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Solving Quadratic Equations

General form: $ax^2+bx+c=a(x-x_1)(x-x_2)=0$, $a\neq 0$

Algebraically:

1) $b=0\Rightarrow ax^2+c=0$

$x^2=-\dfrac{c}{a}$ → $x_{1,2}=\pm\sqrt{\dfrac{c}{a}}$ if $\dfrac{c}{a}<0$

$x^2=-\dfrac{c}{a}$ → no solution if $\dfrac{c}{a}>0$

Example:

$2x^2-8=0$

$x=\pm\sqrt{\dfrac{8}{2}}=\pm\sqrt{4}=\pm 2$

Example:

$2x^2+8=0$

$x^2=-\dfrac{8}{2}=-4\Rightarrow$ no solutions

2) $c=0\Rightarrow ax^2+bx=0$

$x(ax+b)=0$ $\left[\begin{array}{l} x_1=0 \\ x_2=-\dfrac{b}{a}\end{array}\right.$

Example:

$2x^2-8x=0$

$x(2x-8)=0$ $\left[\begin{array}{l} x_1=0 \\ x_2=\dfrac{8}{2}=4\end{array}\right.$

3) $b\neq 0$ and $c\neq 0$

a) General formula: $x_{1,2}=\dfrac{-b\pm\sqrt{D}}{2a}$, $D=b^2-4ac$

$D>0\Rightarrow$ two different real roots

$D=0\Rightarrow$ two coincide real roots

$D<0\Rightarrow$ no real roots, two complex

Example:

$x^2-4x-5=0$

$D=(-4)^2-4\cdot 1\cdot(-5)=36=6^2>0$

$x_{1,2}=\dfrac{-(-4)\pm\sqrt{6^2}}{2\cdot 1}=\dfrac{4\pm 6}{2}$=5 and -1

Example:

$x^2+4x+4=0$

$D=4^2-4\cdot 1\cdot 4=0$

$x_{1,2}=\dfrac{-4\pm\sqrt{0}}{2\cdot 1}=-\dfrac{4}{2}=-2$

Example:

$x^2+4x+5=0$

$D=4^2-4\cdot 1\cdot 5=-4<0$

no real roots

b) $b=2k$ $x_{1,2}=\dfrac{-k\pm\sqrt{D/4}}{a}$, $D/4=k^2-ac$

Example:

$x^2-4x-5=0$ $-4=2k\Rightarrow k=-2$

$D/4=(-2)^2-1\cdot(-5)=9=3^2$

$x_{1,2}=\dfrac{-(-2)\pm\sqrt{3^2}}{1}=2\pm 3$=5 and -1

c) Vieta's Theorem:

$$\begin{cases} x_1 + x_2 = -\dfrac{b}{a} \\ x_1 \cdot x_2 = \dfrac{c}{a} \end{cases}$$

Example:

$x^2 - 4x - 5 = 0$

$$\begin{cases} x_1 + x_2 = -\dfrac{-4}{1} = 4 \\ x_1 \cdot x_2 = \dfrac{-5}{1} = -5 \end{cases} \Rightarrow \left[\begin{array}{l} x_1 = 5 \\ x_2 = -1 \end{array} \right.$$

d) Complete the square

$x^2 - 4x - 5 = 0$

$(x^2 - 2 \cdot 2 \cdot x + 4) - 4 - 5 = 0$

$(x-2)^2 = 9$

$$|x-2| = 3 \Rightarrow \left[\begin{array}{l} x - 2 = 3 \\ x - 2 = -3 \end{array} \right. \Leftrightarrow \left[\begin{array}{l} x = 5 \\ x = -1 \end{array} \right.$$

e) $a + b + c = 0 \Rightarrow \left[\begin{array}{l} x_1 = 1 \\ x_2 = \dfrac{c}{a} \end{array} \right.$

Example:

$x^2 - 4x + 3 = 0$

$$1 + (-4) + 3 = 0 \Rightarrow \left[\begin{array}{l} x_1 = 1 \\ x_2 = \dfrac{3}{1} = 3 \end{array} \right.$$

$$a + c = b \Rightarrow \left[\begin{array}{l} x_1 = -1 \\ x_2 = -\dfrac{c}{a} \end{array} \right.$$

Example:

$x^2 - 4x - 5 = 0$

$$1 - 5 = -4 \Rightarrow \left[\begin{array}{l} x_1 = -1 \\ x_2 = -\dfrac{-5}{1} = 5 \end{array} \right.$$

Graphically:

1) Draw a parabola $y = ax^2 + bx + c$, the x-intercepts will be the roots of the equation $ax^2 + bx + c = 0$

2) From $ax^2 + bx + c = 0$ we can get $x^2 = -\dfrac{b}{a}x - \dfrac{c}{a}$ and then $\begin{cases} y = x^2 \\ y = -\dfrac{b}{a}x - \dfrac{c}{a} \end{cases}$ The points of intersection of these two functions will be the roots of $ax^2 + bx + c = 0$