

Limits

1) At a point: $\lim_{x \rightarrow c} f(x) = f(c)$

Example:

$$\lim_{x \rightarrow 3} x^3 - 2x^2 + 6 = 3^3 - 2 \cdot 3^2 + 6 = 15$$

2) Infinite limits: $\lim_{x \rightarrow c} f(x) = \infty$ plug numbers that approach c from the left and then from the right in $f(x)$: limit exists if the function behaves in the same manner and it does not exist if the values of the function go in the opposite directions.

Example:

$$\lim_{x \rightarrow -2} \frac{1}{(x+2)^2} = \infty \quad \lim_{x \rightarrow -2} \frac{1}{x+2} = \text{undefined}$$

3) One-sided limits:

$\lim_{x \rightarrow c^+} f(x) = \pm\infty$ plug numbers that approach c from the right in $f(x)$: values of the function indicate the direction of the limit

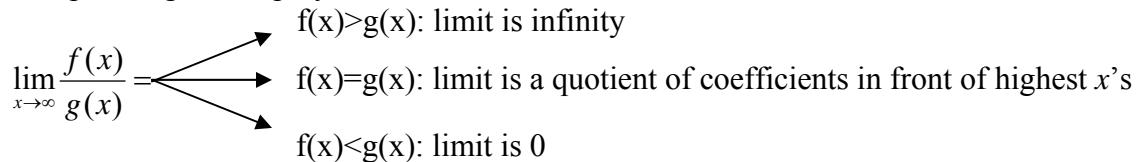
$\lim_{x \rightarrow c^-} f(x) = \pm\infty$ plug numbers that approach c from the left in $f(x)$: values of the function indicate the direction of the limit

Example:

$$\lim_{x \rightarrow 1^+} \frac{x+2}{x-1} = \infty \quad \lim_{x \rightarrow 1^-} \frac{x+2}{x-1} = -\infty$$

4) Limits at infinity: $\lim_{x \rightarrow \infty} f(x) = a$ or $\lim_{x \rightarrow \infty} f(x) = \infty$

Compare degrees of polynomials in the numerator and the denominator:



Examples:

$$\lim_{x \rightarrow \infty} \frac{x^2}{x+2} = \infty \quad \lim_{x \rightarrow \infty} \frac{4x^3 + 3x + 1}{2x^3 + 5} = \frac{4}{2} = 2 \quad \lim_{x \rightarrow \infty} \frac{1}{x+2} = 0$$

OR Divide each term in the numerator and denominator by the highest degree of x in the denominator:

Examples:

$$\lim_{x \rightarrow \infty} \frac{x^2}{x+2} = \lim_{x \rightarrow \infty} \frac{x^2/x}{x/x + 2/x} = \lim_{x \rightarrow \infty} \frac{x}{1 + 2/x} = \infty$$

$$\lim_{x \rightarrow \infty} \frac{4x^3 + 3x + 1}{2x^3 + 5} = \lim_{x \rightarrow \infty} \frac{4x^3/x^3 + 3x/x^3 + 1/x^3}{2x^3/x^3 + 5/x^3} = \lim_{x \rightarrow \infty} \frac{4 + 3/x^2 + 1/x^3}{2 + 5/x^3} = \frac{4}{2} = 2$$

$$\lim_{x \rightarrow \infty} \frac{1}{x+2} = \lim_{x \rightarrow \infty} \frac{1/x}{x/x + 2/x} = \lim_{x \rightarrow \infty} \frac{1/x}{1 + 2/x} = 0$$

Other strategies:

Cancellation

$$\lim_{x \rightarrow -1} \frac{x^2 + 4x + 3}{x + 1} = \lim_{x \rightarrow -1} \frac{(x+1)(x+3)}{x+1} = \lim_{x \rightarrow -1} x + 3 = -1 + 3 = 2$$

Rationalization

$$\lim_{x \rightarrow 0} \frac{\sqrt{x+1} - 1}{x} = \lim_{x \rightarrow 0} \frac{\sqrt{x+1} - 1}{x} \cdot \left(\frac{\sqrt{x+1} + 1}{\sqrt{x+1} + 1} \right) = \lim_{x \rightarrow 0} \frac{x+1-1}{x(\sqrt{x+1} + 1)} = \lim_{x \rightarrow 0} \frac{1}{\sqrt{x+1} + 1} = \frac{1}{\sqrt{0+1} + 1} = \frac{1}{2}$$

Useful formulas:

- a) $ax^2 + bx + c = a(x - x_1)(x - x_2)$
- b) $(a+b)^2 = a^2 + 2ab + b^2$
- c) $(a-b)^2 = a^2 - 2ab + b^2$
- d) $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$
- e) $(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$
- f) $a^2 - b^2 = (a-b)(a+b)$
- g) $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$
- h) $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$

L'Hopital's Rule:

If the limit of a function as x approaches c produces indeterminate form $0/0$, ∞/∞ , $0^*\infty$, 1^∞ , 0^0 , or $\infty-\infty$, then

$$\lim_{x \rightarrow c} \frac{f(x)}{g(x)} = \lim_{x \rightarrow c} \frac{f'(x)}{g'(x)}$$

The Special Trigonometric Limits:

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1 \quad \lim_{x \rightarrow 0} \frac{1 - \cos x}{x} = 0$$