
$$X_{11} = x^2x^3\frac{\partial}{\partial W},\ X_{12} = x^1f(x^3)\frac{\partial}{\partial \Phi},\ X_{13} = x^2g(x^3)\frac{\partial}{\partial \Phi},\ X_{14} = h(x^3)\frac{\partial}{\partial \Phi},$$

where f, g, *and* h *are arbitrary functions depending on the time only.*

As for the symmetries of the nonhomogeneous MvK equations, we proved that:
Proposition 3 *A nonhomogeneous time-independent MvK system is invariant under a vector field* Y *iff* $Y = c^jY_j$ $(j = 1, \ldots, 10)$*, where* c^j *are real constants, and*

$$2P\xi^\mu_{,\mu} + \xi^\mu P_{,\mu} = 0,\ 2Q\xi^\mu_{,\mu} + \xi^\mu Q_{,\mu} = 0, \tag{5}$$

for $\xi^\alpha = Y(x^\alpha)$*,* Y *being regarded as an operator acting on the functions* $\zeta : \Theta \to \mathbf{R}$*,* $\Theta \subset \mathbf{R}^2$*.*
Proposition 4 *A nonhomogeneous time-dependent MvK system is invariant under a vector field* X *iff* $X = C^jX_j$ $(j = 1, \ldots, 14)$*, where* C^j *are real constants, and*

$$P\xi^i_{,i} + \xi^i P_{,i} = 0,\ Q\xi^i_{,i} + \xi^i Q_{,i} = 0, \tag{6}$$

for $\xi^i = X(x^i)$*,* X *being regarded as an operator acting on the functions* $\chi : \Theta\times T \to \mathbf{R}$*,* $\Theta \subset \mathbf{R}^2$*,* $T \subset \mathbf{R}$*.*

The above Propositions imply the following group classification results.
Theorem 1 *The time-independent MvK equations (3) admit a group* G *iff* G *is generated by a vector field* $Y = c^jY_j$ $(j = 1, \ldots, 10)$ *and the right-hand sides* P *and* Q *are invariants of* G *(when* $c^{10} = 0$*) or eigenfunctions (when* $c^{10} \neq 0$*) of its generator* Y*.*
Theorem 2 *The time-dependent MvK equations (4) admit a group* G *iff* G *is generated by a vector field* $X = C^jX_j$ $(j = 1, \ldots, 14)$ *and the right-hand sides* P *and* Q *are invariants of* G *(when* $C^5 = 0$*) or eigenfunctions (when* $C^5 \neq 0$*) of its generator* X*.*

[1]For the basic notions, statements and techniques used in the group analysis of differential equations and variational problems see, e.g., the book by Olver [4].

3 Conservation laws

Both the time-independent and the time-dependent MvK equations constitute self-adjoint systems and are the Euler-Lagrange equations associated with the functionals

$$I^{(S)}[W,F] = \int\int\int L^{(S)}\,dx^1dx^2,\ L^{(S)} = \Pi,$$

and

$$I^{(D)}[W,F] = \int\int\int L^{(D)}dx^1dx^2dx^3,\ L^{(D)} = (\mathrm{T}-\Pi),$$

respectively, where

$$\begin{aligned}\Pi &= (D/2)\left\{(\Delta W)^2 - (1-\nu)e^{\alpha\mu}e^{\beta\gamma}W_{,\alpha\beta}W_{,\mu\nu}\right\}\\ &- (1/2Eh)\left\{(\Delta\varPhi)^2 - (1+\nu)e^{\alpha\mu}e^{\beta\nu}\varPhi_{,\alpha\beta}\varPhi_{,\mu\nu}\right\} + (1/2)\,e^{\alpha\mu}e^{\beta\nu}\varPhi_{,\alpha\beta}W_{,\mu}W_{,\nu}\\ &- PW - Q\varPhi,\end{aligned}$$

is the strain energy per unit area of the shell middle-surface and

$$\mathrm{T} = (\rho/2)\,(W_{,3})^2\,,$$

is the kinetic energy per unit area of the shell middle-surface.

In [6], the variational symmetries of the above functionals with $P = Q = 0$ are established and all Noether's conservation laws admitted by the smooth solutions of the homogeneous MvK equations are presented (see also Table 1 in [7] where the conservation laws associated with the time-dependent MvK equations are listed). The following statements hold for the nonhomogeneous MvK equations.

Proposition 5 *A conservation law of flux* $A^\alpha_{(j)}$ *and characteristic* $\Lambda^\alpha_{(j)}$ $(j = 1, ..., 9)$ *admitted by the smooth solutions of the homogeneous time-independent MvK equations takes the form*

$$A^\mu_{(j),\mu} + S_{(j)} = 0,\ S_{(j)} = -\Lambda^1_{(j)}P - \Lambda^2_{(j)}Q, \tag{7}$$

on the smooth solutions of the non-homogeneous time-independent MvK equations;

$$S_{(j)} = \widetilde{A}^\mu_{(j),\mu},$$

iff (5) hold, and then (7) can be written as a divergence free expression (i.e. it becomes a proper conservation law in the sense appropriated in the group analysis of differential equations, see e.g. [4]), otherwise it has supply (production) $S_{(j)}$.

Proposition 6 *Each conservation law of density* $\varPsi_{(i)}$, *flux* $P^\alpha_{(i)}$ *and characteristic* $\Lambda^\alpha_{(i)}$ $(i = 1, ..., 14)$ *admitted by the smooth solutions of the homogeneous time-dependent MvK equations takes the form*

$$\varPsi_{(i),3} + P^\mu_{(i),\mu} + S_{(i)} = 0,\ S_{(i)} = -\Lambda^1_{(i)}P - \Lambda^2_{(i)}Q, \tag{8}$$

on the smooth solutions of the non-homogeneous time-dependent MvK equations;

$$S_{(i)} = \widetilde{\varPsi}_{(i),3} + \widetilde{P}^\mu_{(i),\mu},$$

iff (6) hold, and in this case (8) becomes a proper conservation, otherwise it has supply (production) $S_{(i)}$.

Note that the source therms in (7) and (8) appear due to the curvature of the shell.

4 Balance laws

Given a region Θ in the shell middle-surface with sufficiently smooth boundary Σ of outward unit normal n_α, a balance law

$$\int_\Sigma A^\alpha_{(j)} n_\alpha d\Sigma + \int_\Theta S_{(j)} dx^1 dx^2 = 0, \tag{9}$$

corresponds to each of the nine basic conservation laws of fluxes $A^\alpha_{(j)}$ characteristic $\Lambda^\alpha_{(j)}$ $(j = 1, ..., 9)$ admitted by the smooth solutions of the homogeneous time-independent MvK equations (these conservation laws are listed in Appendix B [6]).

The same holds true for the fourteen basic conservation laws of densities $\Psi_{(i)}$, fluxes $P^\alpha_{(i)}$ and characteristics $\Lambda^\alpha_{(i)}$ $(i = 1, ..., 14)$ admitted by the smooth solutions of the homogeneous time-dependent MvK equations (see Appendix A [6] and Table 1 [7]). Namely, to each of them it corresponds a balance low

$$\frac{d}{dt}\int_\Theta \Psi_{(i)} dx^1 dx^2 + \int_\Sigma P^\alpha_{(i)} n_\alpha d\Sigma + \int_\Theta \int_T S_{(i)} dx^1 dx^2 dx^3 = 0, \tag{10}$$

where T is a certain time interval.

Both (9) and (10) hold, just as the respective conservation laws, for every smooth solution of the nonhomogeneous MvK equations.

In the static case the balance laws (9) provide a set of path-independent integrals inherent to Marguerre's shell theory. Among them are the counterparts of the well-known and widely used in fracture mechanics J-, L- and M-integrals. The applicability of the latter integrals in the analysis of cracked plates is discussed in [8] (see also the references therein). In the similar way, the path-independent integrals corresponding to the balance laws (9) can be used in the analysis of equilibrium and stability of shells undergoing stress concentrations near the tips of cracks and notches since they allow to compute the stress intensity factors and energy release rates, the former characterizing the distribution of the stress field in a vicinity of a certain singular point, say the crack tip, and the latter characterizing the propagation of the crack through the shell.

In the dynamic case, the balance laws (10) provide the theoretical background for studying the propagation of waves of discontinuity in shallow shells since they are applicable in the domains where some important physical quantities suffer jump discontinuities at a certain curve. Using the balance laws (10) one can extend the "continuous" Marguerre's shell theory in the same manner as it is done in [7] for the "continuous" von Kármán plate theory. At that Definition 1 [7] should be changed by supplying the integrals (4) with right-hand sides

$$\int_\Theta \int_T P dx^1 dx^2 dx^3, \quad \int_\Theta \int_T Q dx^1 dx^2 dx^3.$$

Then, Definition 1, Propositions 1 to 4 and the jump conditions listed in Table 2 remain the same under the assumption that P and Q are smooth functions.

5 Group-invariant solutions

In the cases when the MvK equations (3) or (4) admit a curtain subgroup of $G_{(S)}$ or $G_{(D)}$, respectively, it is worth looking for the corresponding group-invariant solutions. To obtain such solution one should follow the procedure described in details in [4]. Here, we would like only to notice that the group-invariant solutions to the homogeneous time-dependent MvK equations obtained in [6] and shown to determine acceleration waves in plates can be used for the same purpose in Marguerre's shell theory provided (according to Theorem 2) that P and Q are at the same time invariants of the group generated by X_6 and eigentfunctions of X_5. In this case the reduced system reads

$$\begin{aligned} D(u'' - 4u' + 4u)'' + (De^{4s}/4 - \varphi')u'' + (De^{4s}/2 - \varphi'' + 2\varphi')u' &= P, \\ (\varphi'' - 4\varphi' + 4\varphi)'' + Eh(u'' - u')u' &= Q, \end{aligned}$$

where $s = (1/2)\ln(\sqrt{\rho/D}r^2/t)$, $r^2 = (x^1)^2 + (x^2)^2$, $u(s)$ and $\varphi(s)$ are the new dependent variables, and the prime denotes differentiation with respect to the argument s.

As for the group-invariant solutions describing traveling waves in plates discussed in Section 5 [7], now, according to Theorem 1, P and Q are to be joined invariants of the group generated by X_3 and $X_2 + (1/c)X_4$.

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