M CHANNEL CAUSAL STABLE IIR PERFECT RECONSTRUCTION FILTER BANK DESIGN USING MINIMAL FACTORIZATION

*Viswanadha Reddy P.**

Imaging Technologies Lab
GE Global Research, JFWTC
Banaglore, INDIA
E-mail: *ViswanadhaReddy.P@geind.ge.com*

*Anamitra Makur **

Electrical and Electronic Engineering
Nanyang Technological University
SINGAPORE
E-mail: *EAMakur@ntu.edu.sg*

ABSTRACT

In this work, the problem of design of M channel IIR causal stable perfect reconstruction filter banks is addressed and a design approach based on the minimal factorization of the state space form of the polyphase matrix are presented based on the concept of factorization of rational matrix functions. The analysis and synthesis polyphase matrices are constrained to be minimal. Simulation results are presented for 3 channel and 4 channel cases.

1. INTRODUCTION

Perfect reconstruction filter banks (PRFBs) are widely used for signal decomposition, subband coding, subband adaptive filtering etc. A PRFB design involves designing its analysis and synthesis polyphase matrices, $\mathbf{E}(z)$ and $\mathbf{R}(z)$ [1]. $\mathbf{E}(z)$ with minimum phase determinant leads to the design of IIR PRFBs where the analysis as well as the synthesis bank is causal stable. Causal stable IIR PRFBs have both merits in that they have good responses unlike FIR case, and additional processing for anti-causal filtering is not necessary unlike anti-causal stable IIR case. 2 Channel designs have been proposed in [3, 4, 5, 6]. General M channel causal stable IIR PRFB designs have been proposed in [7] and [8]. In [7], first an analysis low pass filter is designed, then the remaining analysis filters are designed such that $\mathbf{E}(z)$ becomes unimodular, using the *Quillen-Suslin theorem* on the completion of unimodular matrix polynomials. The design method is complex. It is highly likely that other analysis filters may not have good responses as they are constrained by the first filter. In [8] M channel causal stable IIR PRFB is designed by assuming a diagonal structure for $\mathbf{E}(z)$ with each diagonal filter being minimum phase.

The problem of interest of the present work is design methods for M channel causal stable IIR PRFBs using state space form. Minimal factorization of the analysis polyphase $\mathbf{E}(z)$ is used to achieve such design. The paper is organized as follows. Preliminaries, necessary assumptions and useful results regarding minimal factorization of rational matrix functions are presented in section (2). Proposed design using minimal factorization is given in section (3). Simulation results are given in section (4) and conclusions in section (5).

*work done at Department of Electrical Communication Engineering, Indian Institute of Science, Bangalore

1.1. Notations and terminology

Boldface small letters represent vectors. Boldface capital letters represent matrices. $\mathbf{I}_r$ represents $r \times r$ identity matrix, and subscript r is omitted when the size is evident from the context. Calligraphic letters represent matrix functions. $\mathcal{N}(\mathbf{X})$ and $\mathcal{R}(\mathbf{X})$ represent nullspace and range of the matrix $\mathbf{X}$. Vectorspaces, fields and subspaces are represented by the letters of the type $\mathbb{L}$, such as $\mathbb{C}$ represents the space of complex numbers. $rank()$, $det()$, and $adj()$ denote rank, determinant, and adjoint of a matrix. $dim()$, $span()$ denote dimension of a space and space spanned by vector(s).

2. PRELIMINARIES AND DESIGN APPROACH

In this section we describe the preliminaries and summarize the design approach taken in this work. If $\mathbf{E}(z)$ and $\mathbf{R}(z)$ are the $M \times M$ analysis and synthesis polyphase matrices, perfect reconstruction is achieved (neglecting scaling and delay) when

$$\mathbf{R}(z) = \mathbf{E}^{-1}(z) = \frac{adj(\mathbf{E}(z))}{det(\mathbf{E}(z))}. \qquad (1)$$

It is obvious that $\mathbf{E}(z)$ must be invertible. For IIR case, causal stable synthesis filters are obtained if $det(\mathbf{E}(z))$ is minimum phase with $\mathbf{E}(z)$ being causal, which is obvious from equation (1). Constraining $\mathbf{E}(z)$ with minimum phase determinant is very difficult. This problem is tackled in this work by taking $\mathbf{E}(z)$ in state space form.

As it is already known [9], any rational matrix function $\mathbf{E}(z)$ can be expressed in state space form,

$$\mathbf{E}(z) = \mathbf{D} + \mathbf{C}'(z\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}'$$

where $\mathbf{D}$, $\mathbf{A}$, $\mathbf{B}'$ and $\mathbf{C}'$ are $M \times M$, $m \times m$, $m \times M$ and $M \times m$ matrices respectively. $\mathbf{A}$ is called the *state transition matrix*.

The explicit inverse formula for $\mathbf{E}(z)$ [10] can be given if $\mathbf{E}(z)$ is invertible at $z = \infty$, i.e., $\mathbf{D}$ is invertible. Rewriting the above equation we have

$$\begin{aligned} \mathbf{E}(z) &= \mathbf{D}(\mathbf{I} + \mathbf{C}(z\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}) \\ &= \mathbf{D}\mathbf{E}'(z) \end{aligned} \quad (2)$$

The inverse of $\mathbf{E}(z)$ is given by

$$\begin{aligned} \mathbf{R}(z) &= (\mathbf{D}\mathbf{E}'(z))^{-1} = (\mathbf{I} - \mathbf{C}(z\mathbf{I} - \mathbf{A}^*)^{-1}\mathbf{B})\mathbf{D}^{-1} \\ &= \mathbf{R}'(z)\mathbf{D}^{-1} \end{aligned} \quad (3)$$

where

$$\mathbf{A}^* = \mathbf{A} - \mathbf{B}\mathbf{C}. \quad (4)$$

2.1. Minimal systems

While designing filters using state space form, minimal systems are preferred so that non-unique representations of $\mathbf{E}(z)$ are avoided, and that its implementation involves minimum number of delays. From [9, 10], if $\mathbf{E}(z)$ is minimal then following conditions are satisfied, where k is the index of nil-potency of $\mathbf{A}$.

- The $m \times Mk$ matrix $\mathcal{C}(\mathbf{A}, \mathbf{B}) = [\mathbf{B}\ \mathbf{A}\mathbf{B}\ \mathbf{A}^2\mathbf{B}\ \cdots\ \mathbf{A}^{(k-1)}\mathbf{B}]$ must be of rank m; this condition is called *controllability condition*.

- The $m \times Mk$ matrix $\mathcal{O}(\mathbf{C}, \mathbf{A}) = [\mathbf{C}^T\ (\mathbf{C}\mathbf{A})^T\ (\mathbf{C}\mathbf{A}^2)^T\ \cdots\ (\mathbf{C}\mathbf{A}^{(k-1)})^T]$ must be of rank m; this condition is called *observability condition*.

To keep $\mathbf{E}(z)$ minimal, additional constraints are imposed in the present design. It can be seen that if $m \le M$, full rank matrices $\mathbf{B}$ and $\mathbf{C}$ are enough to satisfy both minimality conditions discussed above, irrespective of the rank of $\mathbf{A}$. So, in the present design methods we assume $m \le M$ (the dimension of the matrix $\mathbf{A}$ never exceeds the number of channels M) and $rank(\mathbf{B}) = rank(\mathbf{C}) = m$.

2.2. Factorization of rational matrix functions

Cascade approach for the design of filter banks received lot of attention in the filter bank design community. Filter banks based on factorization of rational lossless systems [2] is popular in filter bank designs. The propositions discussed in this section are general, i.e., $\mathbf{B}$ and $\mathbf{C}$ need not be full rank and size of $\mathbf{A}$ is unrestricted. The design problem boils down to factorization of *rational matrix functions*, exhaustively dealt in [10]. All the propositions described in this chapter are taken from [10]. Here the main idea is to decompose a minimal system of degree m into a product of minimal systems with degree one. Now a question arises that on what conditions do a larger system, which is minimal, can be decomposed into product of degree one minimal systems. The following proposition gives the conditions under which a minimal system can be factorized into product of degree one minimal systems as shown in figure (1).

Proposition 1 *Let m be the size of $\mathbf{A}$ in $\mathbf{E}'(z)$, and let*

$$\mathbb{C}^m = \mathbb{L}_1 + \cdots + \mathbb{L}_m \quad (5)$$

where the chain

$$\mathbb{L}_1 \subset \mathbb{L}_1 + \mathbb{L}_2 \subset \cdots \subset \mathbb{L}_1 + \mathbb{L}_2 + \cdots + \mathbb{L}_{m-1} \quad (6)$$

consists of $\mathbf{A}$-invariant subspaces[1] *whereas the chain*

$$\mathbb{L}_m \subset \mathbb{L}_m + \mathbb{L}_{m-1} \subset \cdots \subset \mathbb{L}_m + \mathbb{L}_{m-1} + \cdots + \mathbb{L}_2 \quad (7)$$

consists of $\mathbf{A}^*$-invariant subspaces. Then $\mathbf{E}'(z)$ admits the minimal factorization

$$\begin{aligned} \mathbf{E}'(z) = &[\mathbf{I} + \mathbf{C}\pi_1(z\mathbf{I} - \mathbf{A})^{-1}\pi_1\mathbf{B}] \\ &\cdots[\mathbf{I} + \mathbf{C}\pi_m(z\mathbf{I} - \mathbf{A})^{-1}\pi_m\mathbf{B}] \end{aligned} \quad (8)$$

where π_j is the projector[2] *on $\mathbb{L}_j$ along $\mathbb{L}_1 + \cdots + \mathbb{L}_{j-1} + \mathbb{L}_{j+1} + \cdots + \mathbb{L}_m$.*
Conversely, for every minimal factorization

$$\mathbf{E}'(z) = \mathbf{E}'_1(z)\mathbf{E}'_2(z)\cdots\mathbf{E}'_m(z) \quad (9)$$

where $\mathbf{E}'_j(z)$ are rational $M \times M$ matrix functions with $\mathbf{E}'_j(\infty) = \mathbf{I}$, there exists a unique direct sum decomposition (5) with the property that the chains (6) and (7) consist of invariant subspaces for $\mathbf{A}$ and $\mathbf{A}^*$, respectively, such that for $j = 1, \cdots, m$

$$\mathbf{E}'_j(z) = \mathbf{I} + \mathbf{C}\pi_j(z\mathbf{I} - \mathbf{A})^{-1}\pi_j\mathbf{B}.$$

Detailed proof of the above proposition is given in [10]. The factorization (8) implies the minimal factorization for $\mathbf{E}'(z)^{-1}$ as

$$\begin{aligned} \mathbf{E}'(z)^{-1} = &[\mathbf{I} - \mathbf{C}\pi_m(z\mathbf{I} - \mathbf{A}^*)^{-1}\pi_m\mathbf{B}] \\ &\cdots[\mathbf{I} - \mathbf{C}\pi_1(z\mathbf{I} - \mathbf{A}^*)^{-1}\pi_1\mathbf{B}] \end{aligned} \quad (10)$$

In the propositions discussed above, since $\mathbf{E}'(z)$ is assumed to be a minimal system, the matrices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ must satisfy the conditions for minimality discussed earlier.

[1] A subspace $\mathbb{L}$ is $\mathbf{A}$-invariant if for any $\mathbf{x} \in \mathbb{L}$, $\mathbf{A}\mathbf{x} \in \mathbb{L}$

[2] A matrix $\mathbf{P}$ is a projector onto a space $\mathbb{L}$, if $\mathbf{P}^2 = \mathbf{P}$ and $\mathcal{R}(\mathbf{P}) = \mathbb{L}$

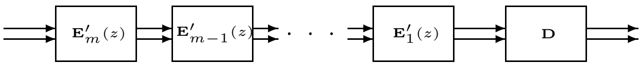


Fig. 1. Factorization structure of $\mathbf{E}(z)$

3. MINIMAL FACTORIZATION BASED DESIGN

3.1. Construction of invariant subspaces

In this section we discuss the design method for decomposing a minimal system into degree one systems. In the previous sections it was discussed that minimal factorization requires information regarding the chain of invariant spaces with respect to $\mathbf{A}$ and $\mathbf{A}^*$. In order to simplify the calculation of invariant subspaces, we propose to constrain $\mathbf{A}$ and $\mathbf{A}^*$ to be triangular.

Theorem 1 *If* $\mathbf{A}$ *is an* $m \times m$ *upper triangular matrix and* $\mathbf{e}_j$ *is an* m *dimensional vector with* j^{th} *element as unity and others being zero, then there exists an* $\mathbf{A}$*-invariant chain of subspaces*

$$\mathbb{S}_{\mathbf{e}_1} \subset \mathbb{S}_{\mathbf{e}_1} + \mathbb{S}_{\mathbf{e}_2} \subset \cdots \subset \mathbb{S}_{\mathbf{e}_1} + \mathbb{S}_{\mathbf{e}_2} + \cdots + \mathbb{S}_{\mathbf{e}_m} \quad (11)$$

where $\mathbb{S}_{\mathbf{e}_j}$ *is the* $span\{\mathbf{e}_j\}$.

Proof: Let $\mathbf{A}$ be an $m \times m$ upper triangular matrix of the form

$$\mathbf{A} = \begin{bmatrix} \lambda_1 & k_1^2 & k_1^3 & \cdots & k_1^m \\ 0 & \lambda_2 & k_2^3 & \cdots & k_2^m \\ 0 & 0 & \lambda_3 & \cdots & k_3^m \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_m \end{bmatrix} \quad (12)$$

Here the diagonal elements, which are the eigen values, need not be distinct. Let $\mathbf{x} \in \mathbb{M}_i$, where $\mathbb{M}_i$ is the $span\{\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_i\}$, then

$$\mathbf{x} = \sum_{j=1}^{i} \alpha_j \mathbf{e}_j$$

where $\alpha_j \in \mathbb{C}$. Now the vector

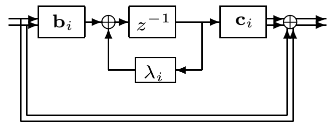
$$\mathbf{Ax} = \sum_{j=1}^{i} \alpha_j \{\mathbf{A}\mathbf{e}_j\} = \sum_{j=1}^{i} \alpha_j \mathbf{a}_j$$

where $\mathbf{a}_j$, the j^{th} column of the matrix $\mathbf{A}$, can be written as

$$\mathbf{a}_j = \sum_{m=1}^{j-1} k_m^j \mathbf{e}_m + \lambda_j \mathbf{e}_j.$$

Using this in the previous expression,

$$\mathbf{Ax} = \sum_{j=1}^{i} \lambda_i \alpha_i \mathbf{e}_i + \sum_{j=1}^{i} \sum_{m=1}^{j-1} k_m^j \alpha_j \mathbf{e}_m$$



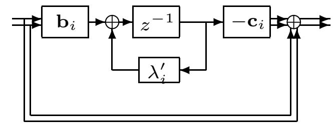


Fig. 2. Realization of $\mathbf{E}'_i(z)$ **Fig. 3**. Realization of $\mathbf{R}'_i(z)$

a chain of $\mathbf{A}^*$-invariant subspaces $\mathbb{S}_{\mathbf{e}_m} \subset \mathbb{S}_{\mathbf{e}_m} + \mathbb{S}_{\mathbf{e}_{m-1}} \subset \cdots \subset \mathbb{S}_{\mathbf{e}_m} + \mathbb{S}_{\mathbf{e}_{m-1}} + \cdots + \mathbb{S}_{\mathbf{e}_2}$. $\square$

Using propositions (1) and above choice of matrices $\mathbf{A}$ and $\mathbf{A}^*$, there exists projector matrices π_j on the space $\mathbb{S}_{\mathbf{e}_j}$ along the space $\mathbb{S}_{\mathbf{e}_1} + \cdots + \mathbb{S}_{\mathbf{e}_{j-1}} + \mathbb{S}_{\mathbf{e}_{j+1}} + \cdots + \mathbb{S}_{\mathbf{e}_m}$. Since the spaces $\mathbb{S}_{\mathbf{e}_j}$ are defined on the standard ordered basis, the projector matrix π_j is given by

$$\begin{aligned} \{\pi_j\}_{k,m} &= 1 \;\text{ if } k = m = j \\ &= 0 \;\text{ otherwise.} \end{aligned} \quad (13)$$

From the structure of the projector matrices given in equation (13), each degree 1 factor of $\mathbf{E}'(z)$ from equation (8) can be simplified as

$$\mathbf{E}'_i(z) = (\mathbf{I} + \mathbf{c}_i (z - \lambda_i)^{-1} \mathbf{b}_i)$$

where $\mathbf{c}_i$ is the i^{th} column of $\mathbf{C}$ and $\mathbf{b}_i$ is the i^{th} row of $\mathbf{B}$. The analysis polyphase matrix is then given as

$$\mathbf{E}(z) = \mathbf{D}(\mathbf{I} + \mathbf{c}_1 (z - \lambda_1)^{-1} \mathbf{b}_1) \cdots (\mathbf{I} + \mathbf{c}_m (z - \lambda_m)^{-1} \mathbf{b}_m) \quad (14)$$

and the synthesis polyphase matrix, from equation (10), is decomposed as

$$\mathbf{R}(z) = (\mathbf{I} - \mathbf{c}_m (z - \lambda'_m)^{-1} \mathbf{b}_m) \cdots (\mathbf{I} - \mathbf{c}_1 (z - \lambda'_1)^{-1} \mathbf{b}_1) \mathbf{D}^{-1}$$

where λ'_i are the eigen values (diagonal elements) of $\mathbf{A}^*$. The factors $\mathbf{E}'_i(z)$ and $\mathbf{R}'_i(z)$ are realized as shown in the figures (2) and (3) respectively.

3.2. Design approach

The proposed design approach is as follows. For causal stable analysis and synthesis filters, poles of $\mathbf{E}'(z)$ and $\mathbf{R}'(z)$ must be inside unit circle. From system theory it is known that poles of $\mathbf{E}'(z)$ are same as the eigen values of $\mathbf{A}$. Similarly, the poles of $\mathbf{R}'(z)$ are same as the eigen values of $\mathbf{A}^*$. Thus, eigen values of $\mathbf{A}$ and $\mathbf{A}^*$ must be inside unit circle for causality and stability. $\mathbf{A}$ and $\mathbf{A}^*$ are taken as triangular matrices to simplify the choice of invariant subspaces. As mentioned earlier, in the present design, minimality of $\mathbf{E}'(z)$ is ensured if $\mathbf{B}$ and $\mathbf{C}$ are full rank and $m \leq M$. Even if $\mathbf{B}$ and $\mathbf{C}$ are full rank (m here), the product of the matrices ($\mathbf{BC} = \mathbf{A} - \mathbf{A}^*$) can have a rank less than m. The bounds on rank of a matrix, generated by the product of two full rank matrices is discussed in the following theorem.

Theorem 2 *If $\mathbf{B}$ and $\mathbf{C}$ are $m \times M$ and $M \times m$ full rank matrices, and if r is the rank of their product, then r is bounded as $\max\{0, 2m-M\} \leq r \leq m$.*

Proof: The rank of $\mathbf{BC}$ is given by [11]

$$rank(\mathbf{BC}) = rank(\mathbf{C}) - dim(\mathcal{N}(\mathbf{B}) \cap \mathcal{R}(\mathbf{C})).$$

Since $rank(\mathbf{BC}) = r$,

$$dim(\mathcal{N}(\mathbf{B}) \cap \mathcal{R}(\mathbf{C})) = m - r. \tag{15}$$

From equation (15), $\mathbf{C}$ has $m-r$ linearly independent columns forming a subspace (say $\mathcal{N}'(\mathbf{B})$) of the space $\mathcal{N}(\mathbf{B})$. The dimension of $\mathcal{N}(\mathbf{B})$ is $M-m$. Since $\mathcal{N}'(\mathbf{B})$ is a subspace of $\mathcal{N}(\mathbf{B})$, $dim(\mathcal{N}'(\mathbf{B})) \leq dim(\mathcal{N}(\mathbf{B}))$ which implies

$$\begin{aligned} m - r &\leq M - m \\ r &\geq 2m - M \end{aligned} \tag{16}$$

which is the lower bound on r. For some m, this lower bound can become negative. Since rank cannot be negative, lower bound is then modified to zero. The upper bound on r is m which is obvious from equation (15). □

Choosing $m \leq M$, with triangular matrices $\mathbf{A}$ and $\mathbf{A}^*$ satisfying rank bound on ($\Delta_{\mathbf{A}} = \mathbf{A} - \mathbf{A}^*$) set by theorem (2), full rank matrices $\mathbf{B}$, $\mathbf{C}$ are constructed from $\mathbf{A}$ and $\mathbf{A}^*$ using the following theorem.

Theorem 3 *If $\Delta_{\mathbf{A}}$ is an $m \times m$ matrix with rank r and it is known to be the product of two full rank matrices $\mathbf{B}$ and $\mathbf{C}$, with sizes $m \times M$ and $M \times m$ respectively, and if r lies in the bounds given by theorem (2), then $\mathbf{B}$ and $\mathbf{C}$ are given by*

$$\begin{aligned} \mathbf{B} &= \mathbf{UX} \\ \mathbf{C} &= \left[\mathbf{X}^\dagger \begin{bmatrix} \mathbf{\Sigma}_r \\ 0 \end{bmatrix} \quad \mathcal{N}(\mathbf{X})\mathbf{Y} \right] \mathbf{V}^T \end{aligned}$$

where $\mathbf{X}$ is any $m \times M$ full rank matrix, $\mathbf{Y}$ is any $(M-m) \times (m-r)$ full rank matrix, $\mathbf{\Sigma}_r$ is an $r \times r$ diagonal matrix, $\mathbf{X}^\dagger$ is the pseudo inverse of $\mathbf{X}$, and $\mathbf{U}$ and $\mathbf{V}$ are some $m \times m$ unitary matrices.

Proof: Taking the SVD of $\Delta_{\mathbf{A}}$ we have,

$$\Delta_{\mathbf{A}} = \mathbf{U}\mathbf{\Sigma}_{\mathbf{X}}\mathbf{V}^T$$

where $\mathbf{U}$ and $\mathbf{V}$ are $m \times m$ unitary matrices and $\mathbf{\Sigma}_{\mathbf{X}}$ is an $m \times m$ diagonal matrix with first r diagonal entries nonzero. So, $\mathbf{\Sigma}_{\mathbf{X}}$ can be written as $\begin{bmatrix} \mathbf{\Sigma}_r & 0 \\ 0 & 0 \end{bmatrix}$, where $\mathbf{\Sigma}_r$ is the nonzero block of $\mathbf{\Sigma}_{\mathbf{X}}$. If $\mathbf{X}$ is any $m \times M$ full rank matrix and $\mathcal{N}(\mathbf{X})_{M \times (M-m)}$ is the null space of $\mathbf{X}$, then the product $\mathbf{X}\mathcal{N}(\mathbf{X})\mathbf{Y}$ is zero for any full rank matrix $\mathbf{Y}_{(M-m)\times(m-r)}$ (this fact is used below). Now $\Delta_{\mathbf{A}}$ is written as

$$\begin{aligned} \Delta_{\mathbf{A}} &= \mathbf{U} \begin{bmatrix} \mathbf{\Sigma}_r & 0 \\ 0 & 0 \end{bmatrix} \mathbf{V}^T \\ &= \mathbf{U} \left[\mathbf{XX}^\dagger \begin{pmatrix} \mathbf{\Sigma}_r \\ 0 \end{pmatrix} \quad 0_{m \times (m-r)} \right] \mathbf{V}^T \\ &= \mathbf{U} \left[\mathbf{XX}^\dagger \begin{pmatrix} \mathbf{\Sigma}_r \\ 0 \end{pmatrix} \quad \mathbf{X}\mathcal{N}(\mathbf{X})\mathbf{Y} \right] \mathbf{V}^T \\ &= \mathbf{UX} \left[\mathbf{X}^\dagger \begin{pmatrix} \mathbf{\Sigma}_r \\ 0 \end{pmatrix} \quad \mathcal{N}(\mathbf{X})\mathbf{Y} \right] \mathbf{V}^T \end{aligned} \tag{17}$$

It is obvious that $\mathbf{B} = \mathbf{UX}$ is full rank. Let us consider the rank of $\mathbf{C}$. Since $\mathbf{XX}^\dagger = \mathbf{I}_m$, using the rank of product concept,

$$\begin{aligned} rank(\mathbf{XX}^\dagger) &= rank(\mathbf{I}_m) \\ rank(\mathbf{X}^\dagger) - dim(\mathcal{N}(\mathbf{X}) \cup \mathcal{R}(\mathbf{X}^\dagger)) &= m \\ dim(\mathcal{N}(\mathbf{X}) \cup \mathcal{R}(\mathbf{X}^\dagger)) &= 0 \end{aligned} \tag{18}$$

or the space spanned by the columns of $\mathbf{X}^\dagger$ and $\mathcal{N}(\mathbf{X})$ are independent. So the block matrix $\left[\mathbf{X}^\dagger \begin{pmatrix} \mathbf{\Sigma}_r \\ 0 \end{pmatrix} \mathcal{N}(\mathbf{X})\mathbf{Y} \right]$ is full rank if $\begin{pmatrix} \mathbf{\Sigma}_r \\ 0 \end{pmatrix}$ and $\mathbf{Y}$ are full rank. Thus $\mathbf{C} = \left[\mathbf{X}^\dagger \begin{pmatrix} \mathbf{\Sigma}_r \\ 0 \end{pmatrix} \quad \mathcal{N}(\mathbf{X})\mathbf{Y} \right] \mathbf{V}^T$ is also full rank. □

Free variables: In all examples in this work, a matrix of certain rank is parameterized by the rotation angles and the diagonal elements after its SVD decomposition. Thus, a full rank $m_1 \times m_2$ matrix, where $m_1 - m_2 = r \geq 0$, has $\frac{m_1(m_1-1)}{2} + m_2 + \frac{m_2(m_2-1)}{2} = m_1 m_2 + \frac{r^2-r}{2}$ parameters, which becomes $m_1 m_2$ parameters since in all examples r is 0 or 1. For this method, for the invertible matrix $\mathbf{D}$ in $\mathbf{E}(z)$, $M \times M$ free variables are required. mM parameters are required for $\mathbf{X}$, $(M-m) \times (m-r)$ parameters are required for $\mathbf{Y}$. Since $\mathbf{A}$ and $\mathbf{A}^*$ are upper and lower triangular matrices, depending upon the design a maximum of $m(m+1)/2$ parameters are required for each. The diagonal elements of $\mathbf{A}$ and $\mathbf{A}^*$ are constrained to be inside unit circle for causality and stability.

4. SIMULATION RESULTS

In this section the simulation results are presented. The cost function used in the optimization is based on passband and stopband shaping. Matlab constrained optimization routine *fmincon* is used to constrain the eigen values of $\mathbf{A}$ and $\mathbf{A}^*$ inside unit circle. In order to reduce the effect of initialization, optimization is started with only the passband error as the cost function (very small weight for stop band). Thereafter, the cost function is slowly changed by increasing the relative weight for stop band at every step, until equal pass

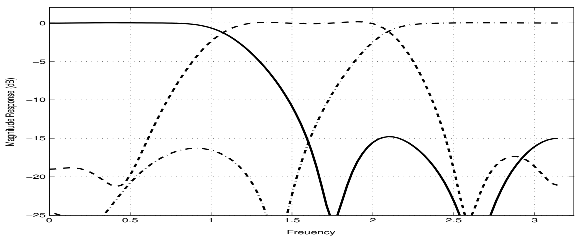


Fig. 4. 3 channel design with $M = 3, m = 3$

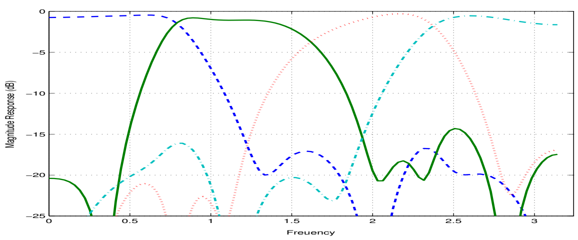


Fig. 5. 4 channel design with $M = 4, m = 3$

band and stop band weight is arrived at. The upper triangular $\mathbf{A}$ and lower triangular $\mathbf{A}^*$ are to be selected such that $rank(\mathbf{A}-\mathbf{A}^*)$ lies within the bounds given by theorem (2). There exist some pairs of $\{\mathbf{A}, \mathbf{A}^*\}$ which satisfy the above condition, but there is no easy way to characterize all such pairs. In the present design, one possible case is assumed for $m = 3$: $\mathbf{A}$ of the form $\begin{bmatrix} \lambda_1 & \alpha_1 & 0 \\ 0 & \lambda_2 & \alpha_2 \\ 0 & 0 & \lambda_3 \end{bmatrix}$ and $\mathbf{A}^*$ of the form $\begin{bmatrix} \lambda_4 & 0 & 0 \\ \beta_1 & \lambda_5 & 0 \\ 0 & \beta_2 & \lambda_3 \end{bmatrix}$. Choosing $\alpha_2 \neq 0$, $\beta_2 \neq 0$ and $\lambda_1 \neq \lambda_4$ ensures $rank(\mathbf{A}-\mathbf{A}^*) = 3$, which satisfies both rank bounds of the examples below. For stability of analysis and synthesis sections, $0 < |\lambda_i| < 1$, $\forall i = 1, \cdots, 5$ is chosen.

Simulations are done for 3 channel and 4 channel cases. First, 3 channel case is assumed with $M = 3, m = 3$, which implies a rank bound of 3 from theorem (2). Figure (4) shows the magnitude responses of analysis filters. The analysis and synthesis filters are of order (11/9). Next, 4 channel case with $M = 4, m = 3$ is assumed, for which the rank bound is [2,3]. Figure (5) shows the magnitude responses of analysis filters. The analysis and synthesis filters are of order (15/12).

5. CONCLUSION

An approach for the design of M channel causal stable IIR PRFB are presented. The state transition matrix $\mathbf{A}$ of the analysis polyphase $\mathbf{E}'(z)$ is assumed to be minimal, and is factorized into degree one systems. This is possible if $\mathbf{A}$-invariant and $\mathbf{A}^*$-invariant chain of subspaces of proposition (1) exist. By choosing $\mathbf{A}$ to be upper triangular and $\mathbf{A}^*$ to be lower triangular satisfying rank condition given by theorem (2), such existance is guaranteed. Full rank matrices $\mathbf{B}$ and $\mathbf{C}$ are constructed using theorem (3). Characterization of the pairs $\{\mathbf{A}, \mathbf{A}^*\}$ satisfying rank bound given by theorem (2) can give a broader class of filter banks.

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