ON THE INVARIANCE OF TIMOSHENKO BEAM EQUATIONS

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ABSTRACT
This study is concerned with the group-invariance properties of a fourth-order linear partial differential equation, arising in the dynamics of Timoshenko beams. All variational symmetries admitted by this equation are obtained and the conservation laws associated with Noether's theorem are constructed. It is shown, that these conservation laws being derived for the single fourth-order differential equation under consideration are also valid for the system of two second-order partial differential equations governing the dynamics of Timoshenko beams (known as Timoshenko beam equations).

INTRODUCTION
The differential equations governing the small vibration of homogeneous Timoshenko beams are [1]:

$$\begin{aligned} E_1 &= EJ\varphi_{11} + kGA(w_1 - \varphi) - \rho J\varphi_{22} = 0 \\ E_2 &= kGA(w_{11} - \varphi_1) - \rho A w_{22} = 0, \end{aligned} \tag{1}$$

where ρ is the constant mass density, G and E are the shear and Young's moduli of the beam material, A and J are the cross section area and inertia moment, respectively, and k is the shear correction factor. The dependent variables are φ – the angle between the deformed and reference states of the beam cross section, and w – the transverse displacement of the beam axis. The subscripts at a dependent variable denote partial derivatives with respect to the independent variables x^1 and x^2 – the coordinate along the beam axis and time, respectively. Eliminating φ from this system, we obtain an equation for w of form

$$E_0 = EJw_{1111} - \rho J\left(1 + 2\frac{1+\nu}{k}\right)w_{1122} + \frac{2\rho^2 J(1+\nu)}{kE}w_{2222} + \rho A w_{22} = 0, \tag{2}$$

where the relation $E = 2G(1+\nu)$ is taken into account, ν being Poisson's ratio.

It is well known (see [1]) that (1) admits an exact variational formulation. The same holds for the single equation (2) as well, and in the present study the action functional

$$A[w] = \frac{1}{2}\int wE_0 dx^1 dx^2,$$

is considered whose Euler-Lagrange equation coincides with (2).

The invariance properties of (1) with respect to local Lie groups of point transformations are established in [2]. The aim of the present study is to explore the invariance properties of equation (2).

LIE POINT SYMMETRIES
The infinitesimal generator of a local one-parameter Lie group of local point transformations acting on some open subset Ω of the space $\mathbf{R}^3$ representing the independent and dependent variables x^1, x^2 and w involved in our basic equation (2) is a vector field X on $\mathbf{R}^3$ of form

$$X = \xi^1(x^1,x^2,w)\frac{\partial}{\partial x^1} + \xi^2(x^1,x^2,w)\frac{\partial}{\partial x^2} + \eta(x^1,x^2,w)\frac{\partial}{\partial w}, \tag{3}$$

whose components $\xi^\mu(x^1,x^2,w)$ and $\eta(x^1,x^2,w)$ are supposed to be functions of class C^∞ on Ω. Applying the standard computational procedure (see, e.g. [3,4]) and omitting the details, one could obtain that the equation (2) admits the point Lie groups generated by the vector fields

$$X_0 = w\frac{\partial}{\partial w}, \quad X_1 = \frac{\partial}{\partial x^1}, \quad X_2 = \frac{\partial}{\partial x^2}, \quad X_u = u(x^1,x^2)\frac{\partial}{\partial w}, \tag{4}$$

where $u(x^1,x^2)$ is an arbitrary solution of the equation considered. It is well known that having obtained the Lie point symmetries of a certain equation, the determination of its variational symmetries is straightforward (see [3,4]). Applying the respective procedure [3,4] to the Lie point symmetries (4) of the equation under consideration, one obtains that the variational (point) symmetries of (2) are linear combinations of the vector fields

$$X_1 = \frac{\partial}{\partial x^1}, \quad X_2 = \frac{\partial}{\partial x^2}, \quad X_u = u(x^1,x^2)\frac{\partial}{\partial w}. \tag{5}$$

CONSERVATION LAWS

In this Section we intend to derive conservation laws, that are relations of form

$$(6)\qquad D_2\Psi + D_1 P = 0,\quad D_\alpha = \frac{\partial}{\partial x^\alpha} + w_\alpha \frac{\partial}{\partial w} + w_{\alpha\beta}\frac{\partial}{\partial w_\beta} + \ldots,\quad \alpha = 1,2,$$

valid on the smooth solutions of equation (2). Here, D_α are the total derivatives, and Ψ and P are functions of the independent and dependent variables involved in equation (2) as well as of the derivatives of the dependent variable.

Following [3,4], the densities and fluxes of the conservation laws associated through Noether's theorem with the variational symmetries (5) are obtained to be

$$\Psi_{(1)} = -\frac{\rho J}{12}\left(1+2\frac{1+\nu}{k}\right)(3w_{11}w_{12} - 8w_1 w_{112} - w_2 w_{111})$$
$$+\frac{\rho^2 J(1+\nu)}{kE}(w_{12}w_{22} - 2w_1 w_{222} - w_2 w_{122}) + \frac{1}{2}\rho A(ww_{12} - w_1 w_2),$$

$$P_{(1)} = \frac{1}{2}EJ(w_{11}w_{11} - 2w_1 w_{111}) + \frac{\rho^2 J(1+\nu)}{kE}w_2 w_{222}$$
$$-\frac{\rho J}{12}\left(1+2\frac{1+\nu}{k}\right)(w_{11}w_{22} + 2w_{12}w_{12} + w_2 w_{112} - 4w_1 w_{122}) - \frac{1}{2}\rho A w w_{22},$$

$$\Psi_{(2)} = \frac{1}{2}EJw_1 w_{111} - \frac{\rho J}{12}\left(1+2\frac{1+\nu}{k}\right)(w_{11}w_{22} + 2w_{12}w_{12} + w_1 w_{122} - 4w_2 w_{112})$$
$$+\frac{\rho^2 J(1+\nu)}{kE}(w_{22}w_{22} - 2w_2 w_{222}) - \frac{1}{2}\rho A w_2 w_2,$$

$$P_{(2)} = \frac{1}{2}EJ(w_{11}w_{12} - w_1 w_{112} - 2w_2 w_{111})$$
$$-\frac{\rho J}{12}\left(1+2\frac{1+\nu}{k}\right)(3w_{12}w_{22} - w_1 w_{222} - 8w_2 w_{122}),$$

$$\Psi_{(u)} = \frac{\rho J}{12}\left(1+2\frac{1+\nu}{k}\right)(-3uw_{112} + 2u_1 w_{12} + u_2 w_{11} - u_{11}w_2 - 2u_{12}w_1 + 3u_{112}w)$$
$$+\frac{\rho^2 J(1+\nu)}{kE}(uw_{222} - u_2 w_{22} + u_{22}w_2 - u_{222}w) + \frac{1}{2}\rho A(uw_2 - u_2 w),$$

$$P_{(u)} = \frac{1}{2}EJ(uw_{111} - u_1 w_{11} + u_{11}w_1 - u_{111}w)$$
$$-\frac{\rho J}{12}\left(1+2\frac{1+\nu}{k}\right)(-3uw_{122} + u_1 w_{22} + 2u_2 w_{12} - 8u_{12}w_2 - u_{22}w_1 + 3u_{122}w).$$

These conservation laws correspond to fundamental physical principles. The conservation law with density and flux $(\Psi_{(1)}, P_{(1)})$ is a representation of the wave momentum conservation law. Its validity is a consequence of the assumption that the beam material is homogeneous (all quantities, associated with the beam properties does not depend on the independent variable. The conservation law associated with $(\Psi_{(2)}, P_{(2)})$ represents the energy conservation law for the equation (2). Here, $\Psi_{(2)}$ is the energy density function and $P_{(2)}$ is the flux. The validity of the energy conservation law for the equation (2) is a result from the basic assumption that neither of the quantities associated with beam properties and involved in this equation depends on the time. Finally, the linearity of the equation under consideration (see [3,4]) leads to the validity of the conservation law with density and flux $(\Psi_{(u)}, P_{(u)})$. This conservation law is a particular representation of the reciprocity relation

$$D_2\Psi_{(u)} + D_1 P_{(u)} = u\left\{EJw_{1111} - \rho J\left(1+2\frac{1+\nu}{k}\right)w_{1122} + \frac{2\rho^2 J(1+\nu)}{kE}w_{2222} + \rho A w_{22}\right\}$$
$$- w\left\{EJu_{1111} - \rho J\left(1+2\frac{1+\nu}{k}\right)u_{1122} + \frac{2\rho^2 J(1+\nu)}{kE}u_{2222} + \rho A u_{22}\right\},$$

associated with equation (2) which is valid for an arbitrary couple of smooth functions (u,w). If u is a solution of (2), this identity reduces to the conservation law with density and flux $(\Psi_{(u)}, P_{(u)})$. The foregoing reciprocity relation is of the same nature as the Betti reciprocal theorem in elasticity, both relations being associated with the linearity of the respective governing equations (see [3,4]).

CHARACTERISTIC FORM OF THE CONSERVATION LAWS

It is well known (see [4]) that given a system in m independent and n dependent variables:

$$E_\mu = 0, \quad \mu = 1,2,\dots,n,$$

each conservation law for its solutions is equivalent to a conservation law in characteristic form

$$D_\alpha P^\alpha = Q^\mu E_\mu,$$

where $P = (P^1, P^2, \dots, P^m)$ is the conserved current and $Q = (Q^1, Q^2, \dots, Q^n)$ is the characteristic of the conservation law in question. The characteristic forms of the conservation laws with densities and fluxes $(\Psi_{(1)}, P_{(1)})$, $(\Psi_{(2)}, P_{(2)})$ and $(\Psi_{(u)}, P_{(u)})$ established in the previous Section are

$$\begin{aligned} &D_1 P_{(\alpha)} + D_2 \Psi_{(\alpha)} = -w_\alpha E_0, \quad \alpha = 1,2, \\ &D_1 P_{(u)} + D_2 \Psi_{(u)} = u E_0. \end{aligned} \tag{7}$$

Using the obvious identity

$$E_0 = -E_2 + D_1 E_1 + \frac{2J(1+\nu)}{kA} D_1 D_1 E_2 - \frac{\rho J}{kGA} D_2 D_2 E_2,$$

one could easily eliminate E_0 from (7) and to obtain

$$\begin{aligned} &D_1\left(P_{(\alpha)} + R^1_{(\alpha)}\right) + D_2\left(\Psi_{(\alpha)} + R^{`2}_{(\alpha)}\right) = Q^\mu_{(\alpha)} E_\mu, \quad \alpha = 1,2, \\ &D_1\left(P_{(u)} + R^1_{(u)}\right) + D_2\left(\Psi_{(u)} + R^{`2}_{(u)}\right) = Q^\mu_{(u)} E_\mu, \end{aligned} \tag{8}$$

where

$$\begin{aligned} &Q^1_{(\alpha)} = w_{1\alpha}, \quad Q^2_{(\alpha)} = w_\alpha - \frac{2J(1+\nu)}{kA} w_{11\alpha} + \frac{\rho J}{kGA} w_{22\alpha}, \quad \alpha = 1,2, \\ &Q^1_{(u)} = -u_1, \quad Q^2_{(u)} = -u + \frac{2J(1+\nu)}{kA} u_{11} - \frac{\rho J}{kGA} u_{22}, \end{aligned} \tag{9}$$

and

$$\begin{aligned} &R^1_{(\alpha)} = w_\alpha E_1 + \frac{2J(1+\nu)}{kA}(w_\alpha D_1 E_2 - w_{1\alpha} E_2), \quad R^2_{(\alpha)} = \frac{\rho J}{kGA}(w_{2\alpha} E_2 - w_\alpha D_2 E_2), \quad \alpha = 1,2, \\ &R^1_{(u)} = -u_\alpha E_1 + \frac{2J(1+\nu)}{kA}(u_1 E_2 - u D_1 E_2), \qquad R^2_{(u)} = \frac{\rho J}{kGA}(u D_2 E_2 - u_2 E_2). \end{aligned}$$

Observing (8) one can deduce that $(\Psi_{(1)}, P_{(1)})$, $(\Psi_{(2)}, P_{(2)})$ and $(\Psi_{(u)}, P_{(u)})$ are densities and fluxes of conservation laws for the solutions of (1) because $R^\mu_{(\alpha)}$ and $R^\mu_{(u)}$ are currents of trivial conservation laws [4], vanishing identically on the solutions of the Timoshenko beam equations (1).

CONCLUDING REMARKS

The obtained conservation laws could be successfully used in analysis of various problems of engineering interest. Indeed, integrating (6) over a certain interval (a,b) of the beam span, a balance law of form

$$\frac{d}{dx^2}\int_a^b \Psi dx^1 = P_a - P_b, \qquad P_a = P\big|_{x^1=a}, \qquad P_b = P\big|_{x^1=b}, \tag{10}$$

is obtained. Balance laws, derived in this manner, hold for every interval (a,b) where the dependent variable is a smooth function. In such intervals, they are equivalent to the respective conservation laws. However, if such a balance law is supposed to hold for an interval (a,b) where the dependent variable or some of its derivatives suffer jump discontinuities at a point $x^1_j \in (a,b)$, then Kochin's theorem [5] provides a relationship for the limit values of discontinuous quantities on both sides of x^1_j. In such a manner, one could obtain nontrivial relations useful in analysis of shock and acceleration waves.

The conservation laws obtained above could also be applied in particular Timoshenko beam problems in the following sense. Suppose the beam ends are $x^1 = a$ and $x^1 = b$, and the boundary conditions are such that $P_a - P_b = 0$ for certain conservation law. Then, the respective balance law (10) implies that the integral over the span of the associated density is a conserved quantity of form

$$\text{(11)} \qquad \frac{d}{dx^2}\int_a^b \Psi dx^1 = 0,$$

whatever motion this Timoshenko beam undergoes.

Finally, it is to be underlined that balance laws similar to (10) or (11) are widely applied in analysis of bodies with cracks, notches, etc., (see [6], where such relations, known as *J*- *L*- and *M*- integrals, are derived within plate theories).

We emphasize that each solution of the system (1) is a solution of the equation (2) as well. Hence, each conservation law for the solutions of (2) holds for the solutions of (1) also, which in particular is valid for conservation laws with densities and fluxes $(\Psi_{(1)}, P_{(1)})$, $(\Psi_{(2)}, P_{(2)})$ and $(\Psi_{(u)}, P_{(u)})$.

The invariance properties of Timoshenko beam equations (1) are studied in [2]. In that paper, the vector fields X_1 and X_2 are identified to be infinitesimal variational symmetries of the system (1) and the conservation laws for wave momentum (associated with X_1) and energy (associated with X_2) are derived therein. The conservation laws with densities and fluxes $(\Psi_{(1)}, P_{(1)})$, $(\Psi_{(2)}, P_{(2)})$ and $(\Psi_{(u)}, P_{(u)})$ which hold for the solutions of the system (1) are different from the conservation laws in [2], because the latter correspond to geometric symmetries while observation of (9) implies that the conservation laws presented here correspond to generalized symmetries of (1). Consequently, the conservation laws derived here for (2) are new conservation laws for Timoshenko beam equations (1) as well.

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