Exercises in Modulo Arithmetic

If a divides b without a remainder, we write $a|b$. Ex: 3 divides 9, so $3|9$. 4 does not divide 9, so $4 \not| 9$.
We say $a \equiv b \pmod{n}$ if $n|(a-b)$. Ex: $5 \equiv 2 \pmod{3}$ because 3—(5-2).
We say that a is congruent to b modulo n if $a \equiv b \pmod{n}$. In the above example, 5 is congruent to 3 modulo 2. We can deduce several facts from the modulo relationship.
For example, suppose $a \equiv a' \pmod{n}$ and $b \equiv b' \pmod{n}$. Then $n|(a-a')$ and $n|(b-b')$, so $n|(a-a'+b-b') = a+b-(a'+b')$, or $a+b \equiv a'+b' \pmod{n}$. Similarly, $a-b \equiv a'-b' \pmod{n}$. Ex: $5 \equiv 1 \pmod{4}$ and $3 \equiv -1 \pmod{4}$, and $5+3 = 8 \equiv 0 = 1+(-1) \pmod{4}$.
A similar result can be shown with multiplication, that is $ab = a'b' \pmod{n}$. However, this is not always true with division. Ex: $10 \equiv 4 \pmod{6}$, but $\frac{10}{2} = 5 \not\equiv 2 = \frac{4}{2} \pmod{6}$.
How is this useful? Modulo arithmetic can be used to calculate remainders of divisions. Ex: What is the remainder of 3^{99} when divided by 10? note that $3^2 = 9 \equiv -1 \pmod{10}$, so $3^{99} = 3^{2*49} * 3 \equiv 9^{49} * 3 \equiv (-1)^{49} * 3 \equiv -3 \pmod{10}$.

1. What is the remainder of 10^{20} when divided by 13?
2. January 1, 2009 is a Thursday. What is October 1, 2009?
3. Prove: if $9|k$, then 9 divides the sum of digits of k. Ex: $9|297$, and $9|2+9+7 = 18$.
4. The supersum of a number n is the result of repeatedly taking the sum of digits of n until you get a single digit. For example, the supersum of 199240 is the supersum of 1+9+9+2+4+0 = 25, and the supersum of 25 is 7, thus the supersum of 199240 is 7. What is the supersum of 2^{64}?
5. Prove: for any number n, m there is a number k such that $n \equiv k$ (mod m) where $0 \leq k \leq m-1$.
6. Prove: if p is prime, then if $p \not| n$ there is a number m such that $p|nm-1$.
7. Prove: if p is prime, then $p|(p-1)!+1$, where $n! = n*(n-1)*...*2*1$.