Limits

1) At a point: $\lim_{x \to c} f(x) = f(c)$

Example:

$$\lim_{x \to 3} x^3 - 2x^2 + 6 = 3^3 - 2 \cdot 3^2 + 6 = 15$$

2) Infinite limits: $\lim_{x \to c} f(x) = \infty$ plug numbers that approach c from the left and then from the right in $f(x)$: limit exists if the function behaves in the same manner and it does not exist if the values of the function go in the opposite directions.

Example:

$$\lim_{x \to -2} \frac{1}{(x+2)^2} = \infty \qquad \lim_{x \to -2} \frac{1}{x+2} = \text{undefined}$$

3) One-sided limits:

$\lim_{x \to c^+} f(x) = \pm\infty$ plug numbers that approach c from the right in $f(x)$: values of the function indicate the direction of the limit

$\lim_{x \to c^-} f(x) = \pm\infty$ plug numbers that approach c from the left in $f(x)$: values of the function indicate the direction of the limit

Example:

$$\lim_{x \to 1^+} \frac{x+2}{x-1} = \infty \qquad \lim_{x \to 1^-} \frac{x+2}{x-1} = -\infty$$

4) Limits at infinity: $\lim_{x \to \infty} f(x) = a$ or $\lim_{x \to \infty} f(x) = \infty$

Compare degrees of polynomials in the numerator and the denominator:

$\lim_{x \to \infty} \frac{f(x)}{g(x)} =$

→ f(x)>g(x): limit is infinity

→ f(x)=g(x): limit is a quotient of coefficients in front of highest x's

→ f(x)<g(x): limit is 0

Examples:

$$\lim_{x \to \infty} \frac{x^2}{x+2} = \infty \qquad \lim_{x \to \infty} \frac{4x^3 + 3x + 1}{2x^3 + 5} = \frac{4}{2} = 2 \qquad \lim_{x \to \infty} \frac{1}{x+2} = 0$$

OR Divide each term in the numerator and denominator by the highest degree of x in the denominator:

Examples:

$$\lim_{x \to \infty} \frac{x^2}{x+2} = \lim_{x \to \infty} \frac{x^2/x}{x/x + 2/x} = \lim_{x \to \infty} \frac{x}{1 + 2/x} = \infty$$

$$\lim_{x \to \infty} \frac{4x^3 + 3x + 1}{2x^3 + 5} = \lim_{x \to \infty} \frac{4x^3/x^3 + 3x/x^3 + 1/x^3}{2x^3/x^3 + 5/x^3} = \lim_{x \to \infty} \frac{4 + 3/x^2 + 1/x^3}{2 + 5/x^3} = \frac{4}{2} = 2$$

$$\lim_{x \to \infty} \frac{1}{x+2} = \lim_{x \to \infty} \frac{1/x}{x/x + 2/x} = \lim_{x \to \infty} \frac{1/x}{1 + 2/x} = 0$$

Other strategies:

Cancellation

$$\lim_{x\to -1}\frac{x^2+4x+3}{x+1}=\lim_{x\to -1}\frac{(x+1)(x+3)}{x+1}=\lim_{x\to -1}x+3=-1+3=2$$

Rationalization

$$\lim_{x\to 0}\frac{\sqrt{x+1}-1}{x}=\lim_{x\to 0}\frac{\sqrt{x+1}-1}{x}\cdot\left(\frac{\sqrt{x+1}+1}{\sqrt{x+1}+1}\right)=\lim_{x\to 0}\frac{x+1-1}{x(\sqrt{x+1}+1)}=\lim_{x\to 0}\frac{1}{\sqrt{x+1}+1}=\frac{1}{\sqrt{0+1}+1}=\frac{1}{2}$$

Useful formulas:

a) $ax^2+bx+c=a(x-x_1)(x-x_2)$
b) $(a+b)^2=a^2+2ab+b^2$
c) $(a-b)^2=a^2-2ab+b^2$
d) $(a+b)^3=a^3+3a^2b+3ab^2+b^3$
e) $(a-b)^3=a^3-3a^2b+3ab^2-b^3$
f) $a^2-b^2=(a-b)(a+b)$
g) $a^3-b^3=(a-b)(a^2+ab+b^3)$
h) $a^3+b^3=(a+b)(a^2-ab+b^3)$

L'Hopital's Rule:
If the limit of a function as *x* approaches *c* produces indeterminate form 0/0, ∞/∞, 0*∞, 1^∞, 0^0, or ∞–∞, then

$$\lim_{x\to c}\frac{f(x)}{g(x)}=\lim_{x\to c}\frac{f'(x)}{g'(x)}$$

The Special Trigonometric Limits:

$$\lim_{x\to 0}\frac{\sin x}{x}=1 \qquad \lim_{x\to 0}\frac{1-\cos x}{x}=0$$