Lie Symmetries and Conservation Laws for 2+1 Dimensional Linear Plate Problems Allowing an Exact Variational Formulation

V. Vassilev

The present notes are concerned with the infinitesimal divergence symmetries of functionals arising in plate theory. The results obtained may be regarded as a first step towards the derivation of conservation laws admitted by the solutions of the associated Euler-Lagrange equations, the latter being the equations of motion for thin elastic plates.

Within the framework of the linear plate theory, a fourth-order linear partial differential equation of the form

$$\Delta[D(x^1,x^2)\Delta w] - [(1-\nu)\epsilon^{\alpha\mu}\epsilon^{\beta\nu}D(x^1,x^2)_{,\alpha\beta} - N^{\mu\nu}(x^1,x^2)]w_{\mu\nu} + k(x^1,x^2)w + \rho(x^1,x^2)w_{33} = 0, \quad (1)$$

in three independent variables – the coordinates of the plate middle plane x^1, x^2 and the time x^3, and one dependent variable – the transversal displacement field w, describes the motion of a thin elastic plate of bending rigidity D, mass density ρ, resting on an elastic foundation with modulus k, and subjected to edge loadings leading to appearance of membrane stresses $N^{\alpha\beta} = N^{\beta\alpha}$, $N^{\alpha\mu}_{,\mu} = 0$. Here: $\Delta \equiv \delta^{\alpha\beta}\partial^2/\partial x^\alpha \partial x^\beta$; $\delta^{\alpha\beta}$ is the Kronecker delta symbol; $\epsilon^{\alpha\beta}$ is the alternating symbol; Greek and Latin indices range over 1, 2 and 1, 2, 3, respectively, and the usual summation rule is employed; ν is Poison's ratio; $w_{i_1\ldots i_k} \equiv \partial^k w/\partial x^{i_1}\ldots\partial x^{i_k}$; if $F = F\left(x^1,x^2,x^3\right)$, then $F_{,i_1\ldots i_k} \equiv \partial^k F/\partial x^{i_1}\ldots\partial x^{i_k}$.

It is easy to see that (1) is the Euler-Lagrange equation associated with the variation functional

$$A[w] = \frac{1}{2}\iiint L dx^1 dx^2 dx^3, \; L = \frac{1}{2}\{D[\delta^{\alpha\beta}\delta^{\mu\nu} - (1-\nu)\epsilon^{\alpha\mu}\epsilon^{\beta\nu}]w_{\alpha\beta}w_{\mu\nu} - N^{\alpha\beta}w_\alpha w_\beta - \rho w_3^2 + kw^2\}. (2)$$

In order to verify this, it suffices to apply the Euler operator $\mathbf{E} = \partial/\partial w - \mathbf{D}_i\partial/\partial w_i + \mathbf{D}_i\mathbf{D}_j\partial/\partial w_{ij} - \ldots$ to the Lagrangian density function L. Here $\mathbf{D}_i = \partial/\partial x^i + w_i\partial/\partial w + w_{ij}\partial/\partial w_j + \ldots$ is the total derivative operator.

The aim of the work is to study, following [1] (see also [2, 3]), the (classical) infinitesimal (i.) divergence (d.) symmetries of the functional $A[w]$. There are at least two reasons for wanting to find these symmetries. First: in virtue of Noether's theorem, a conservation law admitted by the solutions of the associated Euler-Lagrange equation (1) corresponds to each such symmetry. Second: every i.d. symmetry of $A[w]$ is a Lie symmetry of (1).

By definition (see 4.43 [1]), an i.d. symmetry of the functional $A[w]$ is a vector field

$$\mathbf{v} = \xi^j\left(x^1,x^2,x^3,w\right)\partial/\partial x^j + \eta\left(x^1,x^2,x^3,w\right)\partial/\partial w, \quad (3)$$

on $\mathbf{R}^4(x^1,x^2,x^3,w)$ such that the following infinitesimal criterion of invariance (up to a divergence term)

$$\mathrm{pr}^{(2)}\,\mathbf{v}\,(L) + \left(\mathbf{D}_i\xi^i\right)L = \mathbf{D}_i B^i \quad (4)$$

holds for some set of differential functions B^i (i.e., functions of the independent and dependent variables and derivatives of the dependent variable). Here $\mathrm{pr}^{(2)}\,\mathbf{v}$ is the 2nd prolongation of the vector field $\mathbf{v}$. On the other hand, taking into account Theorem 4.7 [1] we observe the following:

T h.*e invariance criterion (4) is equivalent to the following relation*

$$\mathbf{E}\left[\mathrm{pr}^{(2)}\,\mathbf{v}\,(L) + \left(\mathbf{D}_i\xi^i\right)L\right] = 0. \quad (5)$$

Thus, in order to find the system of determining equations for the i.d. symmetries of the functional $A[w]$, we should let the coefficients ξ^j and η of the vector field (3) be unknown functions of x^1, x^2, x^3 and w, then write out in full the left-hand side of (5) by using (2) and the prolongation formulae (2.38) and (2.39) given in [1], and finally we should equate the coefficients of w and it derivatives to zero. In this way, we arrive at the following result.

(i) E.ach i.d. symmetry of the functional $A[w]$ has the form

$$\mathbf{v} = \xi^\mu\left(x^1,x^2\right)\partial/\partial x^\mu + \left(2C_1x^3 + C_2\right)\partial/\partial x^3 + \left[C_3 w + u\left(x^1,x^2,x^3\right)\right]\partial/\partial w, \quad (6)$$

where C_1, C_2 and C_3 are real constants, $u\left(x^1,x^2,x^3\right)$ is a solution of the equation (1), and

$$\delta^{\alpha\mu}\xi^\beta_{,\mu} + \delta^{\mu\beta}\xi^\alpha_{,\mu} = \delta^{\alpha\beta}\xi^\mu_{,\mu}\;; \quad (7)$$

(ii) $A[w]$ possesses i.d. symmetries of the form (6) if and only if the functions $D, N^{\alpha\beta}, k$ and ρ are such that

$$\xi^\mu D_{,\mu} - D\xi^\mu_{,\mu} = -2D(C_1 + C_3), \tag{8}$$

$$\xi^\mu k_{,\mu} + k\xi^\mu_{,\mu} = -2k(C_1 + C_3), \tag{9}$$

$$\xi^\mu \rho_{,\mu} + \rho\xi^\mu_{,\mu} = -2\rho(C_1 - C_3), \tag{10}$$

$$\xi^\mu N^{\alpha\beta}_{,\mu} - N^{\alpha\mu}\xi^\beta_{,\mu} - N^{\mu\beta}\xi^\alpha_{,\mu} - N^{\alpha\beta}\left[\xi^\mu_{,\mu} - 2(C_1 - C_3)\right] = H^{\alpha\beta}, \tag{11}$$

$$H^{\alpha\beta} = (1-\nu)\, D_{,\mu}\left[\epsilon^{\lambda\mu}\epsilon^{\tau\beta}\xi^\alpha_{,\lambda\tau} + \epsilon^{\lambda\mu}\epsilon^{\tau\alpha}\xi^\beta_{,\lambda\tau} - \epsilon^{\lambda\alpha}\epsilon^{\tau\beta}\xi^\mu_{,\lambda\tau}\right]. \tag{12}$$

Note that (7) is the determining equation of the (pseudo) group of conformal transformations of the Euclidean plain. Equations (8) – (12) show that the space of solutions to the whole system of determining equations (7) – (12), i.e., the Lie algebra of i.d. symmetries of $A[w]$, will depend on the form of the functions D, $N^{\alpha\beta}$, k and ρ. At this juncture, we face a rather complicated group classification problem. Its solution will be the topic of another paper.

1. Conservation laws and group-invariant solutions

Let $\mathbf{v}$ be an i.d. symmetry of $A[w]$, i.e., (4) holds for a set of differential functions B^i. Then a conservation law

$$\mathbf{D}_i A^i = 0, \quad A^i = \xi^i L + \left(C_3 w + u - w_j\xi^j\right)\left[\frac{\partial L}{\partial w_i} - \mathbf{D}_k\frac{\partial L}{\partial w_{ik}}\right] + \left[\mathbf{D}_k\left(C_3 w + u - w_j\xi^j\right)\right]\frac{\partial L}{\partial w_{ik}} - B^i, \tag{13}$$

admitted by the solutions of equation (1) may be derived via Noether's theorem (see [1, Sec. 4.4]). A group-invariant solution of (1) may be also found out as $\mathbf{v}$ will be a Lie symmetry of this equation (see Theorem 4.34 [1]).

G i.*ven the equation $D\Delta\Delta w + kw + \rho w_{tt} = 0$, where D, k and ρ are constants, $t \equiv x^3$, and taken u to denote an arbitrary solution to this equation, Theorem 1 implies that the Lie algebra of i.d. symmetries of the corresponding variational functional (2) is spanned over the following five linearly independent vector fields:*

$$\mathbf{v}_1 = \partial/\partial t,\ \mathbf{v}_2 = \partial/\partial x^1,\ \mathbf{v}_3 = \partial/\partial x^2,\ \mathbf{v}_4 = x^2\partial/\partial x^1 - x^1\partial/\partial x^2,\ \mathbf{v}_5 = u\partial/\partial w. \tag{14}$$

Hence, using (2), (13) and (14) we can derive five linearly independent conservation laws related to $\mathbf{v}_1, ..., \mathbf{v}_5$. The conserved densities corresponding to $\mathbf{v}_1, ..., \mathbf{v}_4$ have clear physical meaning; these are the energy, the wave momentum, and the moment of the wave momentum, respectively. The conservation law associated with $\mathbf{v}_5$ is an analog of Betti's reciprocal theorem. If $k = 0$, then two more linearly independent i.d. symmetries appear:

$$\mathbf{v}_6 = \partial/\partial w,\ \mathbf{v}_7 = x^1\partial/\partial x^1 + x^2\partial/\partial x^2 + w\partial/\partial w. \tag{15}$$

In this case, some combinations of the vector fields (14) and (15) lead to interesting group-invariant solutions.

Acknowledgements

The research was supported by the National Scientific Fund, project TH 243/1992.

2. References

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Address: Dr. V. Vassilev, Department of Continuum Mechanics, Institute of Mechanics, Bulgarian Academy of Sciences, Acad. G. Bontchev St., Block 4, 1113 Sofia, BULGARIA. e-mail: vasilvas@imbm.bas.bg