Properties of Matrix Operations

Properties of Addition (Subtraction)

The basic properties of addition for real numbers also hold true for matrices.

Let A, B and C be m x n matrices

$A \pm B = (a_{ij} \pm b_{ij})$ corresponding elements are added or subtracted

Addition/subtraction can be done only with matrices of the same size!

1. $-A = (-1)A$
2. A + B = B + A commutative
3. A + (B + C) = (A + B) + C associative
4. There is a unique m x n matrix O with A + O = A additive identity
5. For any m x n matrix A there is an m x n matrix D (called -A) with
   A + D = O additive inverse

Properties of Matrix Multiplication

Unlike matrix addition, the properties of multiplication of real numbers do not all generalize to matrices. Matrices rarely commute even if AB and BA are both defined. There often is no multiplicative inverse of a matrix, even if the matrix is a square matrix. There are a few properties of multiplication of real numbers that generalize to matrices. We state them now.

Let A, B and C be matrices of dimensions such that the following are defined. Then

Number of columns of the first matrix has to be equal to the number of rows of the second matrix: $[m \times n] \cdot [n \times p] = [m \times p]$. Definition: AB= $\sum_{k=1}^{n} a_{ik} b_{kj}$

1. $AB \neq BA$ (in general)
2. A(BC) = (AB)C = ABC associative
3. A(B + C) = AB + AC distributive
4. (A + B)C = AC + BC distributive
5. There are unique matrices I_m and I_n with
   $I_m A = A I_n = A$ multiplicative identity

We will often omit the subscript and write I for the identity matrix. The identity matrix is a square scalar matrix with 1's along the diagonal. For example

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Properties of Scalar Multiplication

Since we can multiply a matrix by a scalar, we can investigate the properties that this multiplication has. All of the properties of multiplication of real numbers generalize. In particular, we have

Let r and s be real numbers and A and B be matrices. Then

$rA = (ra_{ij})$ multiply each element of a matrix by r

1. rA=Ar

2. $r(sA) = (rs)A$
3. $(r + s)A = rA + sA$
4. $r(A + B) = rA + rB$
5. $A(rB) = r(AB) = (rA)B$

Properties of the Transpose of a Matrix

Recall that the transpose of a matrix is the operation of switching rows and columns. We state the following properties. We proved the first property in the last section.

Let r be a real number and A and B be matrices. Then

Definition: if $A=(a_{ij})$ then $A^T=(a_{ji})$

1. $(A^T)^T = A$
2. $(AB)^T = B^T A^T$
3. $(rA + sB)^T = rA^T + sB^T$
   a) $(rA)^T = rA^T$ (s=0)
   b) $(A + B)^T = A^T + B^T$ (r=s=1)
   c) $(A – B)^T = A^T – B^T$ (r=1, s=-1)

Properties of Unit Matrices

I (a unit matrix) is a square matrix with all its diagonal elements equal to 1.

1. AI=IA=A

Properties of Determinants

Defined only for square matrices.

1. If two rows (or columns) are interchanged then the sign of the determinant is changed.
2. If two rows (or columns) are the same then the determinant is 0.
3. If a row (or column) is multiplied by a constant *k* then the determinant is also multiplied by *k*.
4. Adding a number *p* times one row (or column) to another row (or column) does not change the value of the determinant.
5. $|AB| = |A|\ |B|$
6. $|A^T| = |A|$

Properties of the Inverse Matrices

The inverse of a square matrix A with $|A|\neq 0$ is the matrix defined by $A^{-1} = \frac{1}{|A|}\underline{A}^T$

1. $A\underline{A}^T = |A|\ I = \underline{A}^T A$ (The Cofactor Theorem)
2. $AA^{-1} = I = A^{-1}A$
3. $(A^{-1})^{-1} = A$
4. $(rA)^{-1} = \frac{1}{r} A^{-1}$
5. $(A^T)^{-1} = (A^{-1})^T$
6. $(AB)^{-1} = B^{-1} A^{-1}$

PROPERTIES OF MATRICES

INDEX

BASIC OPERATIONS - addition, subtraction, multiplication

For example purposes, let $\mathbf{A}=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $\mathbf{B}=\begin{bmatrix} e & f \\ g & h \end{bmatrix}$ and $\mathbf{C}=\begin{bmatrix} i \\ j \end{bmatrix}$

then $\mathbf{A}+\mathbf{B}=\begin{bmatrix} a & b \\ c & d \end{bmatrix}\pm\begin{bmatrix} e & f \\ g & h \end{bmatrix}=\begin{bmatrix} a\pm e & b\pm f \\ c\pm g & d\pm h \end{bmatrix}$

and $\mathbf{AB}=\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} e & f \\ g & h \end{bmatrix}=\begin{bmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{bmatrix}$ $\qquad \mathbf{AC}=\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} i \\ j \end{bmatrix}=\begin{bmatrix} ai+bj \\ ci+dj \end{bmatrix}$

a scalar times a matrix is $3\begin{bmatrix} a & b \\ c & d \end{bmatrix}=\begin{bmatrix} 3a & 3b \\ 3c & 3d \end{bmatrix}$

CRAMER'S RULE for solving simultaneous equations

Given the equations:	We express them in matrix form:	Where matrix A is	and vector **y** is
$2x_1+x_2+x_3=3$ $x_1+3x_2-x_3=7$ $x_1+x_2+x_3=1$	$\begin{bmatrix} 2 & 1 & 1 \\ 1 & 3 & -1 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 3 \\ 7 \\ 1 \end{bmatrix}$	$A=\begin{bmatrix} 2 & 1 & 1 \\ 1 & 3 & -1 \\ 1 & 1 & 1 \end{bmatrix}$	$\begin{bmatrix} 3 \\ 7 \\ 1 \end{bmatrix}$

According to Cramer's rule:

$x_1=\dfrac{\begin{vmatrix} 3 & 1 & 1 \\ 7 & 3 & -1 \\ 1 & 1 & 1 \end{vmatrix}}{\lvert A\rvert}=\dfrac{8}{4}=2$	$x_2=\dfrac{\begin{vmatrix} 2 & 3 & 1 \\ 1 & 7 & -1 \\ 1 & 1 & 1 \end{vmatrix}}{\lvert A\rvert}=\dfrac{4}{4}=1$	$x_3=\dfrac{\begin{vmatrix} 2 & 1 & 3 \\ 1 & 3 & 7 \\ 1 & 1 & 1 \end{vmatrix}}{\lvert A\rvert}=\dfrac{-8}{4}=-2$
To find x_1 we replace the first column of A with vector **y** and divide the determinant of this new matrix by the determinant of A.	To find x_2 we replace the second column of A with vector **y** and divide the determinant of this new matrix by the determinant of A.	To find x_3 we replace the third column of A with vector **y** and divide the determinant of this new matrix by the determinant of A.

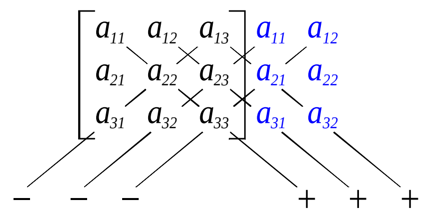
THE DETERMINANT

The determinant of a matrix is a scalar value that is used in many matrix operations. The matrix must be **square** (equal number of columns and rows) to have a determinant. The notation for absolute value is used to indicate "the determinant of", e.g. $|\mathbf{A}|$ means "the determinant of matrix A" and $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ means to take the determinant of the enclosed matrix. Methods for finding the determinant vary depending on the size of the matrix.

The **determinant of a 2×2 matrix** is simply:

$$\text{where } \mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad \det \mathbf{A} = |\mathbf{A}| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

The **determinant of a 3×3 matrix** can be calculated by repeating the first two columns as shown in the figure at right. Then add the products of each of three diagonal rows and subtract the products of the three crossing diagonals as shown.



$$a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}$$
$$- a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}$$

This method used for 3×3 matrices does not work for larger matrices.

The **determinant of a 4×4 matrix** can be calculated by finding the determinants of a group of submatrices. Given the matrix **D** we select any row or column. Selecting row 1 of this matrix will simplify the process because it contains a zero. The first element of row one is occupied by the number 1 which belongs to row 1, column 1.

$$\mathbf{D} = \begin{bmatrix} 1 & 3 & 2 & 0 \\ 4 & 4 & 1 & 1 \\ 2 & 0 & 1 & 3 \\ 3 & 3 & 1 & 5 \end{bmatrix}$$

Mentally blocking out this row and column, we take the determinant of the remaining 3x3 matrix $\mathbf{d}_1$. Using the method above, we find the determinant of $\mathbf{d}_1$ to be 14.

$$\mathbf{d}_1 = \begin{bmatrix} 4 & 1 & 1 \\ 0 & 1 & 3 \\ 3 & 1 & 5 \end{bmatrix}$$

Proceeding to the second element of row 1, we find the value 3 occupying row 1, column 2. Mentally blocking out row 1 and column 2, we form a 3x3 matrix with the remaining elements $\mathbf{d}_2$. The determinant of this matrix is 6.

$$\mathbf{d}_2 = \begin{bmatrix} 4 & 1 & 1 \\ 2 & 1 & 3 \\ 3 & 1 & 5 \end{bmatrix}$$

Similarly we find the submatrices associated with the third and fourth elements of row 1. The determinant of $\mathbf{d}_3$ is -34. It won't be necessary to find the determinant of $\mathbf{d}_4$.

$$\mathbf{d}_3 = \begin{bmatrix} 4 & 4 & 1 \\ 2 & 0 & 3 \\ 3 & 3 & 5 \end{bmatrix}$$

Now we alternately add and subtract the products of the row elements and their cofactors (determinants of the submatrices we found), beginning with adding the first row element multiplied by the determinant $\mathbf{d}_1$ like this:

$$\mathbf{d}_4 = \begin{bmatrix} 4 & 4 & 1 \\ 2 & 0 & 1 \\ 3 & 3 & 1 \end{bmatrix}$$

$$\det \mathbf{D} = (1 \times \det \mathbf{d}_1) - (3 \times \det \mathbf{d}_2) + (2 \times \det \mathbf{d}_3) - (0 \times \det \mathbf{d}_4)$$
$$= 14 - 18 + (-68) - 0 = -72$$

The products formed from row or column elements will be added or subtracted depending on the position of the elements in the matrix. The upper-left element will always be added with added/subtracted elements occupying the matrix in a checkerboard pattern from there. As you can see, we didn't need to calculate $\mathbf{d}_4$ because it got multiplied by the zero in row 1, column 4.

Adding or subtracting matrix elements: $\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$

AUGMENTED MATRIX

A set of equations sharing the same variables may be written as an **augmented matrix** as shown at right.

$$\begin{aligned} y + 3z &= 5 \\ 2x + 2y + z &= 11 \\ 3x + y + 2z &= 13 \end{aligned} \qquad \left[\begin{array}{ccc:c} 0 & 1 & 3 & 5 \\ 2 & 2 & 1 & 11 \\ 3 & 1 & 2 & 13 \end{array}\right]$$

REDUCED ROW ECHELON FORM (rref)

Reducing a matrix to **reduced row echelon form** or **rref** is a means of solving the equations. In this process, three types of **row operations** my be performed. 1) Each element of a row may be multiplied or divided by a number, 2) Two rows may exchange positions, 3) a multiple of one row may be added/subtracted to another.

$$\left[\begin{array}{ccc:c} 0 & 1 & 3 & 5 \\ 2 & 2 & 1 & 11 \\ 3 & 1 & 2 & 13 \end{array}\right]$$

1) We begin by swapping rows 1 and 2.

$$\left[\begin{array}{ccc:c} 2 & 2 & 1 & 11 \\ 0 & 1 & 3 & 5 \\ 3 & 1 & 2 & 13 \end{array}\right]$$

2) Then divide row 1 by 2.

$$\left[\begin{array}{ccc:c} 2 & 2 & 1 & 11 \\ 0 & 1 & 3 & 5 \\ 3 & 1 & 2 & 13 \end{array}\right] \div 2 = \left[\begin{array}{ccc:c} 1 & 1 & .5 & 5.5 \\ 0 & 1 & 3 & 5 \\ 3 & 1 & 2 & 13 \end{array}\right]$$

3) Then subtract row 2 from row 1.

$$\left[\begin{array}{ccc:c} 1 & 1 & .5 & 5.5 \\ 0 & 1 & 3 & 5 \\ 3 & 1 & 2 & 13 \end{array}\right] \begin{array}{l} \text{-II} \\ \\ \\ \end{array}$$

4) And subtract 3 times row 1 from row 3.

$$\left[\begin{array}{ccc:c} 1 & 0 & -2.5 & .5 \\ 0 & 1 & 3 & 5 \\ 3 & 1 & 2 & 13 \end{array}\right] \begin{array}{l} \\ \\ \text{-3(I)} \end{array} = \left[\begin{array}{ccc:c} 1 & 0 & -2.5 & .5 \\ 0 & 1 & 3 & 5 \\ 0 & 1 & 9.5 & 11.5 \end{array}\right]$$

5) Then subtract row 2 from row 3.

$$\left[\begin{array}{ccc:c} 1 & 0 & -2.5 & .5 \\ 0 & 1 & 3 & 5 \\ 0 & 1 & 9.5 & 11.5 \end{array}\right] \begin{array}{l} \\ \\ \text{-II} \end{array}$$

6) And divide row 3 by 6.5.

$$\left[\begin{array}{ccc:c} 1 & 0 & -2.5 & .5 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 6.5 & 6.5 \end{array}\right] \begin{array}{l} \\ \\ \div 6.5 \end{array} = \left[\begin{array}{ccc:c} 1 & 0 & -2.5 & .5 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

7) Add 2.5× row 3 to row 1.

$$\left[\begin{array}{ccc:c} 1 & 0 & -2.5 & .5 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & 1 \end{array}\right] \begin{array}{l} \text{+2.5(III)} \\ \\ \\ \end{array}$$

8) And subtract 3× row 3 from row 2.

$$\left[\begin{array}{ccc:c} 1 & 0 & 0 & 3 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & 1 \end{array}\right] \begin{array}{l} \\ \text{-3(III)} \\ \\ \end{array} = \left[\begin{array}{ccc:c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

The matrix is now in **reduced row echelon form** and if we rewrite the equations with these new values we have the solutions. A matrix is in **rref** when the first nonzero element of a row is 1, all other elements of a column containing a leading 1 are zero, and rows are ordered progressively with the top row having the leftmost leading 1.

$$\begin{aligned} x &= 3 \\ y &= 2 \\ z &= 1 \end{aligned} \qquad \left[\begin{array}{ccc:c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

When a matrix is in reduced row echelon form, it is possible to tell how may solutions there are to the system of equations. The possibilities are 1) **no solutions -** the last element in a row is non-zero and the remaining elements are zero; this effectively says that zero is equal to a non-zero value, an impossibility, 2) **infinite solutions -** a non-zero value other than the leading 1 occurs in a row, and 3) **one solution -** the only remaining option, such as in the example above.

If an invertible matrix **A** has been reduced to rref form then its determinant can be found by $\det(\mathbf{A}) = (-1)^s k_1 k_2 \cdots k_r$, where s is the number of row swaps performed and $k_1, k_2, \cdot\cdot\cdot\ k_r$ are the scalars by which rows have been divided.

RANK

The number of leading 1's is the **rank** of the matrix. Rank is also defined as the **dimension** of the largest square submatrix having a nonzero determinant. The rank is also the number of vectors required to form a **basis of the span** of a matrix.

THE IDENTITY MATRIX

In this case, the **rref** of A is the **identity matrix**, denoted I_n characterized by the diagonal row of 1's surrounded by zeros in a **square matrix**. When a vector is multiplied by an identity matrix of the same dimension, the product is the vector itself, $I_n\mathbf{v} = \mathbf{v}$.

$$\text{rref}(A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

LINEAR TRANSFORMATION

This system of equations can be represented in the form $A\mathbf{x} = \mathbf{b}$. This is also known as a **linear transformation** from **x** to **b** because the matrix A transforms the vector **x** into the vector **b**.

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 3 \\ 2 & 2 & 1 \\ 3 & 1 & 2 \end{bmatrix} \qquad \mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 5 \\ 11 \\ 13 \end{bmatrix}$$

ADJOINT

For a **2×2 matrix**, the adjoint is:

$$\text{where } \mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad \text{adj}\mathbf{A} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

For a **3×3 and higher matrix**, the adjoint is the transpose of the matrix after all elements have been replaced by their **cofactors** (the determinants of the submatrices formed when the row and column of a particular element are excluded). Note the **pattern of signs** beginning with positive in the upper-left corner of the matrix.

$$\text{where } B = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix},$$

$$\text{adj } \mathbf{B} = \begin{bmatrix} \begin{vmatrix} e & f \\ h & i \end{vmatrix} & -\begin{vmatrix} d & f \\ g & i \end{vmatrix} & \begin{vmatrix} d & e \\ g & h \end{vmatrix} \\ -\begin{vmatrix} b & c \\ h & i \end{vmatrix} & \begin{vmatrix} a & c \\ g & i \end{vmatrix} & -\begin{vmatrix} a & b \\ g & h \end{vmatrix} \\ \begin{vmatrix} b & c \\ e & f \end{vmatrix} & -\begin{vmatrix} a & c \\ d & f \end{vmatrix} & \begin{vmatrix} a & b \\ d & e \end{vmatrix} \end{bmatrix}^T = \begin{bmatrix} \begin{vmatrix} e & f \\ h & i \end{vmatrix} & -\begin{vmatrix} b & c \\ h & i \end{vmatrix} & \begin{vmatrix} b & c \\ e & f \end{vmatrix} \\ -\begin{vmatrix} d & f \\ g & i \end{vmatrix} & \begin{vmatrix} a & c \\ g & i \end{vmatrix} & -\begin{vmatrix} a & c \\ d & f \end{vmatrix} \\ \begin{vmatrix} d & e \\ g & h \end{vmatrix} & -\begin{vmatrix} a & b \\ g & h \end{vmatrix} & \begin{vmatrix} a & b \\ d & e \end{vmatrix} \end{bmatrix}$$

INVERTIBLE MATRICES

A matrix is invertible if it is a square matrix with a determinant not equal to 0. The reduced row echelon form of an invertible matrix is the identity matrix rref(**A**) = I_n. The determinant of an inverse matrix is equal to the inverse of the determinant of the original matrix: $\det(\mathbf{A}^{-1}) = 1/\det(\mathbf{A})$. If **A** is an invertible $n \times n$ matrix then rank(**A**) = n, im(**A**) = R^n, ker(**A**) = {0}, the vectors of **A** are linearly independent, 0 is not an eigenvalue of **A**, the linear system $\mathbf{Ax} = \mathbf{b}$ has a unique solution **x**, for all **b** in R^n.

THE INVERSE TRANSFORMATION

If **A** is an invertible matrix, the **inverse matrix** could be used to transform **b** into **x**, $\mathbf{Ax} = \mathbf{b}$, $\mathbf{A}^{-1}\mathbf{b} = \mathbf{x}$. An invertible linear transform such as this is called an **isomorphism**.

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 3 \\ 2 & 2 & 1 \\ 3 & 1 & 2 \end{bmatrix} \qquad \mathbf{A}^{-1} \cong \begin{bmatrix} -0.23 & -0.08 & 0.38 \\ 0.08 & 0.69 & -0.46 \\ 0.31 & -0.23 & 0.15 \end{bmatrix}$$

A matrix multiplied by its inverse yields the identity matrix.
$\mathbf{BB}^{-1} = \mathbf{I}_n$

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 8 & 2 \end{bmatrix} \begin{bmatrix} 10 & -6 & 1 \\ -2 & 1 & 0 \\ -7 & 5 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

FINDING THE INVERSE MATRIX – Method 1

To calculate the inverse matrix, consider the invertible 3×3 matrix **B**.

$$\mathbf{B}=\begin{bmatrix}1&1&1\\2&3&2\\3&8&2\end{bmatrix}$$

1) Rewrite the matrix, adding the identity matrix to the right.

$$\left[\begin{array}{ccc|ccc}1&1&1&1&0&0\\2&3&2&0&1&0\\3&8&2&0&0&1\end{array}\right]$$

2) Perform row operations on the 3×6 matrix to put **B** in rref form. Three types of **row operations** are: 1) Each element of a row may be multiplied or divided by a number, 2) Two rows may exchange positions, 3) a multiple of one row may be added/subtracted to another.

$$\left[\begin{array}{ccc|ccc}1&0&0&10&-6&1\\0&1&0&-2&1&0\\0&0&1&-7&5&-1\end{array}\right]$$

3) The inverse of **B** is now in the 3×3 matrix to the right.

$$\mathbf{B}^{-1}=\begin{bmatrix}10&-6&1\\-2&1&0\\-7&5&-1\end{bmatrix}$$

If a matrix is orthogonal, its inverse can be found simply by taking the transpose.

FINDING THE INVERSE MATRIX – Method 2

To calculate the inverse matrix, consider the invertible 3×3 matrix **B**.

$$\mathbf{B}=\begin{bmatrix}1&1&1\\2&3&2\\3&8&2\end{bmatrix}$$

1) First we must find the **adjoint** of matrix **B**. The adjoint of **B** is the transpose of matrix **B** after all elements have been replaced by their cofactors. (The method of finding the adjoint of a 2×2 matrix is different; see page 4.) The | | notation means "the **determinant** of".

$$\text{adj }\mathbf{B}=\begin{bmatrix}\begin{vmatrix}3&2\\8&2\end{vmatrix}&-\begin{vmatrix}2&2\\3&2\end{vmatrix}&\begin{vmatrix}2&3\\3&8\end{vmatrix}\\-\begin{vmatrix}1&1\\8&2\end{vmatrix}&\begin{vmatrix}1&1\\3&2\end{vmatrix}&-\begin{vmatrix}1&1\\3&8\end{vmatrix}\\\begin{vmatrix}1&1\\3&2\end{vmatrix}&-\begin{vmatrix}1&1\\2&2\end{vmatrix}&\begin{vmatrix}1&1\\2&3\end{vmatrix}\end{bmatrix}^T$$

2) Calculating the determinants we get.

$$\text{adj }\mathbf{B}=\begin{bmatrix}-10&2&7\\6&-1&-5\\-1&0&-1\end{bmatrix}^T$$

3) And then taking the **transpose** we get.

$$\text{adj }\mathbf{B}=\begin{bmatrix}-10&6&-1\\2&-1&0\\7&-5&-1\end{bmatrix}$$

4) Now we need the **determinant** of **B**.

$$\det\begin{bmatrix}1&1&1\\2&3&2\\3&8&2\end{bmatrix}=3\times2+2\times3+2\times8-3\times3-2\times8-2\times2=-1$$

The formula for the inverse matrix is

$$\boxed{\mathbf{B}^{-1}=\frac{\text{adj }\mathbf{B}}{\det\mathbf{B}}}$$

5) Filling in the values, we have the solution.

$$\mathbf{B}^{-1}=\frac{\begin{bmatrix}-10&6&-1\\2&-1&0\\7&-5&-1\end{bmatrix}}{-1}=\begin{bmatrix}10&-6&1\\-2&1&0\\-7&5&1\end{bmatrix}$$

SYMMETRIC MATRICES

A **symmetric matrix** is a square matrix that can be flipped across the diagonal without changing the elements, i.e. **A** = **A**^{T}. All eigenvalues of a symmetric matrix are real. Eigenvectors corresponding to distinct eigenvalues are mutually perpendicular.

$$\begin{bmatrix} 1 & 4 & 5 \\ 4 & 2 & 6 \\ 5 & 6 & 3 \end{bmatrix}$$

A **skew-symmetric** matrix has off-diagonal elements mirrored by their negatives across the diagonal. **A**^{T} = -**A**.

$$\begin{bmatrix} 1 & a & b \\ -a & 2 & c \\ -b & -c & 3 \end{bmatrix}$$

MISCELLANEOUS MATRICES

The **transpose of a matrix** A is written A^T and is the $n \times m$ matrix whose ijth entry is the jith entry of A.

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 9 & 7 & 5 \end{bmatrix} \qquad \mathbf{A}^T = \begin{bmatrix} 1 & 9 \\ 2 & 7 \\ 3 & 5 \end{bmatrix}$$

A **diagonal matrix** is composed of zeros except for the diagonal and is commutative with another diagonal matrix, i.e. $AB = BA$.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

A **diagonal matrix** of equal elements commutes with **any** matrix, i.e. $AB = BA$.

$$\mathbf{A} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

A **lower triangular matrix** has above the diagonal. Similarly an upper triangular matrix has 0's below.

$$\begin{bmatrix} 1 & 0 & 0 \\ 12 & 1 & 0 \\ -5 & 7 & 1 \end{bmatrix}$$

IMAGE OF A TRANSFORMATION

The **image** of a transformation is its possible values. The image of a matrix is the span of its columns. An image has dimensions. For example if the matrix has three rows the image is one of the following:

1) 3-dimensional space, det(A) ≠ 0, rank = 3
2) 2-dimensional plane, det(A) = 0, rank = 2
3) 1-dimensional line, det(A) = 0, rank = 1
4) 0-dimensional point at origin, A = 0

Given the matrix: $\mathbf{A} = \begin{bmatrix} 0 & 1 & 3 \\ 2 & 2 & 1 \\ 3 & 1 & 2 \end{bmatrix}$ of the transformation $A\mathbf{x}$, the **image** consists of all combinations of its (linearly independent) column vectors.

$$x_1 \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$$

SPAN OF A MATRIX

The **span** of a matrix is all of the linear combinations of its column vectors. Only those column vectors which are linearly independent are required to define the span.

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 3 \\ 2 & 2 & 1 \\ 3 & 1 & 2 \end{bmatrix} \quad \text{span} = c_1 \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$$

KERNAL OF A TRANSFORMATION

The **kernal** of a transformation is the set of vectors that are mapped by a matrix to zero. The kernal of an invertible matrix is zero. The **dimension** of a kernal is the number of vectors required to form the kernal.

$$T(\mathbf{x}) = C\mathbf{x} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \qquad \text{kernal } \mathbf{x} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

LINEAR INDEPENDENCE

A collection of vectors is **linearly independent** if none of them are a multiple of another, and none of them can be formed by summing multiples of others in the collection.

BASIS

A **basis** of the span of a matrix is a group of linearly independent vectors which span the matrix. These vectors are not unique. The number of vectors required to form a basis is equal to the rank of the matrix. A basis of the span can usually be formed by incorporating those column vectors of a matrix corresponding to the position of leading 1's in the rref matrix; these are called **pivot columns**. The empty set θ is a basis of the space $\{\mathbf{0}\}$. There is also basis of the kernal, **basis** of the image, **eigenbasis**, **orthonormal basis**, etc. In general terms, *basis* infers a minimum sample needed to define something.

TRACE

A trace is the sum of the diagonal elements of a square matrix and is written tr(A).

ORTHONORMAL VECTORS

Vectors are **orthonormal** if they are all unit vectors (length =1) and are **orthogonal** (perpendicular) to one another. Orthonormal vectors are linearly independent. Their dot product of orthogonal vectors is zero.

ORTHOGONAL MATRIX

An **orthogonal matrix** is composed only of orthonormal vectors; it has a determinant of either 1 or -1. An orthogonal matrix of determinant 1 is a **rotation matrix**. Its use in a linear transformation is called a **rotation** because it rotates the coordinate system. Matrix $\mathbf{A}$ is orthogonal iff $\mathbf{A}^T\mathbf{A} = \mathbf{I}_n$, or equivalently $A^{-1} = \mathbf{A}^T$.

ORTHOGONAL PROJECTION

$\mathbf{V}$ is an $n \times m$ matrix. $\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_m$ are an orthonormal basis of $\mathbf{V}$. For any vector $\mathbf{x}$ in $\Re^n$ there is a unique vector $\mathbf{w}$ in $\mathbf{V}$ such that $\mathbf{x} \perp \mathbf{w}$. The vector $\mathbf{w}$ is called the **orthogonal projection** of $\mathbf{x}$ onto $\mathbf{V}$. see also Gram-Schmidt.pdf

ORTHOGONAL PROJECTION OF **x** ONTO *V*

$$\mathbf{w} = \text{proj}_V \mathbf{x} = (\mathbf{v}_1 \cdot \mathbf{x})\mathbf{v}_1 + \ldots + (\mathbf{v}_m \cdot \mathbf{x})\mathbf{v}_m$$

EIGENVECTORS AND EIGENVALUES

Given a square matrix $\mathbf{A}$, an **eigenvector** is any vector $\mathbf{v}$ such that $A\mathbf{v}$ is a scalar multiple of $\mathbf{A}$. The **eigenvalue** would be the scalar for which this is true. $\mathbf{Av} = \lambda\mathbf{v}$. To determine the eigenvalues, solve the characteristic polynomial $\det(\lambda I_n - \mathbf{A}) = 0$ for values of λ. Then convert to rref form and solve for the coefficients as though it was a matrix of simultaneous equations. This forms a column vector which is an eigenvector. Where there are 0's, you can let the coefficient equal 1.

EIGENSPACE

The **eigenspace** associated with an eigenvalue λ of an $n \times n$ matrix is the kernal of the matrix $\mathbf{A} - \lambda I_n$ and is denoted by E_l. E_l consists of all solutions $\mathbf{v}$ of the equation $\mathbf{Av} = \lambda\mathbf{v}$. In other words, E_l consists of all eigenvectors with eigenvalue λ, together with the zero vector.

EIGENBASIS

An **eigenbasis** of an $n \times n$ matrix $\mathbf{A}$ is a basis of R^n consisting of unit eigenvectors of $\mathbf{A}$. To convert a vector to a unit vector, sum the squares of its elements and take the inverse square root. Multiply the vector by this value.

GEOMETRIC MULTIPLICITY

The **geometric multiplicity** for a given eigenvalue λ is the dimension of the eigenspace E_λ; in other words, the number of eigenvectors of E_λ. The **geometric multiplicity** for a given λ is equal to the number of leading zeros in the top row of rref$(A - \lambda I_n)$..

ALGEBRAIC MULTIPLICITY

The **algebraic multiplicity** for a given eigenvalue λ is the number of times the eigenvalue is repeated. For example if the characteristic polynomial is $(\lambda-1)^2(\lambda-2)^3$ then for $\lambda = 1$ the algebraic multiplicity is 2 and for $\lambda = 2$ the algebraic multiplicity is 3. The algebraic multiplicity is greater than or equal to the geometric multiplicity.

LAPLACE EXPANSION BY MINORS

This is a method for finding the determinant of larger matrices. The process is simplified if some of the elements are zeros. 1) Select the row or column with the most zeros. 2) Beginning with the first element of this selected vector, consider a submatrix of all elements that do not belong to either the row or column that this first element occupies. This is easier to visualize by drawing a horizontal and a vertical line through the selected element, eliminating those elements which do not belong to the submatrix. 3) Multiply the determinant of the submatrix by the value of the element. 4) Repeat the process for each element in the selected vector. 5) Sum the results according to the *rule of signs*, that is reverse the sign of values represented by elements whose subscripts i & j sum to an odd number.

DIAGONALIZABLE

If an $n \times n$ matrix has n distinct eigenvalues, then it is **diagonalizable**.

NULLITY

The **nullity** of a matrix is the number of columns in the result of the matlab command `null(A)`.

SINGULAR MATRIX

A **singular matrix** is not invertible.

SIMILARITY

Matrix **A** is similar to matrix **B** if $S^{-1}\mathbf{A}S = \mathbf{B}$. Similar matrices have the same eigenvalues with the same geometric and algebraic multiplicities. Their determinants, traces, and rank are all equal

REFLECTION

Given that L is a line in $\Re^n$, **v** is a vector in $\Re^n$ and **u** is a unit vector along L in $\Re^n$, the **reflection of v in *L*** is:

$$\text{ref}_L \mathbf{v} = 2(\text{proj}_L \mathbf{v}) - \mathbf{v} = 2(\mathbf{u} \cdot \mathbf{v})\mathbf{u} - \mathbf{v}$$

DOT PRODUCT

The dot product of two matrices is equal to the transpose of the first matrix multiplied by the second matrix.	$\mathbf{A} \cdot \mathbf{B} = \mathbf{A}^T\mathbf{B}$	Example: $\begin{bmatrix}1\\2\\3\end{bmatrix} \cdot \begin{bmatrix}2\\2\\-1\end{bmatrix} = \begin{bmatrix}1 & 2 & 3\end{bmatrix}\begin{bmatrix}2\\2\\-1\end{bmatrix} = 3$

ORTHOGONAL DIAGONALIZATION

A matrix **A** is diagonalizable if and only if **A** is symmetric. $\mathbf{D} = \mathbf{S}^{-1}\mathbf{AS}$ where **D** is a diagonal matrix whose diagonal is composed of the eigenvalues of **A** with the remainder of the elements equal to zero, **S** is an orthogonal matrix whose column vectors form the eigenbasis of **A**. To find **D** we need only find the eigenvalues of **A**. To find **S** we find the eigenvectors of **A**. If **A** has distinct eigenvalues, the unit eigenvectors form **S**, otherwise we have more work to do.

For example if we have a 3 × 3 matrix with eigenvalues 9, 0, 0, we first find a linearly independent eigenvector for each eigenvalue. The eigenvector for $\lambda = 9$ (we'll call it **y**) will be unique and will become a vector in matrix **S**. We must choose eigenvectors for $\lambda = 0$ so that one of them is orthogonal (we'll call it **x**) to the eigenvector **y** from $\lambda = 9$, by keeping in mind that the dot product of two orthogonal vectors is zero. The remaining non-orthogonal eigenvector from $\lambda = 9$ we will call **v**. Now from the eigenspace $\mathbf{x}, \mathbf{v}$ we must find an orthogonal vector to replace **v**. Using the formula for orthogonal projection $\mathbf{w} = \text{proj}_V \mathbf{x} = (\mathbf{v} \cdot \mathbf{x})\mathbf{v}$, we plug in our values for **x** and **v** and obtain vector **w**, orthogonal to **x**. Now matrix **S** = [**w x y**]. We can check our work by performing the calculation $\mathbf{S}^{-1}\mathbf{AS}$ to see if we get matrix **D**.

PRINCIPLE SUBMATRICES

Give a matrix: $\begin{bmatrix}1 & 2 & 3\\4 & 5 & 6\\7 & 8 & 9\end{bmatrix}$, the principle submatrices are: $[1]$, $\begin{bmatrix}1 & 2\\4 & 5\end{bmatrix}$, and $\begin{bmatrix}1 & 2 & 3\\4 & 5 & 6\\7 & 8 & 9\end{bmatrix}$

COORDINATE VECTOR

If we have a basis **B** consisting of vectors $\mathbf{b}_1, \mathbf{b}_2, \cdots \mathbf{b}_n$, then any vector **x** in R^n can be written as:	$\mathbf{x} = c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + \cdots + c_n\mathbf{b}_n$
The vector $\mathbf{c} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$ is the **coordinate vector** of **x** and:	$\mathbf{Bc} = \mathbf{x}$

Determining the Coordinate Vector

Given B and **x**, we find **c** by forming an augmented matrix from B and **x**, taking it to rref form and reading **c** from the right-hand column.

QUADRATIC FORM

A function such as $q(\mathbf{x}) = q(x_1, x_2) = 6x_1^2 - 7x_1x_2 + 8x_2^2$ is called a **quadratic form** and may be written in the form $q(\mathbf{x}) = \mathbf{x} \cdot A\mathbf{x}$. Notice in the example at right how the $-7x_1x_2$ term is split in half and used to form the "symmetric" part of the symmetric matrix.

POSITIVE DEFINITE: Matrix **A** is **positive definite** if all eigenvalues are greater than 0, in which case $q(\mathbf{x})$ is positive for all nonzero **x**, and the determinants of all **principle submatrices** will be greater than 0.

NEGATIVE DEFINITE: Matrix **A** is **negative definite** if all eigenvalues are less than 0, in which case $q(\mathbf{x})$ is negative for all nonzero **x**.

INDEFINITE: Matrix **A** is **indefinite** if there are negative and positive eigenvalues in which case $q(\mathbf{x})$ may also have negative and positive values.

What about eigenvalues which include 0? The definition here varies among authors.

Example:

$$q(\mathbf{x}) = q(x_1, x_2) = 6x_1^2 - 7x_1x_2 + 8x_2^2$$

$$q(\mathbf{x}) = \mathbf{x} \cdot A\mathbf{x}$$

$$q(\mathbf{x}) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \cdot \begin{bmatrix} 6x_1 - \frac{7}{2}x_2 \\ -\frac{7}{2}x_1 + 8x_2 \end{bmatrix}$$

$$= \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \cdot \begin{bmatrix} 6 & -\frac{7}{2} \\ -\frac{7}{2} & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$A = \begin{bmatrix} 6 & -\frac{7}{2} \\ -\frac{7}{2} & 8 \end{bmatrix}$$

DISTANCE OF TWO ELEMENTS OF AN INNER PRODUCT

$$\text{dist}(f,g) = \|f - g\| = \sqrt{\int_a^b [f(t) - g(t)]^2 dt}$$

INNER PRODUCT

An **inner product** in a linear space *V* is a rule that assigns a real scalar (denoted by $\langle f, g \rangle$ to any pair *f*, *g* of elements of *V*, such that the following properties hold for all *f*, *g*, *h* in *V*, and all *c* in R. A linear space endowed with an inner product is called an **inner product space**.

a. $\langle f, g \rangle = \langle g, f \rangle$

b. $\langle f + g, h \rangle = \langle f, h \rangle + \langle g, h \rangle$

c. $\langle cf, g \rangle = c\langle f, g \rangle$

d. $\langle f, f \rangle > 0$ for all nonzero *f* in *V*.

Two elements *f*, *g* of an inner product space are **orthogonal** if: $\langle f, g \rangle = 0$

NORM

The **norm** of a vector is its length:

$$\|\mathbf{v}\| = \sqrt{v_1^{\,2} + v_2^{\,2} + \cdots + v_n^{\,2}}$$

The **norm** of an element f of an **inner product space** is:

$$\|f\| = \sqrt{\langle f,\, f \rangle} = \sqrt{\int_a^b f^2 dt}$$