DESIGN OF M CHANNEL BI ORTHOGONAL FIR PERFECT RECONSTRUCTION FILTER BANKS USING SPECTRAL THEORY OF MATRIX POLYNOMIALS

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1. ABSTRACT

In this work, the problem of design of M channel biorthogonal FIR perfect reconstruction filter banks (FIRPRFB) is addressed using spectral theory of matrix polynomials. The present characterization covers a broader class of filter banks. Some insightful results on reconstruction delay and length of synthesis filters are presented, both of these parameters can be varied in the present characterization. Simulation results are presented for 2,3,4 channel designs.

2. INTRODUCTION

The class (FIRPRFB) has been investigated extensively in the last decade because of its desirable properties. There have been developments in the characterization of such filterbanks with restrictions such as paraunitary, linear phase and cosine modulation. These characterizations impose restrictions on the analysis polyphase matrix constraining the space of filterbanks due complexity of explicit inverse formulation for analysis polyphase. Complete characterization of M channel biorthogonal filerbanks is available for order one polyphse [1, 2, 3]. Attempts were made for a general degree l [4, 5], but complete characterization was not given. In this work we try to provide a very general class of FIRPRFB based on spectral properties of matrix polynomials.

3. SPECTRAL THEORY OF MATRIX POLYNOMIALS

In this section a review of spectral theory of matrix polynomials is given which is an extract of some results developed in [6]. Boldface small letters represent vectors. Boldface capital letters represent matrices. $\mathbf{I}_r$ represents $r \times r$ identity matrix, and subscript r is omitted when the size is evident from the context. Calligraphic letters represent matrix polynomials. $ker(\mathbf{X})$ represents nullspace of the matrix $\mathbf{X}$. $deg()$ represents degree of the matrix polynomial. $det()$ determinant of a matrix. $\mathbf{X} \oplus \mathbf{Y}$ represents the block diagonal matrix with $\mathbf{X}$ and $\mathbf{Y}$ as diagonal elements. Let $\mathcal{E}(\lambda) = \Sigma_{i=0}^{l} \mathbf{E}_i \lambda^i$ be a $M \times M$ matrix polynomial with degree l. Eigen values of the matrix polynomial $\mathcal{E}(\lambda)$ are defined as the roots of $|\mathcal{E}(z)| = 0$. For any eigen value λ_i, an eigenvector $\mathbf{v}$ satisfies $\mathcal{E}(z)\mathbf{v} = 0$. Let $\mathbf{v}_{0,0}^i, \cdots, \mathbf{v}_{n_i,0}^i$ be the set of eigenvectors corresponding to λ_i. $\mathbf{C}_j^i = [\mathbf{v}_{j,0}^i, \cdots, \mathbf{v}_{j,l^i j}^i]$ is called a *Jordan chain* corresponding to λ_i if $\sum_{p=0}^{k} (1/p!) \mathcal{E}^{(p)}(\lambda_i) \mathbf{v}_{j,k-p}^i = 0$ for $k = 0, 1, \cdots, l_j^i$, where $\mathcal{E}^{(p)}(\lambda)$ denotes the p^{th} derivative of $\mathcal{E}(\lambda)$ with respect to λ.

If the sequence $\{l_j^i\}_{j=0}^{n_i}$ is nondecreasing, then the set of Jordan chains $\mathbf{X}_i = [\mathbf{C}_0^i, \cdots, \mathbf{C}_{n_i}^i]$ is called a *canonical set* of Jordan chains corresponding to eigen value λ_i. $\mathbf{X}_i$ is not unique. However, the sequence l_j^i is unique for a given matrix polynomial. Let $\mathbf{J}_i = diag(\mathbf{J}_{l_0^i+1}^i, \cdots, \mathbf{J}_{l_{n_i}^i+1}^i)$, where diag() denotes a block diagonal matrix with matrices in the brackets as the diagonal blocks, and $\mathbf{J}_x^i$ is the Jordan block of size $x \times x$ with λ_i as the eigenvalue.

Let $\mathbf{X}_f = [\mathbf{X}_0, \cdots, \mathbf{X}_m]$ and the Jordan form $\mathbf{J}_f = diag(\mathbf{J}_0, \cdots, \mathbf{J}_m)$, where $m+1$ is the number of eigenvalues (called *finite spectrum*). The dimensions of $\mathbf{X}_f$ and $\mathbf{J}_f$ are are $M \times \mu_f$ and $\mu_f \times \mu_f$ respectively, where μ_f is the sum of multiplicities of all the eigenvalues. The pair $[\mathbf{X}_f, \mathbf{J}_f]$ is called *finite Jordan pair*. But this cannot uniquely represent the matrix polynomial $\mathcal{E}(\lambda)$, since any $\mathcal{U}(\lambda)\mathcal{E}(\lambda)$, with $\mathcal{U}(\lambda)$ being unimodular, gives same spectral data. Spectral data of *dual polynomial* of $\mathcal{E}(\lambda)$, $\tilde{\mathcal{E}}(\lambda) = \lambda^l \mathcal{E}(1/\lambda)$ is considered at zero eigenvalue. The pair $[\mathbf{X}_\infty, \mathbf{J}_\infty]$ is called *infinite Jordan pair*, $\mathbf{J}_\infty$ is a Jordan nilpotent matrix [7] as it corresponds to zero eigenvalue. The dimensions of $\mathbf{X}_\infty$ and $\mathbf{J}_\infty$ are $M \times \mu_\infty$ and $\mu_\infty \times \mu_\infty$ respectively, μ_∞ is the sum of all the multiplicities of the zero eigenvalue of $\tilde{\mathcal{E}}(\lambda)$. For a regular matrix polynomial $\mathcal{E}(\lambda)_{M \times M}$ of degree

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l, $\mu_f + \mu_\infty = Ml$, following matrix

$$S_{l-1} = \begin{bmatrix} \mathbf{X}_f & \mathbf{X}_\infty \mathbf{J}_\infty^{l-1} \\ \mathbf{X}_f \mathbf{J}_f & \mathbf{X}_\infty \mathbf{J}_\infty^{l-2} \\ \vdots & \vdots \\ \mathbf{X}_f \mathbf{J}_f^{l-1} & \mathbf{X}_\infty \end{bmatrix}_{Ml \times Ml} \quad (1)$$

is invertible. The pair $(\mathbf{X}, \mathbf{T})$, called as *decomposable pair of degree l* of the regular matrix polynomial $\mathcal{E}(\lambda)$, completely determines $\mathcal{E}(\lambda)$ up to a constant nonsingualr matrix, where

$$\mathbf{X} = [\mathbf{X}_f \ \mathbf{X}_\infty] \quad (2)$$

$$\mathbf{T} = diag(\mathbf{J}_f, \mathbf{J}_\infty). \quad (3)$$

Conversely, if the pair $(\mathbf{X}, \mathbf{T})$ is decomposable as in equations (2) and (3), the pair satisfies the invertibility condition of (1), then there exists a regular matrix polynomial that has $(\mathbf{Y}, \mathbf{T})$ as its decomposable pair and can be expressed in terms of the pair. It is proved in [6] that if the pair $(\mathbf{X}, \mathbf{T})$ with $\mathbf{Y} = [\mathbf{X}_1 \ \mathbf{X}_2]$ and $\mathbf{T} = (\mathbf{T_1} \oplus \mathbf{T_2})$ is decomposable then the matrix $\mathbf{S}_{l-2} = col(\mathbf{X}_1\mathbf{T}_1^i, \mathbf{X}_2\mathbf{T}_2^{l-2-i})_{i=0}^{l-2}$ is fullrank and the following proposition holds.

Proposition 1 *Let $\mathcal{E}(\lambda)$ be a matrix polynomial with decomposable pair $([\mathbf{X}_1 \ \mathbf{X}_2], \mathbf{T}_1 \oplus \mathbf{T}_2)$ and corresponding decomposable linearization $\mathcal{T}(\lambda) = (\mathbf{I}\lambda - \mathbf{T}_1) \oplus (\mathbf{T}_2\lambda - \mathbf{I})$. If $\mathbf{V} = [\mathbf{E}_1\mathbf{X}_1\mathbf{T}_1^{l-1}, \ -\sum_{i=0}^{l-1} \mathbf{E}_i\mathbf{X}_2\mathbf{T}_2^{l-1-i}]$ and $\mathbf{Z} = [\mathbf{I} \oplus \mathbf{T}_2^{l-1}] \begin{bmatrix} \mathbf{S}_{l-2} \\ \mathbf{V} \end{bmatrix}^{-1} [\mathbf{0} \ \cdots \ \mathbf{0} \ \mathbf{I}]^T$, then inverse of $\mathcal{E}(\lambda)$ is given by,*

$$\mathcal{E}^{-1}(\lambda) = [\mathbf{X}_1 \ \mathbf{X}_2]\mathcal{T}^{-1}(\lambda)\mathbf{Z}. \quad (4)$$

4. INVERSE PROBLEM - FIRPRFB DESIGN

Let $\mathcal{E}(z)_{M \times M}$ and $\mathcal{R}(z)_{M \times M}$ be the analysis and synthesis polyphase matrices. For perfect reconstruction [8]

$$\mathcal{R}(z)\mathcal{E}(z) = z^{-l_0}\mathbf{I} \quad (5)$$

where l_0 is the delay introduced into the system to make synthesis section causal. This leads to $\mathcal{R}(z) = z^{-l_0}\mathcal{E}^{-1}(z)$. For biorthogonal FIRPRFB $det(\mathcal{E}(z)) = z^{-k}$, implies $\mathbf{J}_f$ is Jordan nilpotent of size $k \times k$. It is proved [9] $det(\tilde{\mathcal{E}}(z)) = z^{-(Ml-k)}$, so $\mathbf{J}_\infty$ is a $(Ml-k) \times (Ml-k)$ Jordan nilpotent matrix. For a given structure of $\mathbf{J}_f$ and $\mathbf{J}_\infty$ if the Jordan chains $\mathbf{X}_f$ and $\mathbf{X}_\infty$ are found such that matrix $\mathbf{S}_{l-1}$ referred in (1) is invertible, a regular matrix polynomial can be constructed with the inverse structure given in proposition (1) with $([\mathbf{X}_f, \mathbf{X}_\infty], \mathbf{J}_f \oplus \mathbf{J}_\infty)$ as the decomposable pair of degree l. The following proposition given in [6] deals with the construciton of a regular matrix polynomial from a decomposable pair.

Proposition 2 *Let $([\mathbf{X}_f\mathbf{X}_\infty], \ \mathbf{J}_f \oplus \mathbf{J}_\infty)$ be a decomposable pair of degree l. Then for every $M \times Ml$ matrix $\mathbf{V}$ such that $\begin{bmatrix} \mathbf{S}_{l-2} \\ \mathbf{V} \end{bmatrix}$ is nonsingular, the matrix polynomial*

$$\mathcal{E}(z) = \mathbf{V}(\mathbf{I} - \mathbf{P})[(\mathbf{I}z^{-1} - \mathbf{J}_f) \oplus (\mathbf{J}_\infty z^{-1} - \mathbf{I})] \\ (\mathbf{U}_0 + \mathbf{U}_1 z^{-1} + \cdots + \mathbf{U}_{l-1} z^{-(l-1)}) \quad (6)$$

has $([\mathbf{X}_f, \mathbf{X}_\infty], \ \mathbf{J}_f \oplus \mathbf{J}_\infty)$ as the decomposable pair, where $\mathbf{S}_{l-2} = col(\mathbf{X}_f{\mathbf{J}_f}^i, \mathbf{X}_\infty{\mathbf{J}_\infty}^{l-2-i})_{i=0}^{l-2}$
$\mathbf{P} = (\mathbf{I} \oplus \mathbf{J}_\infty)[col(\mathbf{X}_f{\mathbf{J}_f}^i, \mathbf{J}_\infty{\mathbf{J}_\infty}^{l-1-i})_{i=0}^{l-1}]^{-1} \begin{bmatrix} \mathbf{I} \\ \mathbf{0} \end{bmatrix} \mathbf{S}_{l-2}$
and $[\mathbf{U}_0 \ \mathbf{U}_1 \ \cdots \ \mathbf{U}_{l-1}] = [col(\mathbf{X}_f{\mathbf{J}_f}^i, \mathbf{X}_\infty{\mathbf{J}_\infty}^{l-1-i})_{i=0}^{l-1}]^{-1}$.

Conversely, if $\mathcal{E}(z) = \sum_{i=0}^{l} \mathbf{E}_i z^{-i}$ has $([\mathbf{X}_f, \mathbf{X}_\infty], \ \mathbf{J}_f \oplus \mathbf{J}_\infty)$ as its decomposable pair, then $\mathcal{E}(z)$ admits representation (6) with

$$\mathbf{V} = \mathbf{V}(\mathbf{I} - \mathbf{P}) = \left[\mathbf{E}_l\mathbf{X}_f\mathbf{J}_f^{l-1}, \ -\sum_{i=0}^{l-1} \mathbf{E}_i\mathbf{X}_\infty\mathbf{J}_\infty^{l-1-i} \right].$$

4.1. Characterization of biorthogonal FIRPRFB

It is evident from the above that once degree (l), degree of the determinant of $\mathcal{E}(z)$, (k) are known for FIR analysis polyphase matrix polynomial $\mathcal{E}(z)$, matrices $\mathbf{J}_f$ and $\mathbf{J}_\infty$ become Jordan nilpotent with dimensions $k \times k$ and $(Ml - k) \times (Ml - k)$. For a given structure of $\mathbf{J}_f$ and $\mathbf{J}_\infty$, if matrices $\mathbf{X}_f$ and $\mathbf{X}_\infty$ are constructed satisfying the invertibility of the matrix $\mathbf{S}_{l-1}$, regular analysis polyphase $\mathcal{E}(z)$ can be constructed using the proposition (2). Once $\mathcal{E}(z)$ is so constructed, $\mathcal{E}^{-1}(z)$ is given by the proposition (1). Now it will be shown that rows of the matrix $\mathbf{V}$ used in the proposition (1) span the null space of $\mathbf{S}_{l-2}$.

Theorem 1 *The rows of the matrix $\mathbf{V}$ used in the construction of matrix polynomial $\mathcal{E}(z)$ in (6), span the null space of $\mathbf{S}_{l-2}$.*

Proof: From proposition (2) matrix $\mathbf{V}$ is selected such that the composite matrix $\begin{bmatrix} \mathbf{S}_{l-2} \\ \mathbf{V} \end{bmatrix}$ is invertible. If $\mathbf{X}_f$ and $\mathbf{X}_\infty$ are taken such that $([\mathbf{X}_f \ \mathbf{X}_\infty], \mathbf{J}_f \oplus \mathbf{J}_\infty)$ is the decomposable pair of a regular matrix polynomial $\mathcal{E}(z)$, i.e. then matrix $\mathbf{S}_{l-1}$ is invertible, then it was shown from the additional properties of a decomposable pair that $M(l-1) \times Ml$ matrix, $\mathbf{S}_{l-2}$ is full rank (with rank $M(l-1)$). So, $\mathbf{V}$ must be an $M \times Ml$ full rank matrix for the composite matrix to be invertible. If the rows of matrix $\mathbf{S}_{l-2}$ span a $M(l-1)$ dimensional space, say $\mathbb{W}_1$, rows of $\mathbf{S}_{l-2}$ are taken as the basis vectors of $\mathbb{W}_1$. Since each row of $\mathbf{S}_{l-2}$ is $1 \times Ml$, $\mathbb{W}_1$ can be taken as the $M(l-1)$ dimensional subspace of the Ml dimensional vector space $\mathbb{V}$. The remaining M basis vectors of the space $\mathbb{V}$ are taken from $ker(\mathbf{S}_{l-2})$, which is M dimensional and orthogonal complement to $\mathbb{W}_1$. If $\mathbf{N}_{\mathbf{S}_{l-2}}$ represents the matrix whose rows are the basis of $ker(\mathbf{S}_{l-2})$, rows spanning $\mathbf{V}$ may be taken as linear combinations of the basis of $\mathbb{W}_1$ and $ker(\mathbf{S}_{l-2})$, i.e.

$$\mathbf{V} = [\mathbf{A}_1 \ \mathbf{A}_2] \begin{bmatrix} \mathbf{S}_{l-2} \\ \mathbf{N}_{\mathbf{S}_{l-2}} \end{bmatrix} \tag{7}$$

where $\mathbf{A}_1$ and $\mathbf{A}_2$ are approriate full rank matrices. Thus the term $\mathbf{V}(\mathbf{I}-\mathbf{P})$ in the equation (6) can be written as:

$$\mathbf{V}(\mathbf{I}-\mathbf{P}) = (\mathbf{A}_1\mathbf{S}_{l-2} + \mathbf{A}_2\mathbf{N}_{\mathbf{S}_{l-2}})(\mathbf{I}-\mathbf{P}) \tag{8}$$

Expanding the term $\mathbf{A}_1\mathbf{S}_{l-2}(\mathbf{I}-\mathbf{P})$ in (8) as

$$\mathbf{A}_1\left(\mathbf{S}_{l-2} - \mathbf{S}_{l-2}(\mathbf{I}\oplus\mathbf{J}_\infty)\mathbf{S}_{l-1}^{-1}\begin{bmatrix}\mathbf{I}\\ \mathbf{0}\end{bmatrix}\mathbf{S}_{l-2}\right) \tag{9}$$

The term $\mathbf{S}_{l-2}(\mathbf{I}\oplus\mathbf{J}_\infty)\mathbf{S}_{l-1}^{-1}$ in (9) can be written as:

$$\begin{aligned} &= \begin{bmatrix} \mathbf{X}_f & \mathbf{X}_\infty{\mathbf{J}_\infty}^{l-2} \\ \mathbf{X}_f\mathbf{J}_f & \mathbf{X}_\infty{\mathbf{J}_\infty}^{l-3} \\ \vdots & \vdots \\ \mathbf{X}_f{\mathbf{J}_f}^{l-2} & \mathbf{X}_\infty\mathbf{J}_\infty \end{bmatrix} \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{J}_\infty \end{bmatrix} \mathbf{S}_{l-1}^{-1} \\ &= \begin{bmatrix}\mathbf{I}_{M(l-1)} & \mathbf{0}\end{bmatrix}\mathbf{S}_{l-1}\mathbf{S}_{l-1}^{-1} \\ &= \begin{bmatrix}\mathbf{I}_{M(l-1)} & \mathbf{0}\end{bmatrix} \end{aligned} \tag{10}$$

Using (10) and (9), $\mathbf{A}_1\mathbf{S}_{l-2}(\mathbf{I}-\mathbf{P})$ can be written as:

$$\begin{aligned} &= \mathbf{A}_1\left(\mathbf{S}_{l-2} - \begin{bmatrix}\mathbf{I}_{M(l-1)} & \mathbf{0}\end{bmatrix}\begin{bmatrix}\mathbf{I}_{M(l-1)} \\ \mathbf{0}\end{bmatrix}\mathbf{S}_{l-2}\right) \\ &= \mathbf{0} \end{aligned}$$

Form the above it is obvious that $\mathbf{V} = \mathbf{A}_2\mathbf{N}_{\mathbf{S}_{l-2}}$, thus the rows of matrix $\mathbf{V}$ span the null space of $\mathbf{S}_{l-2}$. □

Having obtained $\mathbf{V}$, the explicit representation for analysis polyphase is $\mathcal{E}(z)$ is

$$\mathcal{E}(z) = \mathbf{A}_2\mathbf{N}_{\mathbf{S}_{l-2}}(\mathbf{I}-\mathbf{P})[(z^{-1}\mathbf{I}-\mathbf{J}_f)\oplus(z^{-1}\mathbf{J}_\infty-\mathbf{I})] \mathbf{S}_{l-1}^{-1}col(z^{-i}\mathbf{I})_{i=0}^{l} \tag{11}$$

and for synthesis polyphase $\mathcal{R}(z) = z^{-l_0}\mathcal{E}^{-1}(z)$ is

$$= z^{-l_0}[\mathbf{X}_f \ \mathbf{X}_\infty]\begin{bmatrix} (z^{-1}\mathbf{I}-\mathbf{J}_f) & 0 \\ 0 & (z^{-1}\mathbf{J}_\infty-\mathbf{I}) \end{bmatrix}^{-1} [\mathbf{I}\oplus\mathbf{J}_\infty^{l-1}]\begin{bmatrix}\mathbf{S}_{l-2} \\ \mathbf{A}_2\mathbf{N}_{\mathbf{S}_{l-2}}\end{bmatrix}^{-1}[0\ \cdots\ 0\ \mathbf{I}]^T \tag{12}$$

4.1.1. Reconstruction delay

From equation (5), a delay of l_0 is introduced in the synthesis section to make synthesis filters causal. Using the expressions (11) and (12) for analysis and synthesis polyphase matrices the reconstruction delay to be used in synthesis section is given by the following theorem.

Theorem 2 *The minimum delay introduced for synthesis section to be causal is equal to $z^{-\kappa_f}$, where κ_f is the index of nil potency of $\mathbf{J}_f$.*

Proof: If κ_f and κ_∞ are the nilpotent indices of $\mathbf{J}_f$ and $\mathbf{J}_\infty$, then using the term responsible for the polynomial expansion in the expression (12) we have,

$$\begin{aligned} &\begin{bmatrix} (\mathbf{I}z^{-1}-\mathbf{J}_f) & \mathbf{0} \\ \mathbf{0} & (z^{-1}\mathbf{J}_\infty-\mathbf{I}) \end{bmatrix}^{-1}[\mathbf{I}\oplus\mathbf{J}_\infty^{l-1}] \\ &= \begin{bmatrix} (\mathbf{I}z^{-1}-\mathbf{J}_f)^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{J}_\infty^{l-1}(z^{-1}\mathbf{J}_\infty-\mathbf{I})^{-1} \end{bmatrix} \end{aligned}$$

Expanding the terms

$$(\mathbf{I}z^{-1}-\mathbf{J}_f)^{-1} = z\mathbf{I} + z^2\mathbf{J}_f + \cdots + z^{\kappa_f}\mathbf{J}_f^{\kappa_f-1} \tag{13}$$

$$\mathbf{J}_\infty^{l-1}(z^{-1}\mathbf{J}_\infty-\mathbf{I})^{-1} = -(\mathbf{J}_\infty^{l-1}+z^{-1}\mathbf{J}_\infty^{l}+\cdots+z^{-(\kappa_\infty-l)}\mathbf{J}_\infty^{\kappa_\infty-1}) \tag{14}$$

$(\mathbf{I}z^{-1}-\mathbf{J}_f)^{-1}$ is the term that contributes to the noncausality of $\mathcal{E}^{-1}(z)$. So, a delay of z^{-l_0} is introduced to make synthesis system causal. The term (13) becomes causal by introducing a delay of κ_f, so l_0 used in the expression (12) must be equal to κ_f, index of nilpotency of $\mathbf{J}_f$. □

4.1.2. Length of the synthesis filters

Since analysis polyphase matrix has a degree l, length of the analysis filters is $M(l+1)$. Length of the synthesis filters depend on the nilpotent indices κ_f and κ_∞ as follows:

- If $\kappa_\infty < l$, then the term $(\mathbf{J}_\infty z^{-1}-\mathbf{I})^{-1}{\mathbf{J}_\infty}^{l-1} = 0$ in the expression (14) , so the degree of the synthesis polyphase matrix is $\kappa_f - 1$ and the length of the synthesis filters is $M\kappa_f$.
- If $\kappa_\infty \geq l$, then the degree of the synthesis polyphase matrix is $\kappa_f + \kappa_\infty - l$ and the length of the synthesis filters is $M(\kappa_f + \kappa_\infty - l + 1)$.

So, for a given degree l of analysis polyphase, synthesis polyphase can have a degree different from l.

4.1.3. Freevariables

Since the matrix $\mathbf{S}_{l-1}$ must be full rank given $\mathbf{J}_f$ and $\mathbf{J}_\infty$, characterization of $\mathbf{X}_f$ and $\mathbf{X}_\infty$ such that $\mathbf{S}_{l-1}$ is full rank is very difficult. It was observed in simulations that in most of the cases the matrix $\mathbf{S}_{l-1}$ becomes invertible for arbitrary selection of $\mathbf{X}_f$ and $\mathbf{X}_\infty$, which requires Mk and $M(Ml-k)$ free variables. The full rank matrix used in the construction of matrix $\mathbf{V}$, given by $\mathbf{V} = \mathbf{A}_2\mathbf{N}_{\mathbf{S}_{l-2}}$, requires M^2 free variables for $\mathbf{A}_2$. So, for characterizing an M channel degree l analysis polyphase matrix, we require to optimize $M^2(l+1)$ free variables.

4.2. Simulation Results

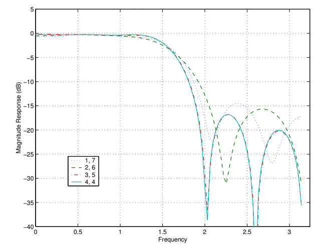
The design procedure is briefly recalled below. Given number of channels M and analysis polyphase order l, some k and a certain structure of $\mathbf{J}_f$ and $\mathbf{J}_\infty$ are assumed. The finite and infinite Jordan chains $\mathbf{X}_f$ and $\mathbf{X}_\infty$ are taken arbitrarily as free variables to be optimized, because constructing them with the constraint that $\mathbf{S}_{l-1}$ is full rank is difficult. In simulations it was seen that $\mathbf{S}_{l-1}$ is invertible for most of the cases. The regular matrix polynomial $\mathcal{E}(z)$ with $([\mathbf{X}_f\ \mathbf{X}_\infty], diag\{\mathbf{J}_f, \mathbf{J}_\infty\})$ as the decomposable pair is constructed from proposition (2). $\mathbf{V} = \mathbf{A}_2\mathbf{N}_{\mathbf{S}_{l-2}}$ is optimized. The full rank $\mathbf{A}_2$ is obtained from SVD method, $\mathbf{A}_2 = \mathbf{UDV}^T$, where $\mathbf{U}$ and $\mathbf{V}$ are the real orthogonal matrices and $\mathbf{D}$ is a diagonal matrix. The orthogonal matrices are obtained form Givens rotations [8]. Optimization may be run for any appropriate cost function, such as frequency selectivity. Once $\mathcal{E}(z)$ is obtained from optimization, $\mathcal{R}(z)$ is found as mentioned with l_0, the delay, taken such that the synthesis filters are causal. Simulations are done for 2,3 and 4 channel cases. Matlab unconstrained optimization routine *fminunc* is used for optimizing the filters.

For 2 channel case the analysis polyphase matrix is considered to be of degree 4. Some possible cases of $\mathbf{J}_f$ and $\mathbf{J}_\infty$ are assumed. As the $deg(det(\mathcal{E}(z)))$ increases from 1 to 4, the filter responses improve, especially the stop band attenuation is improved. But for further values of the $deg(det(\mathcal{E}(z)))$, the results are not consistent. For example, for degree 5 the stop band attenuation is better compared to degree 4, but transition band is more when compared to former one. For degree 6, two cases are possible. First one is having a finite Jordan structure [1] $[5, 1]$, which leads to $l_s = 10$ and better stopband attenuation at the cost of transition band performance when compared to degree 5 case. For the latter case (having a finite Jordan structure 6) $l_s = 12$, and the transition band performance is good compared to $[5, 1]$ case. The reason for the performance of the filters becoming bad after degree 5, could be because the term $(\mathbf{J}_\infty z^{-1} - \mathbf{I})^{-1}{\mathbf{J}_\infty}^{l-1}$ becomes zero in the synthesis polyphase expression for $\mu_\infty < l$, so this leads to zeroing of coefficients in synthesis filters. Since the FB is PR, the effects on synthesis section can be seen in the analysis. The zeroing of coefficients in synthesis filters has been clearly observed. The responses for different Jordan structures are shown in figure (1).

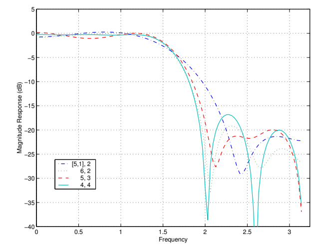
Next simulations are done for 3 channel case with $l = 2$. Regarding the spectral data, following cases are taken.

1. $\mathbf{J}_f$ and $\mathbf{J}_\infty$ have same Jordan structure, i.e. [2,1], so $l_a = l_s = 9$.

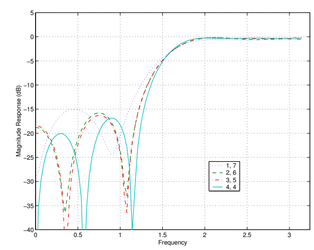
[1] If $\mathbf{J}$ is matrix with Jordan structure $[3, 2]$ then it has the form $diag\left(\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right)$



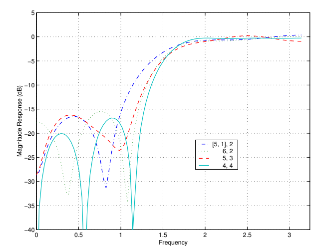
(a) Analysis Low pass filter responses



(b) Synthesis Low pass filter responses



(c) Analysis High pass filter responses



(d) Synthesis High pass filter responses

Fig. 1. 2 channel analysis filter responses for different $(\mathbf{J}_f, \mathbf{J}_\infty)$

2. $\mathbf{J}_f$ and $\mathbf{J}_\infty$ have same Jordan structure, i.e. 3, so $l_a = 9$ and $l_s = 15$.

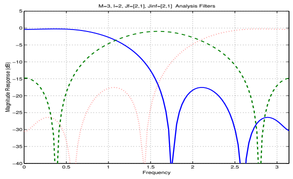
3. $\mathbf{J}_f$ and $\mathbf{J}_\infty$ having 3 and $[2, 1]$ as Jordan structures, so $l_a = 9$ and $l_s = 12$.

It is clearly seen in figure (2) that as the length of the synthesis filters increase improvement can be seen in the responses. This improvement can be seen in analysis filters also. For the above cases (2) and (3), reconstruction delay is same but synthesis filter length changes and responses are better for case (3), which is expected.

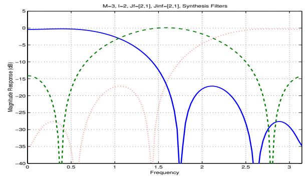
For 4 channel case, simulations are done for $l = 2$. Regarding the spectral data, following cases are taken.

- $\mathbf{J}_f$ and $\mathbf{J}_\infty$ have same Jordan structure, i.e. [2,2], so $l_a = l_s = 12$.
- $\mathbf{J}_f$ and $\mathbf{J}_\infty$ having $[3, 1]$ and $[2, 2]$ as Jordan structures, so $l_a = 12$ and $l_s = 16$.

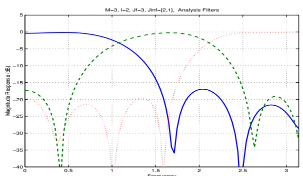
For the latter case, the finite Jordan structure is changed from $[2, 2]$ to $[3, 1]$. The analysis filters show the same behavior. Synthesis filter coefficients for the latter case are shifted version of the former by a factor of 4, padded with zeros for first 4 coefficients. As the magnitude response is considered for the cost function formulation, the optimized



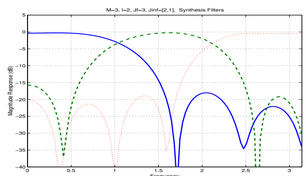
(a) Analysis filter responses with $\mathbf{J}_f = [2,1]$ and $\mathbf{J}_\infty = [2,1]$



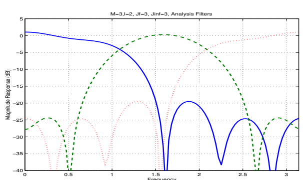
(b) Synthesis filter responses with $\mathbf{J}_f = [2,1]$ and $\mathbf{J}_\infty = [2,1]$



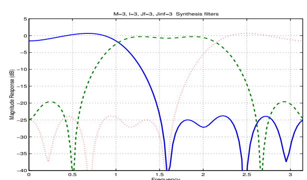
(c) Analysis filter responses with $\mathbf{J}_f = 3$ and $\mathbf{J}_\infty = [2,1]$



(d) Analysis filter responses with $\mathbf{J}_f = 3$ and $\mathbf{J}_\infty = [2,1]$



(e) Analysis filter responses with $\mathbf{J}_f = 3$ and $\mathbf{J}_\infty = 3$



(f) Analysis filter responses with $\mathbf{J}_f = 3$ and $\mathbf{J}_\infty = 3$

Fig. 2. 3 channel responses for different $(\mathbf{J}_f, \mathbf{J}_\infty)$

cost is same for both the cases. It was clearly observed for both cases, synthesis filters phase responses are different and the magnitude responses are same.

5. CONCLUSION

A characterization for the design of M channel bi orthogonal FIRPRFB is given using spectral theory of matrix polynomials. A regular matrix polynomial is constructed with inverse using the known spectral data $(deg(det(\mathcal{E}(z))))$. It is a challenging task to derive explicit expressions for $\mathbf{X}_f$ and $\mathbf{X}_\infty$.

6. REFERENCES

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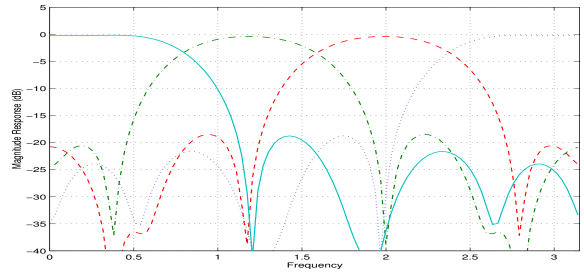

Fig. 3. Magnitude responses of analysis filters of 4 channel filter bank