Chapter 2. Linear Spaces

2.1 Field and Linear Space

Definition: Field

F is a field, if the following axioms hold

1. + is associative

$\forall\ a,b,c\ \in F$ (a + b) + c = a + (b + c)

2. O is the neutral element under addition

$\forall a \in F$ a + 0 = 0 + a = a

3. Every element has an apposite

$(\forall a \in F\ \ and\ \ \exists b \in F)$ a + b $\neq$ b + a = 0

4. + is commutative

$\forall\ a,b\ \in F$ a + b = b + a

There fore $\langle F;+\rangle$ is an abelian group.

5. $\bullet$ is associative

$\forall\ a,b,c\ \in F$ a . (b . c) = (a . b) . c

6. 1 is the neutral element

$\forall\ a \in F$ a . 1 = 1 . a = a

7. Every non-zero element has an inverse

$(\forall a \in F)\ \ (a \neq 0 \rightarrow (\exists b \in F))$ a . b = b . a = 1

8. $\bullet$ is commutative

$(\forall ab \in F)$ $a\ .\ b = b\ .\ a$

There fore $\langle$ F;$\bullet$ $\rangle$ is an abelian group.

9. Distributive Laws

$\forall\ a,b,c\ \in F$ Left distributy a . (b+c) = ab + ac

right distributy (b+c) . a = ab + ac

Example: IR is the set of real numbers, and C is the set of complex numbers are fields.

2.2 Definition: Linear space

Let $u,v,\omega ..... \in V,\quad a_1,a_2 .... \in F, F$ is either IR or C, then V is said to be linear space (vector space) if and only if the properties satisfied

VS1 – V is closed under scalar multiplication

$a\nu \in V$

VS2 – V is closed under vector addition

$u + \nu \in V$

Here, it is required that the properties of vector addition and scalar multiplication.

I. $u + (u + \omega) = (u + \nu) + \omega \qquad u, \nu, \omega \in V$

II. $u + \nu = \nu + u$

III. $u + \phi = \phi + u = u, \ \phi \in V$

IV. $u + \nu = \nu + u = \phi$ if and only if $u = -\nu$

V. $a\,(bu) = (a\,b)\,\mathrm{u} \qquad a\,,b \in F \quad and \quad u \in V$

VI. $1 \cdot u = u \cdot 1 = u \qquad u \in V \quad 1 \in F$

$$\left.\begin{array}{ll} \text{VII.} & u + v \in V \\ \text{VIII.} & av \in V \end{array}\right\} \text{Closure addition and multiplication}$$

Example: IR^3 is a vector space, since it is closed under vector addition and scales multiplication together with the properties of vector addition and scales multiplication satisfied.

NOTE: Closure property can be written as

$a_1 u + a_2 \nu \in V$

2.2.1 Definition: Linear Subspace

Let V be a vector space over a field IR or C. A non-empty subset of W of V is called subspace of V if W itself is a vector space.

Lemma (Subspace Criterion): Let V be a vector space a over a field IR or C and let W be a non-empty subset of V. Then W is subspace if and only if

i) $w_1 + w_2 \in W$ for all $w_1, w_2 \in W$

ii) $a\,w \in W$ for all $a \in IR \quad or \quad a \in C,\ w \in W$ or we can simply say that W is subspace of V if and only if

$a\,w_1 + b\,w_2 \in W$ for all w_1 , $w_2 \in W$ and $a\,,b \in IR$ or $a,\,b \in C$

2.2.2 Definition: Dependency Relation

Let $v_1, v_2, \ldots, v_n \in V$ (real or complex vector space) and let $a_1, a_2, \ldots\ldots a_n \in F$ $(F = IR \, or \, C)$. Then the relation

$a_1 v_1 + a_2 v_2 + \ldots\ldots + a_n v_n = 0$

is called a dependency relation for $v_1, v_2, \ldots, v_n$ if the dependence relation is satisfied only when

$a_1 = a_2 = \ldots\ldots = a_n = 0$, then $v_1, v_2, \ldots, v_n$ are said to linearly in dependent vector space, otherwise they are said to be linearly dependent.

2.2.3 Definition: Linear Span

Let W be a subset of V, then we call the set of all possible linear combinations of vectors of W by the linear span of W and denote it by < w >

$< w > = \{a_1 w_1 + \ldots\ldots + a_n w_n : \quad \forall a_1, \ldots\ldots, a_n \in F \quad \forall w_1, \ldots, w_n \in W\}$

One also says that < w > in the subspace spanned.

2.2.4 Definition: Basis

A subset $W \subset V$ of linearly independent vectors is called a basis for V if spanW = V ; that is , all the elements of V can be generated by the proper linear combinations of the vector of W.

Example: The set X = $\{1, x, x^2, \ldots\ldots, x^n\}$ is a basis for the set of polynomials up to the *n* th order.

$a_0 + a_1 x + a_2 x^2 + \ldots\ldots + a_n x^n = 0 \Rightarrow a_0 = a_1 = a_2 = \ldots\ldots\ldots = a_n = 0$

2.3 Norm and Normed Vector Space

The idea of the length of a vector is intuitive and can be easily extended to any real vector space R^n. Its properties are

1. a vector always has a strictly positive length. The only exception is the zero vector
2. Multiplying a vector by a positive number has the same effect on the length.

3. The triangle inequality holds. The distance from A through B to C is never shorter than going directly from A to C

2.3.1 Definition: Norm

If v is a vector space over a field F (IR or C), a norm on V is a function from V to IR. It associates to each vector v in V a real number, which is usually denoted $\|v\|$. The norm must satisfying the following conditions.

For all a in K and all u and v in V

1. $\|v\| \geq 0$ with equality if and only if v = 0

2. $\|av\| = |a| \cdot \|v\|$

3. $\|u + v\| \leq \|u\| + \|v\|$

   $\|u + v\| = \|u\| + \|v\|$ if $v = a\,u,\ u > 0$

2.3.2 Definition: Normed Vector Space

A vector space on which a norm is defined is called as normed vector space or only normal space.

Example:

i) Euclidean norm

On R^n, the intuitive notion of length of vector

$x = (x_1, x_2, \ldots, x_n)$ is captured by the formula

$$\|x\| = \sqrt{|x_1|^2 + |x_2|^2 + \ldots + |x_n|^2}$$

This gives the ordinary distance from the origin to the point x, a consequence of the Pythagorean theorem.

ii) p – norm

Let $p \geq 1$ be a real number

$$\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p}$$

Note that for p = 1 we get the Manhattan Norm and p = 2 we get the Euclidean norm.

Example:

i) Sequence space I^{∞}: the space of all bounded sequences of complex numbers $v = (x_1, x_2, \ldots, x_i, \ldots) \in V$, $x_i \in C$, $|x_i| = \text{finite}$, $\|v\| = \sup_{i E N^+} |x_i|$

ii) Space C [a, b] is defined as the space of real continous functions with a norm $\|v\| = \max_{t \in I} |v(t)| \quad t \in I\ [a,b]$

iii) The space of real continous function v (t), $t \in [a,b]$, $\|v\| = \left(\int_a^b (v(t))^2\, dt \right)^{1/2}$ is a norm for this space; but $\|v\| = \int_a^b v(t)\, dt$ is not

2.4 Metric and Metric Space

2.4.1 Definition: Metric

A non-negative function d (x,y) describing the distance between neighbourhoods of points for a given set. Metric is a generalization of the concept of distance. A metric satisfies the axioms below:

i) d (x,y) = 0 iff x = y

ii) d (x,y) = d (y,x) , symmetry

iii) d (x,y) + d (y,z) $\geq$ d (x,z) , triangle inequality

2.4.2 Definition: Metric Space

A metric space is a set X together with a function d (called a metric or distance function) which assings a real number d (x,y) to every pair x, y $\in$ X

i) d (x,y) $\geq$ 0 and d (x,y) = 0 if and only if x = y

ii) d (x,y) = d (y,x), symtery

iii) d (x,y) + d (y,z) $\geq$ d (x,z) , triangle inequality

Example: The metric on spaces of functions. Let C [0, 1] be the set of all continues real valued function on the inter [0, 1]

$$d_1(f,g)=\int_D^1 |f(x)-g(x)|\, d_x$$

So the distance between function is the area between their graphs.

2.5 Complete Metric Space

2.5.1 Definition: A sequence $\{v_n\}$ converges to the limit u if for all $\in > 0$ there exist $N \in IV$ such that $|v_n - u| < \in$ for all $n \geq N$. Then $\{v_n\}$ is said to be convergent sequence.

2.5.2 Definition: A sequence $\{v_n\}$ is a Cauchy sequence if for any $\in > 0$ there exist $N \in N$ such that $|v_n - v_m| < \in$ for any m,n$\geq$ N

2.5.3 Definition: The point that Cauchy sequence converges is called the limit point (or accumulation point).

2.5.4 Definition: A complete metric space is a metric space in which every Cauchy Sequence is convergent.

Example: a metric space in IR

$$d(x,y)=|x-y|$$

All Cauchy sequences are bounded and every bounded subset of IR must have a limit point.

Every Cauchy sequences has a limit point in IR

So, IR is complete.

Example: a metric space in C

$$|Z_n - Z_m| \leq \varepsilon \quad n,m \geq N(\varepsilon)$$

$z_n = x_n + i\, y_n$

$$\{z_n\} \to \{x_n\}, \{y_n\} \leq IR$$

$$|x_n - x_m| \leq |z_n - z_m| < \varepsilon$$

$$|y_n - y_m| \leq |z_n - z_m| < \varepsilon$$

Let $\{x_n\} \to x$ and $\{y_n\} \to y$

$z = x + i\, y \quad |z_n - z| \leq |x_n - x| + |y_n - y|$

$$\lim_{n\to\infty}|z_n - z| = 0 \Rightarrow \lim_{n\to\infty}|z_n - z| = \left|\lim_{n\to\infty} z_n - z\right| = 0$$

$$\lim_{n\to\infty} z_n = 0$$

So, C is complete

2.6 Definition: Banach Space

A complete normed space is called as Banach space.

Example: IR and C are Banach spaces

Example: a metric space in Q

$$d(x, y) = |x - y|$$

The infinite sequence $\left\{\left(1+\frac{1}{n}\right)^n, \ n \in N^+\right\}$

$$\lim_{n\to\infty}\left\{\left(1+\frac{1}{n}\right)^n\right\} = e \quad , \quad e \notin Q$$

So, Q is not complete.

The another example of Q is not complete is that $x_n \in Q$ is infinite sequence that has limit point $\sqrt{2} \notin Q$.

Example: Take a space of continuous functions with the metric

$$d(f, g) = \int_{-1}^{1} [f(x) - g(x)]^2 \, dx$$

Consider a sequence of functions $f_n(x) = \begin{cases} 0 & -1 \le x < 0 \\ \dfrac{2nx}{1+nx} & 0 < x \le \dfrac{1}{n} \\ 1 & \dfrac{1}{n} < x \le 1 \end{cases}$

Then, $\lim_{n\to\infty} d(f_n, g) = \lim_{n\to\infty} \int_{-1}^{1} [fn(x) - g(x)]^2 \, dx$

$$= \lim_{n\to\infty}\left(\int_{-1}^{0} [g(x)]^2 \, dx + \int_{0}^{1/n}\left[\frac{2nx}{1+nx} - g(x)\right]^2 dx + \int_{1/n}^{1} [1 - g(x)]^2 \, dx\right) = C$$

All the integrands are positive. So each term must be zero

$$g(x)=\begin{cases}0 & -1\le x\le 0\\ 1 & 0<x<1\end{cases}$$

It is not complete since the limit is discontinous.

2.7 Inner Product and Inner Product Spaces

Note: Angle Bracket

An angle bracket is the combination of a bra and ket (bra-ket = braket) which represents the inner product of two functions or vectors. In tis notation, a vector is shown by a ket $|\ \rangle$.

2.7.1 Definition: An inner product is a generalization of dot product.

2.7.2 Definition: Inner Product Space

An inner product space, with complex scalars C, is a complex vector space V with a complex-valued function $\langle\cdot|\cdot\rangle$, defined on V x V that has the following properties

v, w, v_1 and v_2 $\in V$, $\forall a\in C$

i) $\langle v|v\rangle\ge 0$

if $\langle v|v\rangle$ then $|v\rangle = 0$

iii) $\langle v|w\rangle=\overline{\langle w|v\rangle}$

iv) $\langle v_1+v_2|w\rangle = \langle v_1|w\rangle+\langle v_2|w\rangle$

v) $\langle a v|w\rangle = a\ \langle v|w\rangle$

If $\langle v|w\rangle$ is assumed to be real-valued, the complex conjugation is dropped in (iii): $\langle v/w\rangle=\langle w/v\rangle$. When (iii) is combined with (iv) and (v)

$iv')$ $\langle w|v_1+v_2\rangle=\langle w|v_1\rangle + \langle w|v_2\rangle$

v') $\langle v|aw\rangle = \bar{a}\ \langle v|w\rangle$

2.7.3 Definitiıon: The Natural Norm

The norm of an element V in an inner product spaces is called natural norm and it is $\|v\|=\sqrt{\langle v|v\rangle}$. It satisfies all the axioms of inner product.

2.7.4 Theorem: The Schwarz Inequality

$\left| \langle v | w \rangle \right| \leq \|v\| \, \|w\|$, and equality holds if and only if one of v and w is a multiple of other.

Suppose that neither of v and w is zero

Let $z \in C$. Consider $\|v - zw\|^2$

z in polar coordinates, $r \in IR$, z = $re^{i\theta}$, and

let f (r) = $\|v - zw\|^2$

$f(r) = \langle v - zw | v - zw \rangle$

= $\langle v | v - zw \rangle - \langle zw | v - zw \rangle$

= $\langle v | v \rangle - \langle v | zw \rangle - \langle zw | v \rangle + \langle zw | zw \rangle$

= $\langle v | v \rangle - (\bar{z} \langle v | w \rangle + z \langle w | v \rangle) |z|^2 \langle w | w \rangle$

= $\|v\|^2 - 2 \operatorname{Re} \bar{z} \langle v / w \rangle + |z|^2 \|w\|^2$

= $\|v\|^2 - 2 r \operatorname{Re} e^{-i\theta} \langle v | w \rangle + r^2 \|w\|^2$

choose $\theta = 0$ so that $e^{-i\theta} \langle v | w \rangle = |\langle v | w \rangle|$

we can write

$f(r) = \|v\|^2 - 2r \, |\langle v | w \rangle| + r^2 \|w\|^2$

If equality holds, then f (r) has zero discriminant.

So v = 2w and equality holds.

2.7.5 Theorem: An inner product space, with $\|v\|$ ar norm, is indeed a norm space.

By the definition of inner product space,

When $\alpha \in C$ $\qquad \|\alpha v\|^2 = \langle \alpha v | \alpha v \rangle = |\alpha|^2 \|v\|^2$

*** The triangle inequality is an application of Schwarz inequality

$\|v + w\|^2 = \|v\|^2 + 2 \operatorname{Re} \langle v | w \rangle + \|w\|^2$

$\leq \|v\|^2 + 2\|v\|\|w\| + \|w\|^2 = (\|v\| + \|w\|)^2$

2.8 Hilbert Space

2.8.1 Definition: A Hilbert Space is a vector space with inner product space which is complete with respect to the norm.

Finite dimensional inner product spaces are Hilbert Space and Hilbert Spaces are always a Banach space, but the converse is not hold.

Example:

i) Square Summable Sequences of Complex Numbers

λ^2 is the space of sequences of complex numbers

$\alpha = \sum_{n=1}^{\infty} \alpha_n$ such that $\sum_{n=1}^{\infty} |\alpha_n|^2 < \infty$

It is a Hilbert Space with the inner product

$$\langle \alpha / \beta \rangle = \sum_{n=1}^{\infty} \bar{\alpha}_n \beta_n$$

ii) Square Integrable Functions on IR

L^2 (IR) is the space of complex valued functions such that

$$\int_{IR} \langle f | x \rangle \langle x | g \rangle \, dx$$

iii) Square Integrable Function on IR^n

Let Ω be an open set in IR^n. The space $L^2(\Omega)$ is the set of complex valued function such that

$\int_{\Omega} |f(x)|^2 \, d_x$ where $x = (x_1, x_2, \ldots.., x_n)$

$dx = dx_1, dx_2, \ldots., dx_n$

It is a Hilbert space with inner product

$$\langle f | g \rangle = \int_{\Omega} \overline{\langle f | x \rangle} \langle x | g \rangle \, dx$$

2.9 Function Space

2.9.1 Definition: Any $| f \rangle$ in a vector space whose basis vectors are $|x>$ and all such vectors constitute a vector space called as function space.

2.10 Orthogonality

2.10.1 Definition: A basis $E = \{|e_1>, |e_2>, ..., |e_i>, ...\}$ for a vector space V is said to be an orthogonal basis if $\langle e_i | e_j \rangle = 0, i \neq j$.

2.10.2 Definition: The vectors u, v $\in$ V are said to be orthogonal, if $\langle u | v \rangle$ = 0,

* Orthogonality is symmetric $\langle u | v \rangle = \langle v | u \rangle = 0$

* θ vector is orthogonal to every v$\in$V $\langle 0 | v \rangle = 0$

* if $\langle u | v \rangle$= 0, every scalar multiple of u is also orthogonal to v

$\langle ku | v \rangle = 0 \Rightarrow \langle ku | v \rangle = k \langle u | v \rangle = k \cdot 0 = 0$

2.10.3 Definition: Consider a set $S = \{|u_1\rangle, |u_2\rangle, ..., |u_n\rangle\}$ of vectors in an inner product space V . S is said to be orthogonal set if each vectors in S are non-zero and if the vectors in S are mutually orthogonal.

$i.e \quad if \quad \langle u_i | u_j \rangle \neq 0 \quad but \quad \langle u_i | u_j \rangle = 0 \quad for \quad i \neq j$

2.10.4 Definition: S is said to be orthonormal if S is orthogonal and if each vectors in S have unit length.

$$\langle u_i | u_j \rangle = S_{ij} = \begin{Bmatrix} 1 & if & i = j \\ 0 & if & i \neq j \end{Bmatrix}$$

Normalization of a vector is that

$$|u|_0 = \frac{|u>}{\sqrt{\langle u | u \rangle}} = \frac{|u>}{\|u\|}$$

An orthogonal set is linear independent, it is a basis for V Suppose $\{|u_1>, |u_2>, ..., |u_n>\}$ is an orthogonal basis for V Then, for any $|v>$ in V

$$v = \frac{\langle v | u_1 \rangle}{\langle u_1 | u_1 \rangle} |u_1> + \frac{\langle v | u_2 \rangle}{\langle u_2 | u_2 \rangle} |u_2> + ... + \frac{\langle v | u_n \rangle}{\langle u_n | u_n \rangle} |u_n\rangle$$

Since, $v = k_1 u_1 + k_2 u_2 + \dots + k_n u_n$

Taking the inner product of both sides with u_1 yield

$$\langle u | u_1 \rangle = \langle ku_1 + ku_2 + ... + ku_n | u_1 \rangle$$
$$= k_1 \langle u_1 | u_1 \rangle + k_2 \langle u_2 | u_1 \rangle + ... + k_n \langle u_n | u_1 \rangle$$
$$= k_1 \langle u_1 | u_1 \rangle + k_2 \ O + ... + k_n \ O$$

$$k_1 = \frac{\langle v | u_1 \rangle}{\langle u_1 | u_1 \rangle} = \frac{\langle v | u_1 \rangle}{\| u_1 \|^2}$$

generalized this result

$$k_i = \frac{\langle v | u_i \rangle}{\langle u_i | u_i \rangle} = \frac{\langle v | u_i \rangle}{\| u_i \|^2}$$

The above scalar k_i is called the component of v along u_i or the Fourier Coefficient of v with respect to u_i.

Suppose v_1, v_2, …, v_n form a basis for a subspace U of an inner product space V. The **Gramm – Schmidt algorithm(2.10.5)** which yield an orthogonal basis (and by normalization of an orthonormal basis) of U.

Suppose $\{|w_1 >, |w_2 >, ..., |w_n >\}$ is an orthogonal set of vectors in V

Set $w_1 = v_1$

$$w_2 = v_2 - k_{21} \ w_1 = |v_2\rangle \Big| - \frac{< v_2 / w_1 >}{< w_1 / w_1 >} \ |w_1\rangle$$

$w_3 = v_3 - k_{31}w_1 - k_{32}w_2$

$$= \ v_3 - \frac{\langle v_3 | w_1 \rangle}{\langle w_1 | w_1 \rangle} \ |w_2\rangle \ - \frac{\langle v_3 | w_2 \rangle}{\langle w_2 | w_2 \rangle} \ |w_2\rangle$$

$w_n = v_n - k_{n1} \ w_1 - k_{n2} \ w_2 - …. - k_{n-1} \ w_{n-1}$

where

$$k_{ni} = \frac{\langle v_n | w_i \rangle}{\langle w_i | w_i \rangle}$$

The set $\{|w_1 >, |w_2 >, ..., |w_n >\}$ is the required arthogonal basis of U.

Ex: L^2 (-1,1) is the space of real continous functions with an inner product

$$\langle f \mid g \rangle = \int_{-1}^{1} f(x)\, g(x)\, dx$$

a sequence 1, x, x^2, …., x^i, …… $i \in N^+$, $x \in [-1,1]$

Identifying $x^n \to |n>$ ($x^n = <x/n>$)

$$\langle n | m \rangle = \int_{-1}^{1} x^n \; x^m \; dx = \int_{-1}^{1} x^{n+m} \, dx = \begin{cases} 0 & , \quad n+m = odd \\ \dfrac{2}{n+m+1} & , \quad n+m = even \end{cases}$$

$\Rightarrow \langle 0|1 \rangle = \langle 0|3 \rangle = \langle 1|2 \rangle = \langle 2|3 \rangle = 0$, others nonzero.

The sequence is not orthogonal

Apply the Gram-Schmidt Algorithm for the first few term:

Take 1, x, x^2, x^3, ……

Orthogonalize $|2>$ and $|3>$

$$|2>_0 = |2> - \frac{<0/2>}{<0/0>} |0> = |2> - \frac{1}{3} |0>$$

$$\Rightarrow \; |2>_0 \Rightarrow x^2 - \frac{1}{3} \left(<x/2>_0 - \frac{1}{3} <x/0> = x^2 - \frac{1}{3} \right)$$

$$|3>_0 = |3> - \frac{<1/3>}{<1/1>} |1> = |3> - \frac{3}{5} \; |\, 1>$$

$$\Rightarrow |3>_0 \to x^3 - \frac{3}{5}\, x$$

The orthogonal sequence is then 1, x, $x^2 - \frac{1}{3}, x^3 - \frac{3}{5} x$, …………

Normalization of the sequence

$$\frac{1}{\sqrt{2}}, \left(\frac{\sqrt{3}}{2}\right) x \,, \quad \left(\frac{\sqrt{5}}{2}\right) \left(\frac{1}{2}(3x^2 - 1)\right), \quad \left(\frac{\sqrt{7}}{2}\right) \left(\frac{5}{2}(x^3 - 3x)\right), \;\; \ldots\ldots$$

$$\to \left\{ \sqrt{\frac{2n+1}{2}} \; P_n(x) \right\} \leftrightarrow \{ In >_0 \}_{\text{normalized}}$$

Ex:

Let V be the vector space of real continous function on the interval $-\pi \le x \le \pi$ with inner product defined

$$< f / g >= \int_{-\pi}^{\pi} f(x) g(x) \, dx$$

The following set S of functions plays a fundamental role in the theory of Fourier Series

S = { 1, sinx, cosx, sin2x, cos2x, ….}

S is orthogonal since for any function f, g $\in$ S we have

$$< f / g >= \int_{-\pi}^{\pi} f(x) \, g(x) \, dx = 0$$

On the other hand S is not orthonormal, since

For example $< \cos x / \cos x > = \int_{-\pi}^{\pi} f(x) \, g(x) \, dx = \pi$

Chapter 3. Linear Operators

3.1 Linear Operators

Let us first clarify the notion of mapping. A mapping is a transformation of a scalar or a vector from some spaces of scalars or vectors.

1) $y = f(x)$ $\quad f : x \in K \to y \in K$ scalar function

2) $| y >=| f(x-) >$ $\quad | f >: x \in K \to | y > \in V$ vector function; e.g.,

$\vec{f}(x) = f_1(x) \hat{i} + f_2(x) \hat{j}$

3) $y = f(| x >)$ $\quad f :| x > \in V \to y \in K$ linear or nonlinear functional;

e.g., $f(| x >) =< c | x >$

4) $| y >=| f(| x >) >$ $f :| x > \in V \to | y > \in V$ $\quad ? \to$ operators

$\left| y \right\rangle = \left| f\left(\left| x \right\rangle \right) \right\rangle = L\left| x \right\rangle$ which we call as "operator". The set $\{ \left| x \right\rangle \}$ is called the domain of the operator and the set $\{ | \left| y \right\rangle \}$ is called the range of operator.

An operator can be linear or nonlinear.

3.1.1 Definition: Linear Operators

If an operator L satisfies

i) $L(|\nu>+|w>)=L|\nu>+L|w>$

ii) $L(a|\nu>)=aL|\nu>$

where $a\in k, |\nu>\in V$, then it is called linear. The two linearity conditions can also be written in a unique way as $L(a|v\rangle+b|w\rangle)=aL|\nu\rangle+bL|w\rangle$ and they can be extended to scalar functions, vector functions and functionals.

Example: Given $|\nu\rangle=\nu_1|1\rangle+\nu_2|2\rangle, |\nu\rangle\in V^2$, what is the domain and the range of the operator A if $A|\nu\rangle=\dfrac{\nu_1}{\nu_1-\nu_2}|1\rangle+\dfrac{\nu_2}{\nu_1-\nu_2}|2\rangle, |\nu\rangle\in V^2$.

$D=V^2-\{|\nu\rangle \,|\, |\nu\rangle=\alpha(|1\rangle+|2\rangle), \alpha\in K\}, R=V^2$. Is it linear? (exercise; the answer is NO)

If A and B are two linear operators in the space of vectors, $|\ \rangle$, then

i) $(A+B)|\ \rangle=A|\ \rangle+B|\ \rangle$; the sum of two linear operators is again an operator;

ii) $(A\cdot B)|\ \rangle=A|\ \rangle\cdot B|\ \rangle$; the multiplication of two linear operators is again an operator;

iii) A = B iff $A|\ \rangle=B|\ \rangle$;

iv) $A\cdot B\neq B\cdot A$ in general.

Since $A\cdot B\neq B\cdot A$ in general, $A\cdot B-B\cdot A\neq 0$ and $[A,B]=A\cdot B-B\cdot A$ is called the commutator of A and B . If [A, B] = 0, and B are said to be commuting.

v) The n times multiplication of an operator A by it self is A . A . A . …… A = A^n .

vi) $(A+O)|\ \rangle=(O+A)|\ \rangle=A|\ \rangle$, where O is called the null (or, neutral) operator.

vii) $I|\ \rangle=|\ \rangle$, I is called the identiy operator and for any A, AI = IA = A.

viii) $(|\nu\rangle A)|w\rangle=(|\nu\rangle A)|w\rangle=(|\nu\rangle A)|w\rangle$, the operator may act in both ways.

ix) $A_L^{-1}\ A=I$, the left inverse of $A; AA_R^{-1}=I$, the right inverse of A.

If both A_L^{-1} and A_R^{-1} exist, $A_L^{-1}\ (AA_R^{-1})=(A_L^{-1}\ A)\ A_R^{-1}\Rightarrow A_L^{-1}=A_R^{-1}\equiv A^{-1}$.

x) $\langle w|A^+|\nu\rangle=\langle\nu|A|w\rangle$, A^+ is called the “adjoint” of the operator A.

So, $\big(A|\ \rangle\big)=\langle\ |A^+$ is the dual of the vector $A|\ \rangle$.

For example, $\langle \nu | I^{+} | w \rangle = \langle \nu | I | w \rangle = \langle w | \nu \rangle = \langle w | I | \nu \rangle \Rightarrow I^{+} = I.$

One can also show that

xi) There is very special class of operators, which has remarkable properties:

An operator is called self – adjoint or Hermitian if it is equal to its own adjoint, $H = H^{+}$.

xii) If the inverse of an operator equals to its own adjoint, then it is called unitary, $U^{+}=U^{-1}$.

The special Hermitian and Unitary operators are commonly encountered in applications.

Norm of a Linear Operator and Boundedness:

Consider a vector $|\nu\rangle \in V, \| \nu \rangle \| \neq 0$, where V is afinite or infinite dimensional vector space and an operator B which acts in this space. The operator is said to be bounded if there exists a real constant β such that $\| B|\nu\rangle \| \leq \beta \| \nu\rangle \|$ and the smallest value of this constant is called the norm of the operator. Then, the norm of B is defined as

$\|B\| = \sup \frac{\|B|\nu\rangle\|}{\|\nu\rangle\|} = \sup \frac{\|B|\nu\rangle\|}{\sqrt{\langle \nu|\nu\rangle}}$, where sup denotes the “least upper bound”.

3.2 Eigenvalue Equations

Equations of type $A|\nu\rangle = \lambda|\nu\rangle$ are called eigenvalue equations, where λ is an arbitrary scalar.

The complex or real scalar λ is called an eigenvalue of A and $|\nu\rangle$ is called an eigenvector of A.

Given a particular operator, the solution, if any, to the eigenvalue equation give rise to a set of eigenvalues $\{\lambda_n\}$ and a set of eigenvectors $\{|\nu_n\rangle\}$, whose number of elements dependt on the dimension of the vector space.

.

$$U|\nu\rangle = \lambda|\nu\rangle \Rightarrow \langle \nu| U^{+} = \lambda^{*} \langle \nu| \Rightarrow \langle \nu|U^{+}U|\nu\rangle = \langle \nu|\lambda^{*}\lambda|\nu\rangle \Rightarrow \langle \nu|\nu\rangle = |\lambda|^{2} \langle \nu|\nu\rangle \Rightarrow |\lambda|^{2} = 1 \Rightarrow \lambda = e^{i\beta}$$

, where $\beta \in IR$ is a free parameter.

3.2.1 Theorem : All eigenvalues of a Hermitian operator are real and the eigenvectors that

correspond to different eigenvalues are orthogonal.

Proof:

Let $|h_1\rangle$ and $|h_2\rangle$ be the two eigenvectors for any two eigenvalues $h_1 \neq h_2$ respectively.

Then,

$H|h_1\rangle = h_1|h_1\rangle$

$H|h_2\rangle = h_2|h_2\rangle$

$\Rightarrow \langle h_1|H|h_1\rangle = \langle h_1|h_1|h_1\rangle = h_1\langle h_1|h_1\rangle$

$\Rightarrow \langle h_1|H|h_1\rangle = h_1\langle h_1|h_1\rangle$. But, since $\langle h_1|H|h_1\rangle(= h_1\langle h_1|h_1\rangle)$ $= \langle h_1|H|h_1\rangle = h_1\langle h_1|h_1\rangle \Rightarrow h_1 = \bar{h}_1$.

On the other had,

$\langle h_2|H|h_2\rangle = \langle h_2|h_1|h_1\rangle = h_1\langle h_2|h_1\rangle$ and

$\langle h_1|H|h_2\rangle = \langle h_1|h_2|h_2\rangle = h_2\langle h_1|h_2\rangle \Rightarrow \langle h_2|H|h_1\rangle = \langle h_1|H|h_2\rangle = h_2\langle h_1|h_2\rangle = h_2\langle h_2|h_1\rangle$

$\Rightarrow (h_1 - h_2)\langle h_2|h_1\rangle = 0 \Rightarrow \langle h_2|h_1\rangle = 0$, since $h_1 \neq h_2$.

Above each eigenvalue there corresponds a single eigenvector.

But, there are some cases where to a single eigenvalue, there correspond many eigenvectors.

In this case, the particular eigenvalue and the corresponding eigenvectors are called degenerate .

Then, the last equality above does not necessarily imply the orthogonality of the eigenvectors.

However, by Gramm - Schmidtt process, it is seen that Hermitian operators can always admit an orthogonal set as the set of eigenvectors.

The number of linearly independent eigenvectors of a Hermitian operator is exactly the dimension of the space.

3.3 Linear Differential Operators

3.3.1 Definition: An operator which acts on a function space as

$$\left\langle x\middle|D^n\middle|f\right\rangle \equiv D_x^n\ f(x) = a_n(x)\frac{d^n f}{dx^n}(x) + a_{n-1}(x)\ \frac{d^{n-1}f}{dx^{n-1}}(x) + a_{n-2}(x)$$

$\frac{d^{n-2}f}{dx^{n-2}}(x) + ... + a_1(x)\frac{df}{dx}(x) + a_0(x)f(x)$ is called a linear differential operator. In $L^2(a,b)$,

$$\left\langle x\middle|D^n\middle|f\right\rangle \equiv \int_a^b \left\langle x\middle|D\middle|x'\right\rangle\left\langle x'\middle|f\right\rangle dx' = \int_a^b \delta(x-x')D_{x'}f(x')dx' = D_x f(x),\text{ where } a < x < b.$$

Example: Show that D^2 f(x) = a_1 (x) $\frac{df}{dx}(x)\frac{df}{dx}(x)$ + a_0 (x) f(x) is not linear.

$$D^2[f(x)+g(x)] = a_1(x)\frac{d(f+g)}{dx}\ (x)\frac{d(f+g)}{dx}(x) + a_0(x)(f(x)+g(x)) \neq D f(x) + D g(x).$$

3.3.2 Adjoint of a Differential Operator D in L^2 (a,b):

$$D \equiv a_1(x)\frac{d}{dx} + a_0(x) \rightarrow \left\langle g\middle|D^+\middle|f\right\rangle = \left\langle f\middle|D\middle|g\right\rangle$$

$$\left\langle f\middle|D\middle|g\right\rangle = \int_a^b\int_a^b \left\langle f\middle|x'\right\rangle\left\langle x'\middle|D\middle|x\right\rangle\left\langle x\middle|g\right\rangle dx = \int_a^b f^*\left(a_1\frac{dg}{dx} + a_0 g\right)dx\ .$$

$$\int_a^b f(a_1\frac{dg}{dx} + a_0 g)\,dx = \int_a^b \left(\frac{d(f^* a_1 g)}{dx} - g\frac{d(f^* a_1)}{dx} + a_0 fg\right)dx =$$

$$\int_a^b g - \frac{d(f^* a_1)}{dx} + a_0 f^*)\,dx + (a_1 f^* g)\Big|_a^b$$

Taking the complex conjugate of both sides

$$\left\langle f\middle|D\middle|g\right\rangle = \int_a^b g^*\left(-\frac{d(fa_1^*)}{dx} + a_0^* f\right)dx + (a_1^* f g^*)\Big|_a^b = \left\langle g\middle|D^+\middle|f\right\rangle$$

$$\Rightarrow D^+ f \equiv -\frac{d(a_1^* f)}{dx} + a_0^* f$$

3.3.3 Second Order Linear Differential Operators

In general, a second order linear differential operator is given by

$D = a_2(x)\frac{d^2}{dx^2} + a_1\frac{d}{dx} + a_0(x)$ with real coefficients.

Subject to an inner product

$\langle g|f\rangle = \int_a^b g^*(x) f(x)\, dx$, in L^2 (a, b);

its adjoint form is determined by

$\langle f|D^+|g\rangle = \langle g|D|f\rangle$. But, due to the integration, some integration constants may pop up and hence one considers another definition which suits to these cases:

$\langle f|D^+|g\rangle = \langle g|D|f\rangle + I(f^*(x), g(x), a_2(x), a_1(x), a_0(x))\Big|_a^b$.

$$\langle g|D|f\rangle = \int_a^b dxg\left(a_2 \frac{d^2 f^*}{dx^2} + a_1 \frac{df^*}{dx} + a_0 f^*\right);\text{ by partial integration}$$

$$\langle g|D|f\rangle = \int_a^b dx\left(\frac{d}{dx}\left(ga_2 \frac{df^*}{dx}\right) - \frac{d}{dx}(ga_2)\frac{df^*}{dx} + \frac{d}{dx}(ga_1 f^*) - \frac{d}{dx}(a_1 g)f^* + ga_0 f^*\right)$$

$$= \left(ga_2 \frac{df^*}{dx} + ga_1 f^*\right)\Bigg|_a^b + \int_a^b dx\left(-\frac{d}{dx}\frac{d(ga_2)}{dx} f^*\right) + f^* \frac{d^2}{dx^2}(ga_2) - \frac{d}{dx}(a_1 g)f^* + ga_0 f^*$$

$$= \left(ga_2 \frac{df^*}{dx} - \frac{d(ga_2)}{dx} f^* + ga_1 f^* - ga_2 f^*\right)\Bigg|_a^b + \int_a^b dxf^*\left(\frac{d^2}{dx^2}(a_2 g) - \frac{d}{dx}(a_1 g) + a_0 g\right)$$

$$= \langle f|D^+|g\rangle + I(f^*(x) g(x), a_2(x), a_1(x), a_0(x))\Big|_a^b$$

$$\Rightarrow D^+ = \frac{d^2}{dx^2}(a_2 .) - \frac{d}{dx}(a_1 .) + a_0$$

$$I(f^*(x), g(x), a_2(x), a_1(x), a_0(x))\Bigg|_a^b = \left(ga_2 \frac{df^*}{dx} - a_2 f^* \frac{dg}{dx} - f^* g \frac{da_2}{dx} + ga_1 f^*\right)\Bigg|_a^b$$

3.3.4 Self Adjoint Differential Operators

A differential operators is said to be self – adjoint if $D^+ = D$. Consider again the second order linear differential operator. If it is self – adjoint

$$D^+ = D \Rightarrow a_2 \frac{d^2 f}{dx^2} + a_1 \frac{df}{dx} + a_0 f = \frac{d^2}{dx^2}(a_2 f) - \frac{d}{dx}(a_1 f) + a_0 f$$

$$= a_2 \frac{d^2 f}{dx^2} + \left(2\frac{da_2}{dx} - a_1\right)\frac{df}{dx} + \left(\frac{d^2 a_2}{dx^2} - \frac{da_1}{dx} + a_0\right) f$$

$$\Rightarrow 2\frac{da_2}{dx} - a_1 = a_1 \text{ and } \frac{d^2 a_2}{dx^2} - \frac{da_1}{dx} + a_0 = a_0 .$$

$a_1 = \frac{da_2}{dx} \Rightarrow a_2(x) = \int a_1(x)\, dx$. Second equation is the differentiation of the first one.

Therefore, the self – adjoint differential operator is given by

$$D = D^{+} = a_2 \frac{d^2}{dx^2} + \frac{da_2}{dx}\frac{d}{dx} + a_0 = \frac{d}{dx}\left(a_2 \frac{d}{dx} \right) + a_0 .$$

The boundary terms now become

$$I(f^*(x), g(x), a_2(x), a_1(x))\Big|_a^b = \left(a_2 g \frac{df^*}{dx} - a_2 f^* \frac{dg}{dx} - f^* g \frac{da_2}{dx} + g a_1 f^* \right)\Bigg|_a^b$$

$$= \left[a_2 \left(g \frac{df^*}{dx} - f^* \frac{dg}{dx} \right) \right]\Bigg|_a^b .$$

Precisely talking,

When I $\neq 0$, the differential operator is said to be formally self – adjoint;

When I = 0, it is said to be totally or completely self – adjoint.

The operator D, by definition, acts on L^2 (a, b).

The procedure can be carried to L^2_w (a, b) in the following way:

Inner product for L^2_w (a, b) is given by

$\langle g|f \rangle = \int_a^b g^*(x) f(x) w(x) dx,$ where w(x)>0. Then, the adjoint of the second order linear

differenti al operators are given by

$$\langle f|D^{+}|g\rangle = \langle g|D|f \rangle + I(f^*(x), g(x), a_2(x), a_1(x), a_0(x))\Big|_a^b$$

$$= \int_a^b dxwg\left(a_2 \frac{d^2 f^*}{dx^2} + a_1 \frac{df^*}{dx} + a_0 f^* \right) = \int_a^b dxg\left((wa_2)\frac{d^2 f^*}{dx^2} + (wa_1)\frac{df^*}{dx} + (wa_0) f^* \right)$$

$= \int_a^b dxg\left(b_2 \frac{d^2 f^*}{dx^2} + b_1 \frac{df^*}{dx} + b_0 f^* \right)$; similar calculation ns give

$$= \left(gb_2 \frac{df^*}{dx} - \frac{d(gb_2)}{dx} f^* + gb_1 f^* - \frac{db_2}{dx} gf^* \right)\Bigg|_a^b + \int_a^b dxwf^* \frac{1}{w}\left(\frac{d^2}{dx^2}(b_2 g) - \frac{d}{dx}(b_1 g) + b_0 g \right)$$

$$\Rightarrow D^{+} = \frac{1}{w}\left(\frac{d^2}{dx^2}(b_2 .) - \frac{d}{dx}(b_1 .) + b_0 \right),$$

$$I(f^*(x), g(x), a_2(x), a_1(x), a_0(x))\Big|_a^b = \left(gb_2 \frac{df^*}{dx} - \frac{d(gb_2)}{dx} f^* + gb_1 f^* - \frac{db_2}{dx} df^* \right)\Bigg|_a^b ,$$

where

$$D = a_2 \frac{d^2}{dx^2} + a_1 \frac{d}{dx} + a_0 = \frac{1}{w}\left(b_2 \frac{d^2}{dx^2} + b_1 \frac{d}{dx} + b_0 \right).$$

The self – adjoint operator is obtained via $D^+ = D$ as

$D^+ = D = \frac{1}{w}\frac{d}{dx}\left(b_2 \frac{d}{dx} \right) + \frac{b_0}{w}$ together with

$$I(f^*(x), g(x), b_2(x))\Big|_a^b = \left(b_2 g \frac{df^*}{dx} - b_2 f^* \frac{dg}{dx} \right)\Bigg|_a^b$$

3.4 Sturm – Liouville Operators

The completely self – adjoint (I=0) second order linear differential operators are called to be the Sturm – Liouville operators.

3.4.1 Sturm – Liouville Equation

Since Sturm – Liouville operators are used commonly in applications, it is better to use a particular notation for convenience:

$S^+ = S = \frac{1}{w}\frac{d}{dx}\left(p \frac{d}{dx} \right) + \frac{q}{w}$. The Sturm – Liouville Equation is the eigenvalue equation given by $S\Psi_\lambda = \lambda \Psi_\lambda$.

Given two arbitrary eigenvalues λ_1 and λ_2,

the eigenvectors are subject to the boundary conditions below:

$$I(\Psi^*_{\lambda_1}, \Psi_{\lambda_2}, p(x))\Big|_a^b = \left(p\Psi^*_{\lambda_1} \frac{d\Psi_{\lambda_2}}{dx} - p\Psi_{\lambda_2} \frac{d\Psi^*_{\lambda_1}}{dx} \right)\Bigg|_a^b = 0.$$

Since the Sturm - Liouville operators are self - adjoint (Hermitian),

all the properties of the Hermitian operators discussed in our previous lectures are also valid for them:

1. The eigenvalues of the Sturm - Liouville operators are all real;
2. Eigenvectors that correspond to different eigenvalues are orthogonal;
3. The eigenvectors of the Sturm - Liouville operators form a basis for L^2_w (a, b).

3.4.2 Boundary Conditions : The boundary conditions for the SL equation are satisfied for the well known types:

1) $\Psi_\lambda(a) = \Psi_\lambda(b) = 0$ **(Dirichlet conditions)**

2) $\left.\frac{d\Psi_\lambda(x)}{dx}\right|_a = \left.\frac{d\Psi_\lambda(x)}{dx}\right|_b = 0$ **(Neumann conditions)**

3) $\Psi_\lambda(a)=\Psi_\lambda(b),\ \left.\frac{d\Psi_\lambda(x)}{dx}\right|_a = \left.\frac{d\Psi_\lambda(x)}{dx}\right|_b$ **(Periodic boundary conditions)**

4) $\alpha\Psi_\lambda(a)-\left.\frac{d\Psi_\lambda(x)}{dx}\right|_a = \beta\Psi_\lambda(b)-\left.\frac{d\Psi_\lambda(x)}{dx}\right|_b$ **(General unmixed conditions)**

These boundary conditions are implied directly by applications one studies.

As an example consider the wave equation $\nabla^2\Psi-\frac{1}{c^2}\frac{\partial^2\Psi}{\partial t^2}=0$. By separation of variables

$$\Psi(\vec{x},t)=X(\vec{x})T(t)\Rightarrow\frac{\nabla^2 X}{X}=\frac{1}{c^2}\frac{1}{T}\frac{d^2T}{dt^2}=-k^2;$$

1) $\frac{d^2T}{dt^2}+\omega^2T=0,\omega=kc$ **(Harmonic Oscillator);**

2) $\nabla^2X+k^2X=0$ **(Helmholtz Equation).**

Let us study now the solutions to the wave equation for some particular boundary conditions.

Example:

i) Harmonic Oscillator: $\frac{d^2T_\omega}{dt^2}=-\omega^2T_\omega,0<\omega,$ e.g., with boundary conditions $T_\omega(0)=T_\omega(l)=0$

The boundary conditions above are of Dirichlet type.

This is of SL type with identification $p=1,q=0,\lambda=-\omega^2,w=1$ and $I\big|_0^l=0$

The general solution is $T_\omega(t)=A\mathrm{Sin}\omega t$. The boundary conditions imply that

$$\mathrm{Sin}\omega l=0\Rightarrow\omega=\frac{n\pi}{l}>0,n=0,\pm1,+\pm2,+3,.......$$

So, the eigenfunctions $T_n(x)=A\mathrm{Sin}\left(\frac{n\pi t}{l}\right)$ form a basis for $L^2(0,l)$ and real continous functions can be expanded in [0,l] in terms of these. The basis is orthogonal since

$$\langle T_m|T_n\rangle=\int_0^l dtT_m{}^*T_n=\begin{cases}0,m\neq n\\ \frac{1}{2}A^2,m=n\end{cases},$$ and one can make it orthonormal

by choosing A = $\sqrt{\dfrac{2}{l}}$

$$\rightarrow T_m(t) = \sqrt{\frac{2}{1}}\,\text{Sin}\left(\frac{m\pi t}{l}\right).$$

$$\left|f\right\rangle = \sum_{n=1}^{\infty} a_n \left|T_n\right\rangle \text{or}\, f(t) = \sum_{n=1}^{\infty} a_n T_n(t)$$

$$\left\langle T_m \middle| f\right\rangle = \sum_{n=1}^{\infty} a_n \left\langle T_m \middle| T_n\right\rangle = \sum_{n=1}^{\infty} a_n \int_0^1 dt \sqrt{\frac{2}{1}}\,\text{Sin}\left(\frac{m\pi t}{l}\right) = \sum_{n=1}^{\infty} a_n \delta_{mn} = \alpha_m$$

$$\alpha_n = \left\langle T_n \middle| f\right\rangle = \sqrt{\frac{2}{1}}\int_0^1 f(t)\,\text{Sin}\left(\frac{n\pi t}{l}\right) dt \text{ or making a redefinition } a_n\sqrt{\frac{2}{1}} = b_n$$

$\Rightarrow f(t) = \displaystyle\sum_{n=1}^{\infty} b_n \,\text{Sin}\left(\frac{n\pi t}{1}\right); b_n = \frac{2}{l}\int_0^1 f(t)\,\text{Sin}\left(\frac{n\pi t}{l}\right) dt$, which is the Finite Fourier Sine Series.

In Practice, the interaction of two fields in this space given by
$\left\langle h \middle| g\right\rangle$ can now be calculated in terms of their Fourier modes

$$\left\langle h \middle| g\right\rangle = \int_0^1 dt \sum_{n=1}^{\infty} c_n \text{Sin}\left(\frac{n\pi t}{l}\right) \sum_{m=1}^{\infty} d_m \text{Sin}\left(\frac{m\pi t}{l}\right)$$

$$= \sum_{n,m=1}^{\infty} c_n d_m \int_0^1 dt\text{Sin}\left(\frac{n\pi t}{l}\right)\text{Sin}\left(\frac{m\pi t}{l}\right) = \sum_{n,m=1}^{\infty} c_n d_m \frac{1}{2}\delta_{nm} = \frac{1}{2}\sum_{n=1}^{\infty} c_n d_n$$

ii) Helmholtz Equation: $\nabla^2 X + k^2 X = 0$. In spherical coordinates, by separation of variables $X(r,\theta,\phi) = R(r)\oplus(\theta)\Phi(\phi)$

$\Rightarrow \dfrac{1}{\Phi}\dfrac{d^2\Phi}{d\phi^2}$ =cons tant t $\equiv -m^2$ (SL type)$\Rightarrow \Phi = e^{\pm im\phi}, 0 \le \phi < 2\pi$, (fundamental solutions); these span the space $L^2(0,2\pi)$ which we have discussed extensively.

$$x \equiv \text{Cos}\theta, 0 \le \theta \le \pi \Rightarrow (1-x^2)\frac{d^2\oplus}{dx^2} - 2x\frac{d\oplus}{dx} + \left[n(n+1) - \frac{m^2}{1-x^2}\right]\oplus = 0, -1 \le x \le 1;$$

associated Legendre equation (Exercise: This is again of SL type (w=?, etc.).)

whose fundemental solutions are

$\oplus = P_n^m(x)$ and $Q_n^m(x)$, the associated Legendre functions and these span L^2 (-1,1).

When m=0, it turns out to be the case we have studied previously. The orthogonality of the associated Legendre functions can be seen via the product

$$\int_{-1}^{1} dx P_n^m(x) P_{n'}^m(x) = \frac{(n+m)!2}{(n-m)!2n+1}\delta_{nn'},$$

$$R(r) \equiv B(r)/\sqrt{r} \Rightarrow \frac{d^2B}{dr^2} + \frac{1}{r}\frac{dB}{dr} + \left[k^2 - \frac{\left(n+\frac{1}{2}\right)^2}{r^2}\right]B = 0,$$ Bessel's equation; (SL type: exercise)

$\rightarrow B(r) = J_{n+\frac{1}{2}}(kr)$ and $Y_{n+\frac{1}{2}}(kr)$, Bessel's functions.

Any function on the interval 0<r<r_o can be expanded in terms of $J_{n+\frac{1}{2}}(kr)$ and the corresponding series is called the Bessel or Fourier-Bessel series. So, these span $L_w^2(0, r_o)$, where $w(r) = r$.

$$f(r) = \int_0^\infty kdkF(k)J_{n+\frac{1}{2}}(kr), F(k) = \int_0^\infty rdrf(r)J_{n+\frac{1}{2}}(kr).$$

Consuquently, if a differential operator is of SL type

$$S^+ = S = \frac{1}{w}\frac{d}{dx}\left(p\frac{d}{dx}\right) + \frac{q}{w},$$

then, the eigenfunctions ψ_λ to the Sturm-Liouville Eigenvalue Equation given by $S\psi_\lambda = \lambda\psi_\lambda$ together with the boundary conditions

$$I(\psi_{\lambda_1}^*, \psi_{\lambda_2}, p(x))\Big|_a^b = \left(p\psi_{\lambda_1}^*\frac{d\psi_{\lambda_2}}{dx} - p\psi_{\lambda_2}\frac{d\psi_{\lambda_1}^*}{dx}\right)\Bigg|_a^b = 0$$

constitute a basis for $L^2_w(a,b)$ and real continous functions can be expanded in terms of these eigenfunctions in the given interval.

$$f(x)=\sum_{n=0}^{\infty}\alpha_n\psi_n(x),\alpha_n=\frac{\int_a^b f(x)\psi_n^*(x)w(x)dx}{\int_a^b \psi_n(x)\psi_n^*(x)w(x)dx}$$