Eigenstructure approach for complete characterization of linear phase FIR perfect reconstruction analysis length $2M$ filterbanks

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Abstract

The eigenstructure based characterization of M-channel FIR PR filterbanks of [1] is extended here to the linear phase case. Some results relating to linear phase filterbanks is derived by finding appropriate restrictions on the eigenstructure of the analysis polyphase matrix. Consequently a complete and minimal characterization for such filterbanks with all analysis length $2M$ and any synthesis length is developed. Parameterization and design examples are also presented.

1 Introduction

In [1] we used the *eigenstructure* representation of the *polyphase matrix* to propose complete characterizations of FIR *perfect reconstruction* M channel filterbanks with first order analysis polyphase matrix. *Linear phase* FIR perfect reconstruction filterbanks (LPFB) find application in many signal and image processing fields. In this correspondence we extend the eigenstructure representation to obtain a complete characterization of linear phase FIR perfect reconstruction M channel filterbanks with all analysis filter length being $2M$ (hence first order analysis polyphase matrix), referred henceforth as FOLPFB. The synthesis filter length in this characterization is not restricted to $2M$ as is conventionally done, but may take value upto M^2.

Characterization of a subclass of LPFB, such as orthogonal, $M = 2$, or $M = 3$ has been reported on several occasions. Design of FOLPFB with a multi-stage structure using DCT such that a fast implementation exists is reported in [2], where it is called the *lapped biorthog-*

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onal transform. Design of LPFB of any order is reported in [3], when first K filters are given such that the part polyphase matrix has rank K for all z^{-1} except $z^{-1} = 0$, and the remaining filters are designed. In [4] symbolic computation is used to characterize LPFB of any order. However, none of the above characterizations are complete. In [5] a lattice structure is used to characterize the LPFB, so that the analysis polyphase matrix of FOLPFB becomes $\frac{1}{2\sqrt{2}}\begin{bmatrix} (1+z^{-1})\mathbf{U} & (1-z^{-1})\mathbf{U} \\ (1-z^{-1})\mathbf{V} & (1+z^{-1})\mathbf{V} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{U}_0 & \mathbf{U}_0\mathbf{J}_{M/2} \\ \mathbf{V}_0\mathbf{J}_{M/2} & -\mathbf{V}_0 \end{bmatrix}$ where $\mathbf{U}$, $\mathbf{V}$, $\mathbf{U}_0$ and $\mathbf{V}_0$ are nonsingular $\frac{M}{2} \times \frac{M}{2}$ matrices, and $\mathbf{J}_k$ is $k \times k$ counter identity matrix. It is shown to be complete for FOLPFB with synthesis length $2M$ [6].

We briefly describe below the characterization of [1]. Replacing z^{-1} by λ, the lth order analysis polyphase matrix $\mathbf{E}(z)$ is seen as a matrix polynomial $\mathbf{E}_l(\lambda)$. Any matrix polynomial may be characterized by the *Jordan pair* (or decomposable pair, or spectral data) $(\mathbf{Y}, \mathbf{T}(\lambda))$ with $\mathbf{Y} = [\mathcal{X}_F \ \mathcal{X}_R]$ and $\mathbf{T}(\lambda) = \text{diag}(\mathbf{I}_\Gamma\lambda - \mathcal{J}_F, \mathcal{J}_R\lambda - \mathbf{I}_{Ml-\Gamma})$, where diag() represents a block diagonal matrix with the arguments as the blocks in sequence, $\mathbf{I}_k$ is $k \times k$ identity matrix, and Γ is the degree of $|\mathbf{E}_l(\lambda)|$. $\mathcal{X}_F$ is the $M \times \Gamma$ canonical set of *Jordan chains* and $\mathcal{J}_F$ is the $\Gamma \times \Gamma$ *Jordan form* of $\mathbf{E}_l(\lambda)$ (finite Jordan pair, or finite spectrum). $\mathcal{J}_F$ is block diagonal with Jordan blocks of size $b_0, ..., b_n$ such that b_i are non-increasing positive integers summing up to Γ. Each Jordan block's diagonal elements are eigenvalues of $\mathbf{E}_l(\lambda)$, upper off-diagonal elements are 1, and remaining elements are 0.[1] $\mathcal{X}_R$ and $\mathcal{J}_R$ are the corresponding $M \times (Ml-\Gamma)$ and $(Ml-\Gamma) \times (Ml-\Gamma)$ matrices of the reversed matrix polynomial $\lambda^l\mathbf{E}_l(\lambda^{-1})$ for the zero eigenvalue (infinite Jordan pair, or infinite spectrum). It follows that [7]

$$\mathbf{E}_l(\lambda) = \mathbf{A}(\mathbf{I}_{Ml} - \mathbf{P})\mathbf{T}(\lambda)\mathbf{S}_{l-1}^{-1}\mathbf{Q}(\lambda) \tag{1}$$

where

$$\mathbf{S}_k = \begin{bmatrix} \mathcal{X}_F & \mathcal{X}_R\mathcal{J}_R^k \\ \mathcal{X}_F\mathcal{J}_F & \mathcal{X}_R\mathcal{J}_R^{k-1} \\ \vdots & \vdots \\ \mathcal{X}_F\mathcal{J}_F^k & \mathcal{X}_R \end{bmatrix}, \qquad \mathbf{Q}(\lambda) = \begin{bmatrix} \mathbf{I}_M \\ \lambda\mathbf{I}_M \\ \vdots \\ \lambda^{l-1}\mathbf{I}_M \end{bmatrix} \tag{2}$$

$\mathbf{A}$ is $M \times Ml$ matrix such that $[\mathbf{S}_{l-2}^T \ \mathbf{A}^T]^T$ is nonsingular, and $\mathbf{P} = \text{diag}(\mathbf{I}_\Gamma, \mathcal{J}_R)\mathbf{S}_{l-1}^{-1}[\mathbf{I}_{Ml-M} \ \mathbf{0}]^T\mathbf{S}_{l-2}$.

For FIR inverse to exist, $\mathbf{E}_l(\lambda)$ has to be a matrix polynomial with monomial determinant. This is equivalent to all eigenvalues of $\mathbf{E}_l(\lambda)$ being zero. Thus, $\mathcal{J}_F$ (as also $\mathcal{J}_R$) should have

[1]For example, for $\Gamma = 4$, $\{b_i\} = \{3, 1\}$ and zero eigenvalue, $\mathcal{J}_F = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

a zero diagonal. Further, for the first order ($l = 1$) case (1) simplifies to the *block diagonal* characterization [1]

$$\mathbf{E}_1(\lambda) = \mathbf{A}\mathbf{T}(\lambda)\mathbf{Y}^{-1} \tag{3}$$

where $\mathbf{A}$ and $\mathbf{Y}$ are any $M \times M$ nonsingular matrices. Note that $\mathcal{J}_F$, $\mathcal{J}_R$ are *nilpotent* matrices with indices of nilpotency n_F and n_R ($n_F = b_0$, size of the largest Jordan block of $\mathcal{J}_F$, etc.). Then the synthesis polyphase becomes $\mathbf{E}_1^{-1}(\lambda) = \mathbf{Y}\text{diag}(\mathbf{I}_\Gamma\lambda^{-1} + \sum_{i=2}^{n_F} \mathcal{J}_F^{i-1}\lambda^{-i}, -\mathbf{I}_{M-\Gamma} - \sum_{i=1}^{n_R-1} \lambda^i \mathcal{J}_R^i)\mathbf{A}^{-1}$. It follows that the maximum length of synthesis filters is $(n_F + n_R)M$, and reconstruction delay is $(n_F + 1)M - 1$. The characterization allows unconstrained parameter optimization and provides control over the length of the synthesis filters and reconstruction delay.

2 Some Results for LPFB

For an M-channel LPFB with analysis filter lengths $Mk_0 + s, ..., Mk_{M-1} + s$ for integer k_i and $0 \le s < M$, the analysis polyphase matrix $\mathbf{E}(z)$ satisfies [8]

$$\mathbf{E}(z) = \mathbf{D}\text{diag}(z^{-k_0+1}, ..., z^{-k_{M-1}+1})\mathbf{E}(z^{-1})\text{diag}(z^{-1}\mathbf{J}_s, \mathbf{J}_{M-s}) \tag{4}$$

where $\mathbf{D}$ is an $M \times M$ diagonal matrix with $\lceil \frac{M}{2} \rceil$ diagonal elements 1 (corresponds to symmetric filters) and rest -1 (antisymmetric filters). Without loss of generality we assume the first $\lceil \frac{M}{2} \rceil$ elements are 1.

Theorem 1: For a LPFB with analysis filter lengths $Mk_0 + s, ..., Mk_{M-1} + s$ and average length L_{avg}, $\Gamma = (L_{avg} - M)/2$.

Proof: Since Γ is the degree of $|\mathbf{E}_l(\lambda)|$, equating the degree of determinant of both sides of (4), $\Gamma = k_0 + ... + k_{M-1} - M - \Gamma + s$. Putting $L_{avg} = k_0 + ... + k_{M-1} + s$ the result is obtained. □

Consider all analysis filter lengths to be $M(l+1)$ ($k_i = l + 1, s = 0$). Then (4) reduces to

$$\mathbf{E}_l(\lambda) = \mathbf{D}\lambda^l\mathbf{E}_l(\lambda^{-1})\mathbf{J}_M \tag{5}$$

and from theorem 1, $\Gamma = \frac{Ml}{2}$.

Theorem 2: For a LPFB with analysis filter lengths $M(l+1)$, $\mathcal{J}_R = \mathcal{J}_F$ and $\mathcal{X}_R = \mathbf{J}_M\mathcal{X}_F$.

Proof: $\mathcal{J}_F$ and $\mathcal{X}_F$ are Jordan form and canonical Jordan chain of $\mathbf{E}_l(\lambda)$ for zero eigenvalue. Further, $\mathcal{J}_R$ and $\mathcal{X}_R$ are Jordan form and chain of $\lambda^l\mathbf{E}_l(\lambda^{-1})$ for zero eigenvalue. Pre-multiplication by a constant nonsingular matrix does not change the Jordan form or chain [7]. Therefore, $\mathcal{J}_R$ and $\mathcal{X}_R$ are Jordan form and chain of $\mathbf{D}\lambda^l\mathbf{E}_l(\lambda^{-1})$. Post-multiplication by a constant nonsingular matrix does not change the Jordan form, but the Jordan chain is pre-multiplied by the same matrix. Therefore, $\mathcal{J}_R$ and $\mathbf{J}_M\mathcal{X}_R$ are Jordan form and chain of $\mathbf{D}\lambda^l\mathbf{E}_l(\lambda^{-1})\mathbf{J}_M$. But from (5), $\mathbf{D}\lambda^l\mathbf{E}_l(\lambda^{-1})\mathbf{J}_M$ is $\mathbf{E}_l(\lambda)$. Therefore, $\mathcal{J}_R = \mathcal{J}_F$ and $\mathbf{J}_M\mathcal{X}_R = \mathcal{X}_F$, or $\mathcal{X}_R = \mathbf{J}_M\mathcal{X}_F$. □

Lemma 1: For a LPFB with analysis filter lengths $M(l+1)$, $\mathbf{A}(\mathbf{I}_{Ml} - \mathbf{P})$ is of the form $\begin{bmatrix} \mathbf{A}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_2 \end{bmatrix} \begin{bmatrix} \mathbf{I}_{Ml/2} & -\mathbf{I}_{Ml/2} \\ \mathbf{I}_{Ml/2} & \mathbf{I}_{Ml/2} \end{bmatrix}$, where $\mathbf{A}_1$, $\mathbf{A}_2$ are of size $\lceil \frac{M}{2} \rceil \times \frac{Ml}{2}$ and $\lfloor \frac{M}{2} \rfloor \times \frac{Ml}{2}$ respectively. This may be shown as follows. Substituting (1) on both sides of (5),

$$\mathbf{A}(\mathbf{I}_{Ml} - \mathbf{P})\mathbf{T}(\lambda)\mathbf{S}_{l-1}^{-1}\mathbf{Q}(\lambda) = \mathbf{D}\mathbf{A}(\mathbf{I}_{Ml} - \mathbf{P})\lambda\mathbf{T}(\lambda^{-1})\mathbf{S}_{l-1}^{-1}\lambda^{l-1}\mathbf{Q}(\lambda^{-1})\mathbf{J}_M. \tag{6}$$

Since $\mathcal{J}_R = \mathcal{J}_F$ and $\Gamma = \frac{Ml}{2}$, $\lambda\mathbf{T}(\lambda^{-1}) = \text{diag}(\mathbf{I}_{Ml/2} - \mathcal{J}_F\lambda, \mathcal{J}_F - \mathbf{I}_{Ml/2}\lambda) = -\mathbf{J}_I\mathbf{T}(\lambda)\mathbf{J}_I$ where $\mathbf{J}_I = \begin{bmatrix} \mathbf{0} & \mathbf{I}_{Ml/2} \\ \mathbf{I}_{Ml/2} & \mathbf{0} \end{bmatrix}$, since pre- and post-multiplication by $\mathbf{J}_I$ swaps two blocks of the diagonal of $\mathbf{T}(\lambda)$. Further, $\lambda^{l-1}\mathbf{Q}(\lambda^{-1})\mathbf{J}_M$ equals

$$\begin{bmatrix} \lambda^{l-1}\mathbf{I}_M \\ \lambda^{l-2}\mathbf{I}_M \\ \vdots \\ \mathbf{I}_M \end{bmatrix} \mathbf{J}_M = \begin{bmatrix} \lambda^{l-1}\mathbf{J}_M \\ \lambda^{l-2}\mathbf{J}_M \\ \vdots \\ \mathbf{J}_M \end{bmatrix} = \mathbf{J}_{Ml} \begin{bmatrix} \mathbf{I}_M \\ \vdots \\ \lambda^{l-2}\mathbf{I}_M \\ \lambda^{l-1}\mathbf{I}_M \end{bmatrix}$$

which is $\mathbf{J}_{Ml}\mathbf{Q}(\lambda)$. Therefore, (6) may be written as

$$\mathbf{A}(\mathbf{I}_{Ml} - \mathbf{P})\mathbf{T}(\lambda)\mathbf{S}_{l-1}^{-1}\mathbf{Q}(\lambda) = -\mathbf{D}\mathbf{A}(\mathbf{I}_{Ml} - \mathbf{P})\mathbf{J}_I\mathbf{T}(\lambda)\mathbf{J}_I\mathbf{S}_{l-1}^{-1}\mathbf{J}_{Ml}\mathbf{Q}(\lambda). \tag{7}$$

The constant matrices on both sides before $\mathbf{T}(\lambda)$ must be equal. Therefore $\mathbf{A}(\mathbf{I}_{Ml} - \mathbf{P}) = -\mathbf{D}\mathbf{A}(\mathbf{I}_{Ml} - \mathbf{P})\mathbf{J}_I$. Let $\mathbf{A}(\mathbf{I}_{Ml} - \mathbf{P}) = \begin{bmatrix} \mathbf{A}_1 & \mathbf{A}_5 \\ \mathbf{A}_2 & \mathbf{A}_6 \end{bmatrix}$ where $\mathbf{A}_1, \mathbf{A}_5$ are $\lceil \frac{M}{2} \rceil \times \frac{Ml}{2}$, and $\mathbf{A}_2, \mathbf{A}_6$ are $\lfloor \frac{M}{2} \rfloor \times \frac{Ml}{2}$. Since $\mathbf{D} = \text{diag}(\mathbf{I}_{\lceil M/2 \rceil}, -\mathbf{I}_{\lfloor M/2 \rfloor})$, it follows that $-\mathbf{D}\begin{bmatrix} \mathbf{A}_1 & \mathbf{A}_5 \\ \mathbf{A}_2 & \mathbf{A}_6 \end{bmatrix}\mathbf{J}_I = \begin{bmatrix} -\mathbf{A}_5 & -\mathbf{A}_1 \\ \mathbf{A}_6 & \mathbf{A}_2 \end{bmatrix}$, or $\mathbf{A}_5 = -\mathbf{A}_1$ and $\mathbf{A}_6 = \mathbf{A}_2$, or $\mathbf{A}(\mathbf{I}_{Ml} - \mathbf{P})$ is of the form mentioned earlier.

A comment on $\mathbf{A}(\mathbf{I}_{Ml} - \mathbf{P})$ is in order. It is known that $\mathbf{S}_{l-2}$ of size $(Ml - M) \times Ml$ is full rank for a valid Jordan pair [7]. Let $\mathbf{N}$ be the basis for the null space of $\mathbf{S}_{l-2}$. $\mathbf{A}$ may be expressed as $\mathbf{A}_7\mathbf{S}_{l-2} + \mathbf{A}_8\mathbf{N}$. Then $\mathbf{A}(\mathbf{I}_{Ml} - \mathbf{P}) = \mathbf{A}_7\mathbf{S}_{l-2}(\mathbf{I}_{Ml} - \mathbf{P}) + \mathbf{A}_8\mathbf{N}(\mathbf{I}_{Ml} - \mathbf{P})$. The first term $\mathbf{A}_7(\mathbf{S}_{l-2} - \mathbf{S}_{l-2}\mathbf{P})$ may be simplified. Since $\mathbf{S}_{l-2}\text{diag}(\mathbf{I}_\Gamma, \mathcal{J}_R)$ equals

$$\begin{bmatrix} \mathcal{X}_F & \mathcal{X}_R\mathcal{J}_R^{l-2} \\ \mathcal{X}_F\mathcal{J}_F & \mathcal{X}_R\mathcal{J}_R^{l-3} \\ \vdots & \vdots \\ \mathcal{X}_F\mathcal{J}_F^{l-2} & \mathcal{X}_R \end{bmatrix} \begin{bmatrix} \mathbf{I}_\Gamma & \mathbf{0} \\ \mathbf{0} & \mathcal{J}_R \end{bmatrix} = \begin{bmatrix} \mathcal{X}_F & \mathcal{X}_R\mathcal{J}_R^{l-1} \\ \mathcal{X}_F\mathcal{J}_F & \mathcal{X}_R\mathcal{J}_R^{l-2} \\ \vdots & \vdots \\ \mathcal{X}_F\mathcal{J}_F^{l-2} & \mathcal{X}_R\mathcal{J}_R \end{bmatrix}$$

which is $\mathbf{S}_{l-1}$ from (2) except the last row, it is equal to $[\mathbf{I}_{Ml-M}\ \mathbf{0}]\mathbf{S}_{l-1}$. It follows that $\mathbf{S}_{l-2}\mathbf{P} = \mathbf{S}_{l-2}\mathrm{diag}(\mathbf{I}_\Gamma, \mathcal{J}_R)\mathbf{S}_{l-1}^{-1}[\mathbf{I}_{Ml-M}\ \mathbf{0}]^T\mathbf{S}_{l-2} = [\mathbf{I}_{Ml-M}\ \mathbf{0}]\mathbf{S}_{l-1}\mathbf{S}_{l-1}^{-1}[\mathbf{I}_{Ml-M}\ \mathbf{0}]^T\mathbf{S}_{l-2} = [\mathbf{I}_{Ml-M}\ \mathbf{0}][\mathbf{I}_{Ml-M}\ \mathbf{0}]^T\mathbf{S}_{l-2} = \mathbf{S}_{l-2}$. Therefore the first term of $\mathbf{A}(\mathbf{I}_{Ml} - \mathbf{P})$ is zero, or $\mathbf{A}(\mathbf{I}_{Ml} - \mathbf{P}) = \mathbf{A}_8\mathbf{N}(\mathbf{I}_{Ml} - \mathbf{P})$ for some $M \times M$ nonsingular matrix $\mathbf{A}_8$. It is known that $\mathbf{P}$ projects a Ml dimension vector to the range space of $\mathbf{S}_{l-2}$ [7].

Further, constant matrices on both sides of (7) between $\mathbf{T}(\lambda)$ and $\mathbf{Q}(\lambda)$ must also be equal. This leads to $\mathbf{S}_{l-1}^{-1} = \mathbf{J}_I\mathbf{S}_{l-1}^{-1}\mathbf{J}_{Ml}$ or $\mathbf{S}_{l-1} = \mathbf{J}_{Ml}\mathbf{S}_{l-1}\mathbf{J}_I$. Now $\mathbf{J}_{Ml}\mathbf{S}_{l-1}\mathbf{J}_I$ may be expressed as

$$\mathbf{J}_{Ml}\begin{bmatrix} \mathcal{X}_F & \mathcal{X}_R\mathcal{J}_R^{l-1} \\ \mathcal{X}_F\mathcal{J}_F & \mathcal{X}_R\mathcal{J}_R^{l-2} \\ \vdots & \vdots \\ \mathcal{X}_F\mathcal{J}_F^{l-1} & \mathcal{X}_R \end{bmatrix}\mathbf{J}_I = \begin{bmatrix} \mathbf{J}_M\mathcal{X}_R & \mathbf{J}_M\mathcal{X}_F\mathcal{J}_F^{l-1} \\ \mathbf{J}_M\mathcal{X}_R\mathcal{J}_R & \mathbf{J}_M\mathcal{X}_F\mathcal{J}_F^{l-2} \\ \vdots & \vdots \\ \mathbf{J}_M\mathcal{X}_R\mathcal{J}_R^{l-1} & \mathbf{J}_M\mathcal{X}_F \end{bmatrix}$$

which should equal to $\mathbf{S}_{l-1}$ of (2). This is automatically satisfied since $\mathcal{J}_R = \mathcal{J}_F$ and $\mathcal{X}_R = \mathbf{J}_M\mathcal{X}_F$. (Alternately, this may be used along with $\mathcal{J}_R = \mathcal{J}_F$ to show that indeed $\mathcal{X}_R = \mathbf{J}_M\mathcal{X}_F$.)

3 Characterization of FOLPFB

The difficulty in finding a structure (leading to characterization) for LPFB of any order from (1), theorems 1, 2 and lemma 1 is in finding a general structure for $\mathbf{S}_{l-1}^{-1}$ for *any* choice of $\mathcal{J}_F$. It is, however, possible to find such a structure for a *given* $\mathcal{J}_F$. For example, if order $l = 2$ and $\mathcal{J}_F = \mathbf{0}$ (i.e., $b_0 = 1$), then $\mathbf{S}_1 = \mathrm{diag}(\mathcal{X}_F, \mathbf{J}_M\mathcal{X}_F)$ whose inverse $\mathrm{diag}(\mathcal{X}_F^{-1}, \mathcal{X}_F^{-1}\mathbf{J}_M)$ may easily be characterized. Thus, any order LPFB restricted to certain class (a given $\mathcal{J}_F$) may be designed and implemented from (1). Note that, even without a structure, design of LPFB of any order may be achieved using explicit inversion of $\mathbf{S}_{l-1}$ at every stage of the numerical optimization.

For FOLPFB, however, the structure of $\mathbf{S}_{l-1}^{-1}$ is independent of the choice of $\mathcal{J}_F$ as seen from (3). This leads to the following complete characterization. It is known that such FOLPFB exists only when M is even.

Theorem 3: $\mathbf{E}_1(\lambda)$ of a FOLPFB can always be expressed as

$$\begin{aligned}\mathbf{E}_1(\lambda) \quad = \quad & \mathrm{diag}(\mathbf{A}_1, \mathbf{A}_2)\begin{bmatrix} \mathbf{I}_{M/2} & -\mathbf{I}_{M/2} \\ \mathbf{I}_{M/2} & \mathbf{I}_{M/2} \end{bmatrix}\mathrm{diag}(\mathbf{I}_{M/2}\lambda - \mathcal{J}_F, \mathcal{J}_F\lambda - \mathbf{I}_{M/2}) \cdot \\ & \begin{bmatrix} \mathbf{I}_{M/2} & \mathbf{I}_{M/2} \\ \mathbf{I}_{M/2} & -\mathbf{I}_{M/2} \end{bmatrix}\mathrm{diag}(\mathbf{A}_3, \mathbf{A}_4)\begin{bmatrix} \mathbf{I}_{M/2} & \mathbf{J}_{M/2} \\ \mathbf{I}_{M/2} & -\mathbf{J}_{M/2} \end{bmatrix}\end{aligned}$$

where $\mathbf{A}_i$ for $i = 1$ to 4 are $\frac{M}{2} \times \frac{M}{2}$ nonsingular matrices and $\mathcal{J}_F$ is $\frac{M}{2} \times \frac{M}{2}$ Jordan form with zero eigenvalue.

Proof: The proof follows from the proof of lemma 1. In case of FOLPFB, it is seen from (3) that $\mathbf{A}(\mathbf{I}_{Ml} - \mathbf{P})$ becomes a nonsingular $M \times M$ matrix $\mathbf{A}$. Thus $\mathbf{A}_1, \mathbf{A}_2$ becomes square submatrices. Since $\begin{bmatrix} \mathbf{I}_{M/2} & -\mathbf{I}_{M/2} \\ \mathbf{I}_{M/2} & \mathbf{I}_{M/2} \end{bmatrix}$ is nonsingular, nonsingularity requirement leads to $\mathbf{A}_1, \mathbf{A}_2$ to be nonsingular. Further, it is also seen from (3) that $\mathbf{S}_{l-1}^{-1}\mathbf{Q}(\lambda)$ becomes $\mathbf{S}_{l-1}^{-1} = \mathbf{Y}^{-1} = [\mathcal{X}_F \ \mathbf{J}_M\mathcal{X}_F]^{-1}$. Express the inverse as $[\mathcal{X}_F \ \mathbf{J}_M\mathcal{X}_F]^{-1} = \begin{bmatrix} \mathbf{A}_9 & \mathbf{A}_{10} \\ \mathbf{A}_{11} & \mathbf{A}_{12} \end{bmatrix}$ where each submatrix is $\frac{M}{2} \times \frac{M}{2}$. Then $\mathcal{X}_F[\mathbf{A}_9 \ \mathbf{A}_{10}] + \mathbf{J}_M\mathcal{X}_F[\mathbf{A}_{11} \ \mathbf{A}_{12}] = \mathbf{I}$. Pre- and post-multiplying both sides by $\mathbf{J}_M$, and using $[\mathbf{A}_9 \ \mathbf{A}_{10}]\mathbf{J}_M = [\mathbf{A}_{10}\mathbf{J}_{M/2} \ \mathbf{A}_9\mathbf{J}_{M/2}]$, we obtain $\mathcal{X}_F[\mathbf{A}_{12}\mathbf{J}_{M/2} \ \mathbf{A}_{11}\mathbf{J}_{M/2}] + \mathbf{J}_M\mathcal{X}_F[\mathbf{A}_{10}\mathbf{J}_{M/2} \ \mathbf{A}_9\mathbf{J}_{M/2}] = \mathbf{I}$, or $\begin{bmatrix} \mathbf{A}_{12}\mathbf{J}_{M/2} & \mathbf{A}_{11}\mathbf{J}_{M/2} \\ \mathbf{A}_{10}\mathbf{J}_{M/2} & \mathbf{A}_9\mathbf{J}_{M/2} \end{bmatrix}$ is also the inverse. It follows that $\mathbf{A}_{11} = \mathbf{A}_{10}\mathbf{J}_{M/2}$ and $\mathbf{A}_{12} = \mathbf{A}_9\mathbf{J}_{M/2}$, or $[\mathcal{X}_F \ \mathbf{J}_M\mathcal{X}_F]^{-1} = \begin{bmatrix} \mathbf{A}_9 & \mathbf{A}_{10} \\ \mathbf{A}_{10}\mathbf{J}_{M/2} & \mathbf{A}_9\mathbf{J}_{M/2} \end{bmatrix}$. Letting $\mathbf{A}_9 = \mathbf{A}_3 + \mathbf{A}_4$ and $\mathbf{A}_{10} = (\mathbf{A}_3 - \mathbf{A}_4)\mathbf{J}_{M/2}$, the inverse can be written as $\begin{bmatrix} \mathbf{I}_{M/2} & \mathbf{I}_{M/2} \\ \mathbf{I}_{M/2} & -\mathbf{I}_{M/2} \end{bmatrix}\text{diag}(\mathbf{A}_3, \mathbf{A}_4)\begin{bmatrix} \mathbf{I}_{M/2} & \mathbf{J}_{M/2} \\ \mathbf{I}_{M/2} & -\mathbf{J}_{M/2} \end{bmatrix}$. Nonsingularity demands $\mathbf{A}_3$, $\mathbf{A}_4$ to be nonsingular since the other two matrices are nonsingular. □

The causal inverse may be readily obtained from earlier $\mathbf{E}_1^{-1}(\lambda)$ result:

$$\begin{aligned} \mathbf{R}(\lambda) &= \lambda^{n_F}/8\begin{bmatrix} \mathbf{I}_{M/2} & \mathbf{I}_{M/2} \\ \mathbf{J}_{M/2} & -\mathbf{J}_{M/2} \end{bmatrix}\text{diag}(\mathbf{A}_3^{-1}, \mathbf{A}_4^{-1})\begin{bmatrix} \mathbf{I}_{M/2} & \mathbf{I}_{M/2} \\ \mathbf{I}_{M/2} & -\mathbf{I}_{M/2} \end{bmatrix}. \\ & \quad \text{diag}(\mathbf{I}_{M/2}\lambda^{-1} + \sum_{i=2}^{n_F} \mathcal{J}_F^{i-1}\lambda^{-i}, -\mathbf{I}_{M/2} - \sum_{i=1}^{n_F-1} \lambda^i \mathcal{J}_F^i)\begin{bmatrix} \mathbf{I}_{M/2} & \mathbf{I}_{M/2} \\ -\mathbf{I}_{M/2} & \mathbf{I}_{M/2} \end{bmatrix}\text{diag}(\mathbf{A}_1^{-1}, \mathbf{A}_2^{-1}) \end{aligned}$$

Since both the Jordan forms are same in a FOLPFB, $n_F = n_R$. So the maximum length of the synthesis filters is $2n_FM$. Note that the characterization of [5] may be shown to be identical (upto scaling, permutation of filters, and flipping of impulse responses) to the proposed characterization for $\mathcal{J}_F = \mathbf{0}$ (synthesis length $2M$). A FOLPFB may be designed in a similar fashion to [1]. All possible $\mathcal{J}_F$ (that is, all possible non-increasing positive integers $b_0, ..., b_n$ summing up to $\frac{M}{2}$) have to be considered. If the synthesis length not more than L_{max} is desirable, then b_0 should be chosen such that $b_0 \leq \frac{L_{max}}{2M}$. Alternately, a restriction on the reconstruction delay may be enforced in a similar fashion. For each choice of $\mathcal{J}_F$, the matrices $\mathbf{A}_1$ to $\mathbf{A}_4$ are optimized, and the best optimized filterbank is chosen. These matrices are parameterized as in [1] using QL factorization: $\mathbf{A}_i = \mathrm{Q}_i\mathrm{L}_i$ for $i = 1, 2$, where Q_i is a $\frac{M}{2} \times \frac{M}{2}$ orthogonal matrix ($\frac{M(M-2)}{8}$ Givens rotations as parameters) and L_i is a $\frac{M}{2} \times \frac{M}{2}$ lower triangular matrix ($\frac{M(M+2)}{8}$ elements as parameters, diagonal elements should be non-zero). Resulting structure of $\mathbf{E}_1(\lambda)$ is shown in Figure 1, where T_i are the diagonal blocks of $\mathbf{T}(\lambda)$. From

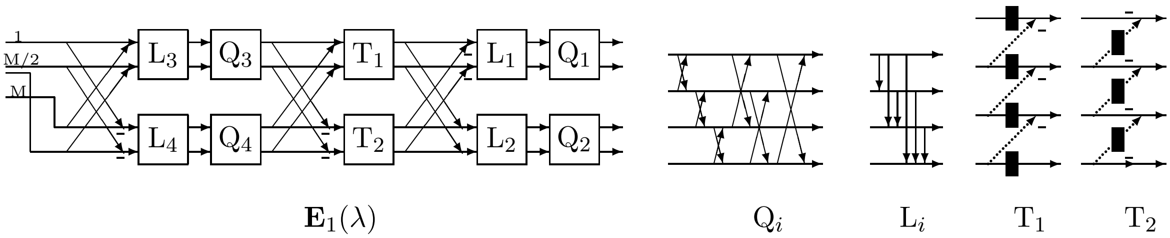

Figure 1: Structure of $\mathbf{E}_1(\lambda)$ (butterflies of Q_i are 2×2 rotations, filled box indicates delay, each dotted line in T_i is present or absent depending on $\mathcal{J}_F$)

theorem 2 of [1], the McMillan degree of a first order analysis polyphase is $\Gamma + R_F$ where R_F is the rank of $\mathcal{J}_F$. A structure is minimal if its number of delays equals the McMillan degree. Noting that T_1 requires $\frac{M}{2} = \Gamma$ delays, and T_2 requires R_F (= number of 1's in the upper off-diagonal of $\mathcal{J}_F = \sum_i (b_i - 1)$) delays, the proposed structure is minimal.

4 Design Examples

In the first example, $M = 6$ channel analysis length 12 LPFB and $M = 4$ channel analysis length 8 LPFB is designed. The filters are optimized for maximum subband coding gain, $G_{sbc} = \sigma_x^2 / \prod_{i=0}^{M-1} [a_i b_i]^{1/M}$, where $a_i = \int_0^\pi S_{xx}(e^{j\omega}) |H_i(e^{j\omega})|^2 d\omega/\pi, \; b_i = \int_0^\pi |F_i(e^{j\omega})|^2 d\omega/\pi$, $S_{xx}(e^{j\omega})$ and σ_x^2 are the one-sided psd and the variance of the source respectively, and $H_i(z)$ and $F_i(z)$ are analysis and synthesis filters. An AR(1) source with $\rho = 0.95$ is considered. Table 1 compares the coding gain obtained for all choices of $\mathcal{J}_F$. For example, if $M = 6$ then $\Gamma = 3$, and possible $b_0, ..., b_n$ sets, as shown in Table 1, are {1,1,1}, {2,1} and {3} (first case has anticausal inverse, remaining cases have non-anticausal inverse [1]). Maximum synthesis length is also given in the table. If synthesis length is restricted to $L_{max} = 24$, say, then only first two of these cases are permissible.

Table 1 shows that longer synthesis filters perform better than synthesis length $2M$ (i.e., $b_0 = 1$) filters for this source. Case {2,1} for $M = 6$ and case {2} for $M = 4$ gives maximum coding gain. Number of iterations taken in each case is also tabulated, where a single iteration takes roughly 2.2×10^5 flops for $M = 6$ and 8.5×10^4 flops for $M = 4$ in Matlab. Figure 2 shows the analysis and synthesis responses for $M = 4$ channel for {2} case.

In the second example, $M = 10$ channel analysis length 20 LPFB with b_i set {1,1,1,1,1} (synthesis length 20) is designed. The filters are optimized for ideal passband and stopband

M	$\{b_i\}$	max. syn. len.	G_{sbc}	#iter.
6	{1,1,1}	12	6.93	1535
	{2,1}	24	8.30	2242
	{3}	36	7.40	4747
4	{1,1}	8	5.65	367
	{2}	16	5.99	465

Table 1: Coding gains of design examples

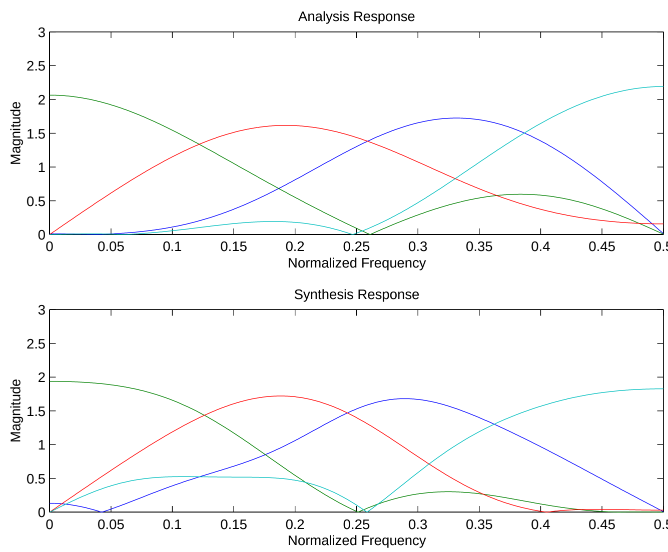

Figure 2: Filter bank responses for first example, $M = 4$

shape in the Chebyshev sense (passband weight 10 times stopband weight). Figure 3 shows the analysis responses.

In conclusion, the approach of [1] is extended to develop a complete and minimal characterization for M-channel linear phase FIR PR filterbank with analysis length $2M$ but unrestricted synthesis length. Design examples illustrate the effectiveness of this characterization. It is possible to restrict the synthesis filter length or reconstruction delay. The results derived here may be used towards designing higher order filterbanks for a given Jordan form. A structure for the higher order case for any choice of Jordan form is still open.

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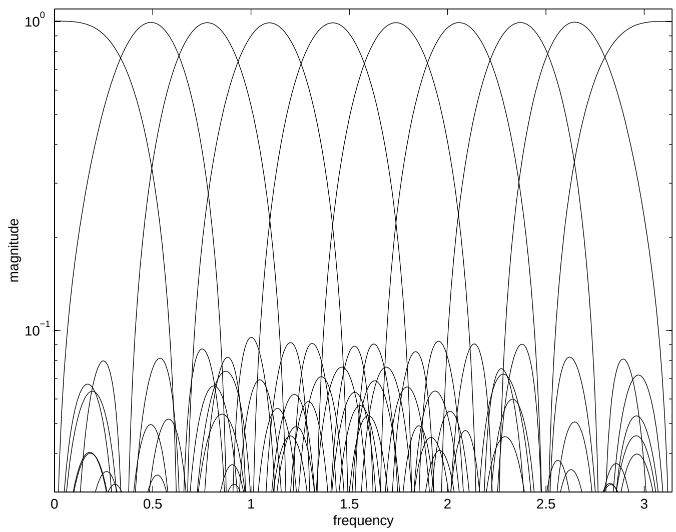

Figure 3: Analysis responses for second example

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